Banach Algebras

Math 212

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Contents

In what follows, all rings will be assumed to be associative and to have an identity element, usually denoted by e. If an element x in the ring is such that the $(e-x)$ has a right inverse, then we may write this inverse as $(e-y)$, and the equation

$$(e-x)(e-y)=e$$

expands out to

$$x+y-xy=0.$$

Following Loomis, we call y the right **adverse** of x and x the left adverse of y. Loomis introduces this term because he wants to consider algebras without identity elements. But it will be convenient to use it even under our assumption that all our algebras have an identity. If an element has both a right and left inverse then they must be equal by the associative law, so if x has a right and left adverse these must be equal. When we say that an element has (or does not have) an inverse, we will mean that it has (or does not have) a two sided inverse. Similarly for adverse.

All algebras will be over the complex numbers. The **spectrum** of an element x in an algebra is the set of all $\lambda \in \mathbf{C}$ such that $(x - \lambda e)$ has no inverse. We denote the spectrum of x by $\mathrm{Spec}(x)$.

Proposition 0.1 *If P is a polynomial then*

$$P(\mathrm{Spec}(x)) = \mathrm{Spec}(P(x)). \tag{1}$$

Proof. The product of invertible elements is invertible. For any $\lambda \in \mathbf{C}$ write $P(t) - \lambda$ as a product of linear factors:

$$P(t) - \lambda = c \prod (t - \mu_i).$$

Thus

$$P(x) - \lambda e = c \prod (x - \mu_i e)$$

in A and hence $(p(x) - \lambda e)^{-1}$ fails to exist if and only if $(x - \mu_i e)^{-1}$ fails to exist for some i, i.e. $\mu_i \in \mathrm{Spec}(x)$. But these μ_i are precisely the solutions of

$$P(\mu) = \lambda.$$

Thus $\lambda \in \mathrm{Spec}(P(x))$ if and only if $\lambda = P(\mu)$ for some $\mu \in \mathrm{Spec}(x)$ which is precisely the assertion of the proposition. QED

1 Maximal ideals.

1.1 Existence.

Theorem 1.1 *Every proper right ideal in a ring is contained in a maximal proper right ideal. Similarly for left ideals. Also any proper two sided ideal is contained in a maximal proper two sided ideal.*

Proof by Zorn's lemma. The proof is the same in all three cases: Let I be the ideal in question (right left or two sided) and $\mathcal{F}$ be the set of all proper ideals (of the appropriate type) containing I ordered by inclusion. Since e does not belong to any proper ideal, the union of any linearly ordered family of proper ideals is again proper, and so has an upper bound. Now Zorn guarantees the existence of a maximal element. QED

1.2 The maximal spectrum of a ring.

For any ring R we let $\mathrm{Mspec}(R)$ denote the set of maximal (proper) two sided ideals of R. For any two sided ideal I we let

$$\mathrm{Supp}(I) := \{M \in \mathrm{Mspec}(R) : I \subset M\}.$$

Notice that

$$\mathrm{Supp}(\{0\}) = \mathrm{Mspec}(R)$$

and

$$\mathrm{Supp}(R) = \emptyset.$$

For any family I_α of two sided ideals, a maximal ideal contains all of the I_α if and only if it contains the two sided ideal $\sum_\alpha I_\alpha$. In symbols

$$\bigcap_\alpha \mathrm{Supp}(I_\alpha) = \mathrm{Supp}\left(\sum_\alpha I_\alpha\right).$$

Thus the intersection of any collection of sets of the form $\mathrm{Supp}(I)$ is again of this form. Notice also that if

$$A = \mathrm{Supp}(I)$$

then

$$A = \mathrm{Supp}(J) \quad \text{where } J = \bigcap_{M \in A} M.$$

(Here $I \subset J$, but J might be a strictly larger ideal.) We claim that

$$A = \mathrm{Supp}\left(\bigcap_{M\in A} M\right) \text{ and } B = \mathrm{Supp}\left(\bigcap_{M\in B} M\right) \Rightarrow A \cup B = \mathrm{Supp}\left(\bigcap_{M\in A\cup B} M\right). \tag{2}$$

