Lebesgue measure

Math 212a

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1 Lebesgue outer measure.

For any $A \subset \mathbf{R}$ we have defined the Lebesgue outer measure of A as

$$m^*(A) := \inf \sum_i \ell(I_i), \quad A \subset \bigcup I_i \tag{1}$$

where the infimum is over all countable covers of A by intervals. These intervals could be taken as open, closed or half open without changing the definition. If we take them all to be half open, of the form $I_i = [a_i, b_i)$, we can write each I_i as a disjoint union of finite (or countably many) intervals each of length $< \epsilon$. So it makes no difference to the definition if we also require the

$$\ell(I_i) < \epsilon \tag{2}$$

in (1). We will see that when we pass to other types of measures this will make a difference.

We have verified, or can easily verify the following properties:

1.
$$m^*(\emptyset) = 0.$$

2.
$$A \subset B \Rightarrow m^*(A) \leq m^*(B).$$

3.
$$m^*(\bigcup_i A_i) \le \sum_i m^*(A_i).$$

4. If dist $(A,B) > 0$ then
$$m^*(A \cup B) = m^*(A) + m^*(B).$$

5.
$$m^*(A) = \inf\{m^*(U) : \ U \supset A, \quad U \text{ open}\}.$$

6. For an interval
$$m^*(I) = \ell(I).$$

The only items that we have not done in class are items 4 and 5. But these are immediate: for 4 we may choose the intervals in (1) all to have length $< \epsilon$ where $2\epsilon <$ dist (A,B) so that there is no overlap. As for item 5, we know from 2 that $m^*(A) \le m^*(U)$ for any set U, in particular for any open set U which contains A. We must prove the reverse inequality: if $m^*(A) = \infty$ this is trivial. Otherwise, we may take the intervals in (1) to be open and then the union on the right is an open set whose Lebesgue outer measure is less than $m^*(A) + \delta$ for any $\delta > 0$ if we choose a close enough approximation to the infimum.

I should also add that all the above works for $\mathbf{R}^n$ instead of $\mathbf{R}$ if we replace the word "interval" by "rectangle", meaning a rectangular parallelepiped, i.e a set which is a product of one dimensional intervals. We also replace length by volume (or area in two dimensions). What is needed is the following

Lemma 1 *Let $\mathcal{C}$ be a finite non-overlapping collection of closed rectangles all contained in the closed rectangle J. Then*
$$vol\ J \ge \sum_{I \in \mathcal{C}} vol\ I.$$

If $\mathcal{C}$ is any finite collection of rectangles such that
$$J \subset \bigcup_{I \in calC} I$$

then
$$vol\ J \le \sum_{I \in \mathcal{C}} vol\ (I).$$

This lemma occurs on page 1 of Strook, *A concise introduction to the theory of integration* together with its proof. I will take this for granted. In the next few paragraphs I will talk as if we are in $\mathbf{R}$, but everything goes through unchanged if $\mathbf{R}$ is replaced by $\mathbf{R}^n$.

2 Lebesgue inner measure.

Item 5. in the preceding paragraph says that the Lebesgue outer measure of any set is obtained by approximating it from the outside by open sets. The **Lebesgue inner measure** is defined as

$$m_*(A) = \sup\{m^*(K): \ K \subset A, \ K \text{ compact }\}. \tag{3}$$

Clearly

$$m_*(A) \leq m^*(A)$$

since $m^*(K) \leq m^*(A)$ for any $K \subset A$. We also have

Proposition 1 *For any interval I we have*

$$m_*(I) = \ell(I). \tag{4}$$

Proof. If $\ell(I) = \infty$ the result is obvious. So we may assume that I is a finite interval which we may assume to be open, $I = (a, b)$. If $K \subset I$ is compact, then I is a cover of K and hence from the definition of outer measure $m^*(K) \leq \ell(I)$. So $m_*(I) \leq \ell(I)$. On the other hand, for any $\epsilon > 0, \ \epsilon < \frac{1}{2}(b-a)$ the interval $[a+\epsilon, b-\epsilon]$ is compact and $m^*([a-\epsilon, a+\epsilon]) = b - a - 2\epsilon \leq m_*(I)$. Letting $\epsilon \to 0$ proves the proposition. QED

3 Lebesgue's definition of measurability.

A set A with $m^*(A) < \infty$ is said to **measurable in the sense of Lebesgue** if

$$m_*(A) = m^*(A). \tag{5}$$

If A is measurable in the sense of Lebesgue, we write

$$m(A) = m_*(A) = m^*(A). \tag{6}$$

If K is a compact set, then $m_*(K) = m^*(K)$ since K is a compact set contained in itself. Hence all compact sets are measurable in the sense of Lebesgue. If I is a bounded interval, then I is measurable in the sense of Lebesgue by Proposition 1.