Indeed, if N is a maximal ideal belonging to $A \cup B$ then it contains the intersection on the right hand side of so the left hand side contains the right. We must show the reverse inclusion. So suppose the contrary. This means that there is a maximal ideal N which contains the intersection on the right but does not belong either A or B. Since N does not belong to A, the ideal $J(A) := \bigcap_{M\in A} M$ is not contained in N, so $J(A) + N = R$, and hence there exist $a \in J(A)$ and $m \in N$ such that $a + m = e$. Similarly, there exist $b \in J(B)$ and $n \in N$ such that $b + n = e$. But then

$$e = e^2 = (a+m)(b+n) = ab + an + mb + mn.$$

Each of the last three terms on the right belong to N since it is a two sided ideal, and so does ab since

$$ab \in \left(\bigcap_{M\in A} M\right) \cap \left(\bigcap_{M\in B} M\right) = \left(\bigcap_{M\in A\cup B} M\right) \subset N.$$

Thus $e \in N$ which is a contradiction.

The above facts show that the sets of the form $\mathrm{Supp}(I)$ give the closed sets of a topology.

If $A \subset \text{Mspec}(R)$ is an arbitrary subset, its closure is given by

$$\overline{A} = \text{Supp}\left(\bigcap_{M\in A} M\right).$$

(For the case of commutative rings, a major advance was to replace maximal ideals by prime ideals in the preceding construction - giving rise to the notion of $\text{Spec}(R)$ - the prime spectrum of a commutative ring. But the motivation for this development in commutative algebra came from these constructions in the theory of Banach algebras.)

1.3 Maximal ideals in a commutative algebra.

Proposition 1.1 *An ideal M in a commutative algebra is maximal if and only if R/M is a field.*

Proof. If J is an ideal in R/M, its inverse image under the projection $R \to R/M$ is an ideal in R. If J is proper, so is this inverse image. Thus M is maximal if and only if $F := R/M$ has no ideals other than 0 and F. Thus if $0 \neq X \in F$, the set of all multiples of X must be all of F if M is maximal. In particular every non-zero element has an inverse. Conversely, if every non-zero element of F has an inverse, then F has no proper ideals. QED

1.4 Maximal ideals in the ring of continuous functions.

Let S be a compact Hausdorff space, and let $\mathcal{C}(S)$ denote the ring of continuous functions on S. For each $p \in S$, the map of $\mathcal{C}(S) \to \mathbf{C}$ given by

$$f \mapsto f(p)$$

is a surjective homomorphism. The kernel of this map consists of all f which vanish at p. By the preceding proposition, this is then a maximal ideal, which we shall denote by M_p.

Theorem 1.2 *If I is a proper ideal of $\mathcal{C}(S)$, then there is a point $p \in S$ such that*

$$I \subset M_p.$$

In particular every maximal ideal in $\mathcal{C}(S)$ is of the form M_p so we may identify $\text{Mspec}(\mathcal{C}(S))$ *with S as a set. This identification is a homeomorphism between the original topology of S and the topology given above on* $\text{Mspec}(\mathcal{C}(S))$.

Proof. Suppose that for every $p \in S$ there is an $f \in I$ such that $f(p) \neq 0$. Then $|f|^2 = f\overline{f} \in I$ and $|f(p)|^2 > 0$ and $|f|^2 \geq 0$ everywhere. Thus each point of S is contained in a neighborhood U for which there exists a $g \in I$ with $g \geq 0$ everywhere, and $g > 0$ on U. Since S is compact, we can cover S with finitely many such neighborhoods. If we take h to be the sum of the corresponding g's, then $h \in I$ and $h > 0$ everywhere. So $h^{-1} \in \mathcal{C}(S)$ and $e = 1 = hh^{-1} \in I$ so $I = \mathcal{C}(S)$, a contradiction. This proves the first part of the the theorem.