If $m^*(A) = \infty$, we say that A is measurable in the sense of Lebesgue if all of the sets $A \cap [-n, n]$ are measurable.

If F is closed, then $F \cap [-n, n]$ is compact. Hence all closed sets are measurable in the sense of Lebesgue.

Proposition 2 *If $A = \bigcup A_i$ is a (finite or) countable disjoint union of sets which are measurable in the sense of Lebesgue, then A is measurable in the sense of Lebesgue and*

$$m(A) = \sum_i m(A_i).$$

Proof. We may assume that $m(A) < \infty$ - otherwise apply the result to $A \cap [-n, n]$ and $A_i \cap [-n, n]$ for each n. We have

$$m^*(A) \leq \sum_n m^*(A_n) = \sum_n m(A_n).$$

Let $\epsilon > 0$, and for each n choose compact $K_n \subset A_n$ with

$$m^*(K_n) \geq m_*(A_n) - \frac{\epsilon}{2^n} = m(A_n) - \frac{\epsilon}{2^n}$$

since A_n is measurable in the sense of Lebesgue. The sets K_n are pairwise disjoint, hence, being compact, at positive distances from one another. Hence

$$m^*(K_1 \cup \cdots \cup K_n) = m^*(K_1) + \cdots + m^*(K_n)$$

and $K_1 \cup \cdots \cup K_n$ is compact and contained in A. Hence

$$m_*(A) \geq m^*(K_1) + \cdots + m^*(K_n),$$

and since this is true for all n we have

$$m_*(A) \geq \sum_n m(A_n) - \epsilon.$$

Since this is true for all $\epsilon > 0$ we get

$$m_*(A) \geq \sum m(A_n).$$

But then $m_*(A) \geq m^*(A)$ and so they are equal, so A is measurable in the sense of Lebesgue, and $m(A) = \sum m(A_i)$. QED

Any open set can be written as the countable union of open intervals I_i, and $I_n \setminus \bigcup_{i=1}^{n-1} I_i$ is a disjoint union of intervals (some open, some closed, some half open) and so every open set is a disjoint union of intervals hence measurable in the sense of Lebesgue.

Theorem 1 *A is measurable in the sense of Lebesgue if and only if for every $\epsilon > 0$ there is an open set $U \supset A$ and a closed set $F \subset A$ such that*

$$m(U \setminus F) < \epsilon.$$

Proof. Suppose that A is measurable with $m(A) < \infty$. Then there is an open set $U \supset A$ with $m(U) < m^*(A) + \epsilon/2 = m(A) + \epsilon/2$, and there is a compact set F with $m(F) \geq m_*(A) - \epsilon = m(A) - \epsilon/2$. Since $U \setminus F$ is open, it is measurable in the sense of Lebesgue, and so is F as it is compact. Also F and $U \setminus F$ are disjoint. Hence by Proposition 2,

$$m(U \setminus F) = m(U) - m(F) < m(A) + \frac{\epsilon}{2} - \left(m(A) - \frac{\epsilon}{2}\right) = \epsilon.$$

If A is measurable in the sense of Lebesgue and $m(A) = \infty$ we apply the above to each of the sets $A \cap [-n.n]$ to obtain $U_n \supset A \cap [-n, n]$ and $F_n \subset A \cap [-n, n]$

with $m(U_n \setminus F_n) < \epsilon/2^n$. We may also enlarge each F_n if necessary so that $F_n \cap [-n+1, n-1] \supset F_{n-1}$ keeping the F_n compact. We may also decrease the U_n if necessary so that $U_n \cap (-n-1-\delta_n, n-1+\delta_n) \subset U_{n-1}$ for some sufficiently small δ_n. We take $U = \bigcup U_n$. Then U is open. Suppose we take

$$F = \bigcup F_n \cap ([-n, -n+1] \cup [n-1, n]).$$

Then F is closed, $U \supset A \supset F$ and

$$U \setminus F \subset \bigcup (U_n \setminus F_n) \cap ([-n, -n-1] \cup [n-1, n]) \subset \bigcup (U_n \setminus F_n)$$

so $m(U \setminus F) < \epsilon$.

In the other direction, suppose that for each ϵ, there exist $U \supset A \supset F$ with $m(U \setminus F) < \epsilon$. Suppose that $m^*(A) < \infty$. Then $m(F) < \infty$ and $m(U) \leq m(U \setminus F) + m(F) < \epsilon + m(F) < \infty$. Then

$$m^*(A) \leq m(U) < m(F) + \epsilon = m_*(F) + \epsilon \leq m_*(A) + \epsilon.$$

Since this is true for every $\epsilon > 0$ we conclude that $m_*(A) \geq m^*(A)$ so they are equal and A is measurable in the sense of Lebesgue.

If $m^*(A) = \infty$, we have $U \cap (-n-\epsilon, n+\epsilon) \supset A \cap [-n, n] \supset F \cap [-n, n]$ and

$$m((U \cap (-n-\epsilon, n+\epsilon)/(F \cap [-n, n]) < 2\epsilon + \epsilon = 3\epsilon$$

so we can proceed as before to conclude that $m_*(A \cap [-n, n]) = m^*(A \cap [-n, n])$. QED

Several facts emerge immediately from this theorem:

Proposition 3 *If A is measurable in the sense of Lebesgue, so is its complement $A^c = \mathbf{R} \setminus A$.*

Indeed, if $F \subset A \subset U$ with F closed and U open, then $F^c \supset A^c \supset U^c$ with F^c open and U^c closed. Furthermore, $F^c \setminus U^c = U \setminus F$ so if A satisfies the condition of the theorem so does A^c.

Proposition 4 *If A and B are measurable in the sense of Lebesgue so is $A \cap B$.*

For $\epsilon > 0$ choose $U_A \supset A \supset F_A$ and $U_B \supset B \supset F_B$ with $m(U_A \setminus F_A) < \epsilon/2$ and $m(U_B \setminus F_B) < \epsilon/2$. Then $(U_A \cap U_B) \supset (A \cap B) \supset (F_A \cap F_B)$ and $(U_A \cap U_B) \setminus (F_A \cap F_B) \subset (U_A \setminus F_A) \cup (U_B \setminus F_B)$.QED

Putting the previous two propositions together gives

Proposition 5 *If A and B are measurable in the sense of Lebesgue then so is $A \cup B$.*

Indeed, $A \cup B = (A^c \cap B^c)^c$.

Since $A \setminus B = A \cap B^c$ we also get

Proposition 6 *If A and B are measurable in the sense of Lebesgue then so is $A \setminus B$.*

4 Caratheodory's definition of measurability.

A set $E \subset \mathbf{R}$ is said to be **measurable according to Caratheodory** if for any set $A \subset \mathbf{R}$ we have

$$m^*(A) = m^*(A \cap E) + m^*(A \cap E^c) \tag{7}$$

where we recall that E^c denotes the complement of E. In other words, $A \cap E^c = A \setminus E$. This definition has many advantages, as we shall see. Our first task is to show that it is equivalent to Lebesgue's:

Theorem 2 *A set E is measurable in the sense of Caratheodory if and only if it is measurable in the sense of Lebesgue.*

Proof. We always have

$$m^*(A) \leq m^*(A \cap E) + m^*(A \setminus E)$$

so condition (7) is equivalent to

$$m^*(A \cap E) + m^*(A \setminus E) \leq m^*(A) \tag{8}$$

for all A.

Suppose E is measurable in the sense of Lebesgue. Let $\epsilon > 0$. Choose $U \supset E \supset F$ with U open, F closed and $m(U/F) < \epsilon$ which we can do by Theorem 1. Let V be an open set containing A. Then $A \setminus E \subset V \setminus F$ and $A \cap E \subset (V \cap U)$ so

$$\begin{aligned} m^*(A \setminus E) + m^*(A \cap E) &\leq m(V/F) + m(V \cap U) \\ &\leq m(V \setminus U) + m(U \setminus F) + m(V \cap U) \\ &\leq m(V) + \epsilon. \end{aligned}$$

(We can pass from the second line to the third since both $V \setminus U$ and $V \cap U$ are measurable in the sense of Lebesgue and we can apply Proposition 2.) Taking the infimum over all open V containing A, the last term becomes $m^*(A)$, and as ϵ is arbitrary, we have established (8) showing that E is measurable in the sense of Caratheodory.