To prove the last statement, we must show that the closure of any subset $A \subset S$ in the original topology coincides with its closure in the topology derived from the maximal ideal structure. That is, we must show that

$$\text{closure of } A \text{ in the topology of } S = \operatorname{Supp}\left(\bigcap_{M \in A} M\right).$$

Now

$$\bigcap_{M \in A} M$$

consists exactly of all continuous functions which vanish at all points of A. Any such function must vanish on the closure of A in the topology of S. So the left hand side of the above equation is contained in the right hand side. We must show the reverse inclusion. Suppose $p \in S$ does not belong to the closure of A in the topology of S. Then *Urysohn's Lemma* asserts that there is an $f \in \mathcal{C}(S)$ which vanishes on A and $f(p) \neq 0$. Thus $p \notin \operatorname{Supp}(\bigcap_{M \in A} M)$. QED

Theorem 1.3 *Let I be an ideal in $\mathcal{C}(S)$ which is closed in the uniform topology on $\mathcal{C}(S)$. Then*

$$I = \bigcap_{M \in \operatorname{Supp}(I)} M.$$

Proof. $\operatorname{Supp}(I)$ consists of all points p such that $f(p) = 0$ for all $f \in I$. Since f is continuous, the set of zeros of f is closed, and hence $\operatorname{Supp}(I)$ being the intersection of such sets is closed. Let O be the complement of $\operatorname{Supp}(I)$ in S. Then O is a locally compact space, and the elements of $\bigcap_{M \in \operatorname{Supp}(I)} M$ when restricted to O consist of all functions which vanish at infinity. I, when restricted to O is a uniformly closed subalgebra of this algebra. If we could show that the elements of I separate points in O then the *Stone-Weierstrass theorem* would tell us that I consists of all continuous functions on O which "vanish at infinity", i.e. all continuous functions which vanish on $\operatorname{Supp}(I)$, which is the assertion of the theorem. So let p and q be distinct points of O, and let $f \in \mathcal{C}(S)$ vanish on $\operatorname{Supp}(I)$ and at q with $f(p) = 1$. Such a function exists by *Urysohn's Lemma*, again. Let $g \in I$ be such that $g(p) \neq 0$. Such a g exists by the definition of $\operatorname{Supp}(I)$. Then $gf \in I$, $(gf)(q) = 0$, and $(gf)(p) \neq 0$. QED

2 Normed algebras.

A **normed algebra** is an algebra (over the complex numbers) which has a norm as a vector space which satisfies

$$\|xy\| \leq \|x\|\|y\|. \tag{3}$$

Since $e = ee$ this implies that

$$\|e\| \leq \|e\|^2$$

so

$$\|e\| \geq 1.$$

Consider the new norm

$$\|y\|_N := \text{lub}_{\|x\|\neq 0}\|yx\|/\|x\|.$$

This still satisfies (3) and from (3) we have

$$\|y\|_N \leq \|y\|.$$

Under this norm we have

$$\|e\|_N = 1.$$

On the other hand, from its definition

$$\|y\|/\|e\| \leq \|y\|_N.$$

Combining this with the previous inequality gives

$$\|y\|/\|e\| \leq \|y\|_N \leq \|y\|.$$

In other words the norms $\| \quad \|$ and $\| \ \|_N$ are equivalent. So with no loss of generality we can add the requirement

$$\|e\| = 1 \tag{4}$$

to our axioms for a normed algebra.

Suppose we weaken our condition and allow $\| \ \|$ to be only a pseudo-norm. this means that we allow the possible existence of non-zero elements x with $\|x\| = 0$. Then (3) implies that the set of all such elements is an ideal, call it I. Then $\| \ \|$ descends to A/I . Furthermore, any continuous (i.e. bounded) linear function must vanish on I so also descends to A/I with no change in norm. In other words, A^* can be identified with $(A/I)^*$.

If A is a normed algebra which is complete (i.e. A is a Banach space as a normed space) then we say that A is a **Banach algebra**.

3 The Gelfand representation.

Let A be a normed vector space. The space A^* of continuous linear functions on A becomes a normed vector space under the norm

$$\|\ell\| := \sup_{\|x\|\neq 0} |\ell(x)|/\|x\|.$$

Each $x \in A$ defines a linear function on A^* by

$$x(\ell) := \ell(x)$$

and

$$|x(\ell)| \leq \|\ell\|\|x\|$$

so x is a continuous function of ℓ (relative to the norm introduced above on A^*).

Let $B = B_1(A^*)$ denote the unit ball in A^*. In other words $B = \{\ell : \|\ell\| \leq 1\}$. The functions $x(\cdot)$ on B induce a topology on B called the **weak topology**.

Proposition 3.1 *B is compact under the weak topology.*

Proof. For each $x \in A$, the values assumed by the set of $\ell \in B$ at x lie in the closed disk $D_{\|x\|}$ of radius $\|x\|$ in $\mathbf{C}$. Thus

$$B \subset \prod_{x \in A} D_{\|x\|}$$

which is compact by *Tychonoff's theorem* - being the product of compact spaces. To prove that B is compact, if is sufficient to show that B is a closed subset of this product space. Suppose that f is in the closure of B. For any x and y in A and any $\epsilon > 0$, we can find an $\ell \in B$ such that

$$|f(x) - \ell(x)| < \epsilon, \quad |f(y) - \ell(y)| < \epsilon, \text{ and } |f(x+y) - \ell(x+y)| < \epsilon.$$

Since $\ell(x+y) = \ell(x) + \ell(y)$ this implies that

$$|f(x+y) - f(x) - f(y)| < 3\epsilon.$$

Since ϵ is arbitrary, we conclude that

$$f(x+y) = f(x) + f(y).$$

Similarly, $f(\lambda x) = \lambda f(x)$. In other words, $f \in B$. QED

Let $\Delta \subset A^*$ denote the set of all continuous homomorphisms of A onto the complex numbers. In other words, in addition to being linear, we demand of $h \in \Delta$ that

$$h(xy) = h(x)h(y) \quad \text{and} \quad h(e) = 1.$$

Let $E := h^{-1}(1)$. Then E is closed under multiplication. In particular, if $x \in E$ we can not have $\|x\| < 1$ for otherwise x^n is a sequence of elements in E tending to 0, and so by the continuity of h we would have $h(0) = 1$ which is impossible. So $\|x\| \geq 1$ for all $x \in E$. If y is such that $h(y) = \lambda \neq 0$, then $x := y/\lambda \in E$ so

$$\|h(y)| \leq \|y\|,$$

and this clearly also holds if $h(y) = 0$. In other words,

$$\Delta \subset B.$$

Since the conditions for being a homomorphism will hold for any weak limit of homomorphisms (the same proof as given above for the compactness of B), we conclude that Δ is compact.

Once again we can turn the tables and think of $y \in A$ as a function $\hat{y}$ on Δ via

$$\hat{y}(h) := h(y).$$

This map from A into an algebra of functions on Δ is called the **Gelfand representation**.

The inequality $h(y) \leq \|y\|$ for all h translates into

$$\|\hat{y}\|_\infty \leq \|y\|. \tag{5}$$

Putting it all together we get

Theorem 3.1 *Δ is a compact subset of A^* and the Gelfand representation $y \mapsto \hat{y}$ is a norm decreasing homomorphism of A onto a subalgebra $\hat{A}$ of $\mathcal{C}(\Delta)$.*

The above theorem is true for any normed algebra - we have not used any completeness condition. For Banach algebras, i.e. complete normed algebras, we can proceed further and relate Δ to Mspec(A). Recall that an element of Mspec(A) corresponds to a homomorphism of A onto some field. In the commutative Banach algebra case we will show that this field is **C** and that any such homomorphism is automatically continuous. So for commutative Banach algebras we can identify Δ with Mspec(A).