In the other direction, suppose that E is measurable in the sense of Caratheodory. First suppose that

$$m^*(E) < \infty.$$

Then for any $\epsilon > 0$ there exists an open set $U \supset E$ with $m(U) < m^*(E) + \epsilon$. We may apply condition (7) to $A = U$ to get

$$m(U) = m^*(U \cap E) + m^*(U \setminus E) \geq m^*(E) + m^*(U \setminus E)$$

so

$$m^*(U \setminus E) < \epsilon.$$

This means that there is an open set $V \supset (U \setminus E)$ with $m(V) < \epsilon$. But we know that $U \setminus V$ is measurable in the sense of Lebesgue, since U and V are, and

$$m(U) \leq m(V) + m(U \setminus V)$$

so

$$m(U \setminus V) > m(U) - \epsilon.$$

So there is a closed set $F \subset U \setminus V$ with $m(F) > m(U) - \epsilon$. But since $V \supset U \setminus E$, we have $U \setminus V \subset E$. So $F \subset E$. So $F \subset E \subset U$ and

$$m(U \setminus F) = m(U) - m(F) < \epsilon.$$

Hence E is measurable in the sense of Lebesgue.

If $m(E) = \infty$, we must show that $E \cap [-n, n]$ is measurable in the sense of Caratheodory, for then it is measurable in the sense of Lebesgue from what we already know. We know that $[-n, n]$ being measurable in the sense of Lebesgue is measurable in the sense of Caratheodory. So we will have completed the proof of the theorem if we show that the intersection of E with $[-n.n]$ is measurable in the sense of Caratheodory.

More generally, we will show that the union or intersection of two sets which are measurable in the sense of Caratheodory is again measurable in the sense of Caratheodory. Notice that the definition (7) is symmetric in E and E^c so if E is measurable in the sense of Caratheodory so is E^c. So it suffices to prove the next lemma to complete the proof.

Lemma 2 *If E_1 and E_2 are measurable in the sense of Caratheodory so is $E_1 \cup E_2$.*

For any set A we have

$$m^*(A) = m^*(A \cap E_1) + m^*(A \cap E_1^c)$$

by (7) applied to E_1. Applying (7) to $A \cap E_1^c$ and E_2 gives

$$m^*(A \cap E_1^c) = m^*(A \cap E_1^c \cap E_2) + m^*(A \cap E_1^c \cap E_2^c).$$

Substituting this back into the preceding equation gives

$$m^*(A) = m^*(A \cap E_1) + m^*(A \cap E_1^c \cap E_2) + m^*(A \cap E_1^c \cap E_2^c). \tag{9}$$

Since $E_1^c \cap E_2^c = (E_1 \cup E_2)^c$ we can write this as

$$m^*(A) = m^*(A \cap E_1) + m^*(A \cap E_1^c \cap E_2) + m^*(A \cap (E_1 \cup E_2)^c).$$

Now $A \cap (E_1 \cup E_2) = (A \cap E_1) \cup (A \cap (E_1^c \cap E_2)$ so

$$m^*(A \cap E_1) + m^*(A \cap E_1^c \cap E_2) \geq m^*(A \cap (E_1 \cup E_2).$$

Substituting this into the two terms on the right of the previous displayed equation gives

$$m^*(A) \geq m^*(A \cap (E_1 \cup E_2)) + m^*(A \cap (E_1 \cup E_2)^c)$$

which is just (8) for the set $E_1 \cup E_2$. This proves the lemma and the theorem.

We let $\mathcal{M}$ denote the class of measurable subsets of $\mathbf{R}$ -measurability in the sense of Lebesgue or Caratheodory these being equivalent. Notice by induction starting with two terms as in the lemma, that any finite union of sets in $\mathcal{M}$ is again in $\mathcal{M}$

5 Countable additivity.

The first main theorem in the subject is the following description of $\mathcal{M}$ and the function m on it:

Theorem 3 *$\mathcal{M}$ and the function $m : \mathcal{M} \to \mathbf{R}$ have the following properties:*