3.1 Invertible elements in a Banach algebra form an open set.

In this section A will be a Banach algebra.

Proposition 3.2 *If $\|x\| < 1$ then x has an adverse x' given by*

$$x' = -\sum_{n=1}^{\infty} x^n$$

so that $e - x$ has an inverse given by

$$e - x' = e + \sum_{n=1}^{\infty} x^n.$$

Both are continuous functions of x

Proof. Let

$$s_n := -\sum_{1}^{n} x^i.$$

Then if $m < n$

$$\|s_m - s_n\| \leq \sum_{m+1}^{n} \|x\|^i < \|x\|^m \frac{1}{1 - \|x\|} \to 0$$

as $m \to \infty$. Thus s_n is a Cauchy sequence and

$$x + s_n - x s_n = x^{n+1} \to 0.$$

Thus the series $-\sum_1^\infty x^i$ as stated in the theorem converges and gives the adverse of x and is continuous function of x. The corresponding statements for $(e - x)^{-1}$ now follow. QED

The proof shows that the adverse x' of x satisfies

$$\|x'\| \leq \frac{\|x\|}{1 - \|x\|}. \tag{6}$$

Theorem 3.2 *Let y be an invertible element of A and set*

$$a := \frac{1}{\|y^{-1}\|}.$$

Then $y + x$ is invertible whenever

$$\|x\| < a.$$

Furthermore

$$\|(x+y)^{-1} - y^{-1}\| \leq \frac{\|x\|}{(a - \|x\|)a}.$$

Thus the set of elements having inverses is open and the map $x \to x^{-1}$ is continuous on its domain of definition.

Proof. If $\|x\| < \|y^{-1}\|^{-1}$ then

$$\|y^{-1}x\| \leq \|y^{-1}\|\|x\| < 1.$$

Hence $e + y^{-1}x$ has an inverse by the previous proposition. Hence $y + x = y(e + y^{-1}x)$ has an inverse. Also

$$(y+x)^{-1} - y^{-1} = \left((e + y^{-1}x)^{-1} - e\right) y^{-1} = -(-y^{-1}x)'y^{-1}$$

where $(-y^{-1}x)'$ is the adverse of $-y^{-1}x$.

From (6) and the above expression for $(x+y)^{-1}$ we see that

$$\|(x+y)^{-1} - y^{-1}\| \leq \frac{\|x\|\|y^{-1}\|^2}{1 - \|x\|\|y^{-1}\|} = \frac{\|x\|}{a(a - \|x\|)}.$$

QED

Proposition 3.3 *If I is a proper ideal then $\|e - x\| \geq 1$ for all $x \in I$.*

Proof. Otherwise there would be some $x \in I$ such that $e - x$ has an adverse, i.e. x has an inverse which contradicts the hypothesis that I is proper.

Proposition 3.4 *The closure of a proper ideal is proper. In particular, every maximal ideal is closed.*

Proof. The closure of an ideal I is clearly an ideal, and all elements in the closure still satisfy $\|e - x\| \geq 1$ and so the closure is proper. QED

Proposition 3.5 *If I is a closed ideal in A then A/I is again a Banach algebra.*

Proof. The quotient of a Banach space by a closed subspace is again a Banach space. The norm on A/I is given by

$$\|X\| = \min_{x \in X} \|x\|$$

where X is a coset of I in A. The product of two cosets X and Y is the coset containing xy for any $x \in X,\ y \in Y$. Thus

$$\|XY\| = \min_{x\in X,\ y\in Y} \|xy\| \leq \min_{x\in X,\ y\in Y} \|x\|\|y\| = \|X\|\|Y\|.$$

Also, if E is the coset containing e then E is the identity element for A/I and so

$$\|E\| \leq 1.$$

But we know that this implies that $\|E\| = 1$. QED

Suppose that A is commutative and M is a maximal ideal of A. We know that A/M is a field, and the preceding proposition implies that A/M is a normed field containing the complex numbers. The following famous result implies that A/M is in fact norm isomorphic to $\mathbf{C}$. It deserves a subsection of its own:

3.1.1 The Gelfand-Mazur theorem.

A division algebra is a (possibly not commutative) algebra in which every non-zero element has an inverse.