- $\mathbf{R} \in \mathcal{M}$.
- $E \in \mathcal{M} \Rightarrow E^c \in \mathcal{M}$.
- *If $E_n \in \mathcal{M}$ for $n = 1, 2, 3, \ldots$ then $\bigcup_n E_n \in \mathcal{M}$.*
- *If $F_n \in \mathcal{M}$ and the F_n are pairwise disjoint, then $F := \bigcup_n F_n \in \mathcal{M}$ and*

$$m(F) = \sum_{n=1}^{\infty} m(F_n).$$

Proof. We already know the first two items on the list, and we know that a finite union of sets in $\mathcal{M}$ is again in $\mathcal{M}$. We also know the last assertion which is Proposition 2. But it will be instructive and useful for us to have a proof starting directly from Caratheodory's definition of measurablity:

If $F_1 \in \mathcal{M},\ F_2 \in \mathcal{M}$ and $F_1 \cap F_2 = \emptyset$ then taking

$$A = F_1 \cup F_2, \quad E_1 = F_1, \quad E_2 = F_2$$

in (9) gives

$$m(F_1 \cup F_2) = m(F_1) + m(F_2).$$

Induction then shows that if $F_1, \ldots, F_n$ are pairwise disjoint elements of $\mathcal{M}$ then their union belongs to $\mathcal{M}$ and

$$m(F_1 \cup F_2 \cup \cdots \cup F_n) = m(F_1) + m(F_2) + \cdots + m(F_n).$$

More generally, if we let A be arbitrary and take $E_1 = F_1,\ E_2 = F_2$ in (9) we get

$$m^*(A) = m^*(A \cap F_1) + m^*(A \cap F_2) + m^*(A \cap (F_1 \cup F_2)^c).$$

If $F_3 \in \mathcal{M}$ is disjoint from F_1 and F_2 we may apply (7) with A replaced by $A \cap (F_1 \cup F_2)^c$ and E by F_3 to get

$$m^*(A \cap (F_1 \cup F_2)^c)) = m^*(A \cap F_3) + m^*(A \cap (F_1 \cup F_2 \cup F_3)^c),$$

since

$$(F_1 \cup F_2)^c \cap F_3^c = F_1^c \cap F_2^c \cap F_3^c = (F_1 \cup F_2 \cup F_3)^c.$$

Substituting this back into the preceding equation gives

$$m^*(A) = m^*(A \cap F_1) + m^*(A \cap F_2) + m^*(A \cap F_2) + m^*(A \cap (F_1 \cup F_2 \cup F_3)^c).$$

Proceeding inductively, we conclude that if $F_1, \ldots, F_n$ are pairwise disjoint elements of $\mathcal{M}$ then

$$m^*(A) = \sum_1^n m^*(A \cap F_i) + m^*(A \cap (F_1 \cup \cdots \cup F_n)^c). \tag{10}$$

Now suppose that we have a countable family $\{F_i\}$ of pairwise disjoint sets belonging to $\mathcal{M}$. Since

$$\left(\bigcup_{i=1}^n F_i\right)^c \supset \left(\bigcup_{i=1}^\infty F_i\right)^c$$

we conclude from (10) that

$$m^*(A) \geq \sum_1^n m^*(A \cap F_i) + m^*\left(A \cap \left(\bigcup_{i=1}^\infty F_i\right)^c\right)$$

and hence passing to the limit

$$m^*(A) \geq \sum_1^\infty m^*(A \cap F_i) + m^*\left(A \cap \left(\bigcup_{i=1}^\infty F_i\right)^c\right).$$

Now given any collection of sets B_k we can find intervals $\{I_{k,j}\}$ with

$$m(B_k) \leq \sum_j \ell(I_{k,j}) + \frac{\epsilon}{2^k}$$

and so

$$\bigcup_k B_k \subset \bigcup_{k,j} I_{k,j}$$

and hence

$$m^*\left(\bigcup B_k\right) \leq \sum m^*(B_k),$$

the inequality being trivially true if the sum on the right is infinite. So

$$\sum_{i=1}^\infty m^*(A \cap F_k) \geq m^*\left(A \cap \left(\bigcup_{i=1}^\infty F_i\right)\right).$$

Thus

$$m^*(A) \geq \sum_1^\infty m^*(A \cap F_i) + m^*\left(A \cap \left(\bigcup_{i=1}^\infty F_i\right)^c\right) \geq$$

$$\geq m^*\left(A \cap \left(\bigcup_{i=1}^\infty F_i\right)\right) + m^*\left(A \cap \left(\bigcup_{i=1}^\infty F_i\right)^c\right).$$