Theorem 3.3 *Every normed division algebra over the complex numbers is isometrically isomorphic to the field of complex numbers.*

Let A be the normed division algebra and $x \in A$. We must show that $x = \lambda e$ for some complex number λ. Suppose not. Then by the definition of a division algebra, $(x - \lambda e)^{-1}$ exists for all $\lambda \in \mathbf{C}$ and all these elements commute. Thus

$$(x - (\lambda + h)e)^{-1} - (x - \lambda e)^{-1} = h(x - (\lambda + h)e)^{-1}(x - \lambda e)^{-1}.$$

Thus the strong derivative of the function

$$\lambda \mapsto (x - \lambda e)^{-1}$$

exists and is given by the usual formula $(x - \lambda e)^{-2}$. In particular, for any $\ell \in A^*$ the function

$$\lambda \mapsto \ell((x - \lambda e)^{-1})$$

is analytic on the entire complex plane. On the other hand for $\lambda \neq 0$ we have

$$(x - \lambda e)^{-1} = \lambda^{-1}(\frac{1}{\lambda}x - e)^{-1}$$

and this approaches zero as $\lambda \to \infty$. Hence for any $\ell \in A^*$ the function $\lambda \mapsto \ell((x - \lambda e)^{-1})$ is an everywhere analytic function which vanishes at infinity, and hence is identically zero by Liouville's theorem. But this implies that $(x - \lambda e)^{-1} \equiv 0$ by the *Hahn Banach theorem*, a contradiction. QED

3.2 The Gelfand representation for commutative Banach algebras.

Let A be a commutative Banach algebra. We know that every maximal ideal is the kernel of a homomorphism h of A onto the complex numbers. Conversely, suppose that h is such a homomorphism. We claim that

$$|h(x)| \leq \|x\|$$

for any $x \in A$. Indeed, suppose that $|h(x)| > \|x\|$ for some x. Then

$$\|x/h(x)\| < 1$$

so $e - x/h(x)$ is invertible; in particular $h(e - x/h(x)) \neq 0$ or $1 = h(e) \neq h(x)/h(x)$, a contradiction.

In short, we can identify $\text{Mspec}(A)$ with Δ and the map $x \mapsto \hat{x}$ is a norm decreasing map of A onto a subalgebra $\hat{A}$ of $\mathcal{C}(\text{Mspec}(A))$ where we use the uniform norm $\| \ \|_\infty$ on $\mathcal{C}(\text{Mspec}(A))$. A complex number is in the spectrum of an $x \in A$ if and only if $(x - \lambda e)$ belongs to some maximal ideal M, in which case $\hat{x}(M) = \lambda$. Thus

$$\|\hat{x}\|_\infty = \text{l.u.b. } \{|\lambda| : \lambda \in \text{Spec}(x)\}. \tag{7}$$

3.3 The spectral radius.

The right hand side of (7) makes sense in any algebra, and is called the **spectral radius** of x and is denoted by $|x|_{sp}$. We claim that