The extreme right of this inequality is the right hand side of (8) applied to

$$E = \bigcup_i F_i,$$

and so $E \in \mathcal{M}$ and the preceding string of inequalities must be equalities since the middle is trapped between both sides which must be equal. Hence we have proved that if F_n is a disjoint countable family of sets belonging to $\mathcal{M}$ then their union belongs to $\mathcal{M}$ and

$$m^*(A) = \sum_i m^*(A \cap F_i) + m^*\left(A \cap \left(\bigcup_{i=1}^\infty F_i\right)^c\right). \tag{11}$$

If we take $A = \bigcup F_i$ we conclude that

$$m(F) = \sum_{n=1}^\infty m(F_n) \tag{12}$$

if the F_j are disjoint and

$$F = \bigcup F_j.$$

So we have reproved the last assertion of the theorem using Caratheodory's definition. For the third assertion, we need only observe that a countable union of sets in $\mathcal{M}$ can be always written as a countable disjoint union of sets in $\mathcal{M}$. Indeed, set

$$F_1 := E_1,\ F_2 := E_2 \setminus E_1 = E_1 \cap E_2^c$$
$$F_3 := E_3 \setminus (E_1 \cup E_2)$$

etc. The right hand sides all belong to $\mathcal{M}$ since $\mathcal{M}$ is closed under taking complements and finite unions and hence intersections, and

$$\bigcup_j F_j = \bigcup E_j.$$

We have completed the proof of the theorem.

A number of easy consequences follow: The symmetric difference between two sets is the set of points belonging to one or the other but not both:

$$A \Delta B := (A \setminus B) \cup (B \setminus A).$$

Proposition 7 *If $A \in \mathcal{M}$ and $m(A\Delta B) = 0$ then $B \in \mathcal{M}$ and $m(A) = m(B)$.*

Proof. By assumption $A \setminus B$ has measure zero (and hence is measurable) since it is contained in the set $A\Delta B$ which is assumed to have measure zero. Similarly for $B \setminus A$. Also $(A \cap B) \in \mathcal{M}$ since

$$A \cap B = A \setminus (A \setminus B).$$

Thus

$$B = (A \cap B) \cup (B \setminus A) \in \mathcal{M}.$$

Since $B \setminus A$ and $A \cap B$ are disjoint, we have

$$m(B) = m(A \cap B) + m(B \setminus A) = m(A \cap B) = m(A \cap B) + m(A \setminus B) = m(A).$$

QED

Proposition 8 *Suppose that $A_n \in \mathcal{M}$ and $A_n \subset A_{n+1}$ for $n = 1, 2, \ldots$. Then*

$$m\left(\bigcup A_n\right) = \lim_{n\to\infty} m(A_n).$$

Indeed, setting $B_n := A_n \setminus A_{n-1}$ (with $B_1 = A_1$) the B_i are pairwise disjoint and have the same union as the A_i so

$$m\left(\bigcup A_n\right) = \sum_{i=1}^{\infty} m(B_i) = \lim_{n\to\infty} \sum_{i=1}^{n} m(B_n) = \lim_{n\to\infty} m\left(\bigcup_{i=1}^{n} B_i\right) = \lim_{n\to\infty} m(A_n).$$

QED

Proposition 9 *If $C_n \supset C_{n+1}$ is a decreasing family of sets in $\mathcal{M}$ and $m(C_1) < \infty$ then*

$$m\left(\bigcap C_n\right) = \lim_{n\to\infty} m(C_n).$$

Indeed, set $A_1 := \emptyset$, $A_2 := C_1 \setminus C_2$, $A_3 := C_1 \setminus C_3$ etc. The A's are increasing so

$$m\left(\bigcup (C_1 \setminus C_i)\right) = \lim_{n\to\infty} m(C_1 \setminus C_n) = m(C_1) - \lim_{n\to\infty} m(C_n)$$

by the preceding proposition. Since $m(C_1) < \infty$ we have

$$m(C_1 \setminus C_n) = m(C_1) - m(C_n).$$

Also

$$\bigcup_n (C_1 \setminus C_n) = C_1 \setminus \left(\bigcap_n C_n\right).$$

So

$$m\left(\bigcup_n (C_1 \setminus C_n)\right) = m(C_1) - m\left(\bigcap C_n\right) = m(C_1) - \lim_{n\to\infty} m(C_n).$$

Subtracting $m(C_1)$ from both sides of the last equation gives the equality in the proposition. QED