Theorem 3.4 *In any Banach algebra we have*

$$|x|_{sp} = \lim_{n\to\infty} \|x^n\|^{\frac{1}{n}}. \tag{8}$$

Proof. If $|\lambda| > \|x\|$ then $e - x/\lambda$ is invertible, and therefore so is $x - \lambda e$ so $\lambda \notin \text{Spec}(x)$. Thus

$$|x|_{sp} \leq \|x\|.$$

We know from (1) that $\lambda \in \text{Spec}(x) \Rightarrow \lambda^n \in \text{Spec}(x^n)$, so the previous inequality applied to x^n gives

$$|x|_{sp} \leq \|x^n\|^{\frac{1}{n}}$$

and so

$$|x|_{sp} \leq \liminf \|x^n\|^{\frac{1}{n}}.$$

We must prove the reverse inequality with lim sup. Suppose that $|\mu| < 1/|x|_{sp}$ so that $\mu := 1/\lambda$ satisfies $|\lambda| > |x|_{sp}$ and hence $e - \mu x$ is invertible. The formula for the adverse gives

$$(\mu x)' = -\sum_{1}^{\infty} (\mu x)^n$$

where we know that this converges in the open disk of radius $1/\|x\|$. However, we know that $(e-\mu x)^{-1}$ exists for $|\mu| < 1/|x|_{sp}$. In particular, for any $\ell \in A^*$ the function $\lambda \mapsto \ell((\mu x)')$ is analytic and hence its Taylor series

$$-\sum \ell(x^n)\mu^n$$

converges on this disk. Here we use the fact that the Taylor series of a function of a complex variable converges on any disk contained in the region where it is analytic. Thus

$$|\ell(\mu^n x^n)| \to 0$$

for each fixed $\ell \in A^*$ if $|\mu| < 1/|x|_{sp}$. Considered as a family of linear functions of ℓ, we see that

$$\ell \mapsto \ell(\mu^n x^n)$$

is bounded for each fixed ℓ, and hence by the *uniform boundedness principle*, there exists a constant K such that

$$\|\mu^n x^n\| < K$$

for each μ in this disk, in other words

$$\|x^n\|^{\frac{1}{n}} \le K^{\frac{1}{n}}(1/|\mu|)$$

so

$$\limsup \|x^n\|^{\frac{1}{n}} \le 1/|\mu| \quad \text{if } 1/|\mu| > |x|_{sp}.$$

QED

In a commutative Banach algebra we can combine (8) with (7) to conclude that

$$\|\hat{x}\|_\infty = \lim_{n\to\infty} \|x^n\|^{\frac{1}{n}}. \tag{9}$$

We say that x is a **generalized nilpotent element** if $\lim \|x^n\|^{\frac{1}{n}} = 0$. From (8) we see that x is a generalized nilpotent element if and only if $\hat{x} \equiv 0$. This means that x belongs to all maximal ideals. The intersection of all maximal ideals is called the **radical** of the algebra. A Banach algebra is called **semi-simple** if its radical consists only of the 0 element.

3.4 The generalized Wiener theorem.

Theorem 3.5 *Let A be a commutative Banach algebra. Then $x \in A$ has an inverse if and only if $\hat{x}$ never vanishes.*

Proof. If $xy = e$ then $\hat{x}\hat{y} \equiv 1$. So if x has an inverse, then $\hat{x}$ can not vanish anywhere. Conversely, suppose x does not have an inverse. Then Ax is a proper ideal. So x is contained in some maximal ideal M (by Zorn's lemma). So $\hat{x}(M) = 0$.

Example. Let G be a countable commutative group, and we equip G with the counting measure. Thus $L^1(G)$ consists of all complex valued functions on G which are absolutely summable, i.e. such that

$$\sum_{a\in G} |f(a)| < \infty.$$

Make $L^1(G)$ into an algebra via convolution:

$$f \star g)(x) := \sum_{y\in G} f(xy^{-1})g(y).$$

Since $L^1(G) \subset L^2(G)$ this sum converges and

$$\begin{aligned}
\sum_{x\in G} |(f\star g)(x)| &\leq \sum_{x,y\in G} |f(xy^{-1}||g(y)| \\
&= \sum_{y\in G} |g(y)| \sum_{x\in G} |f(xy^{-1})| \\
&= \sum_{y\in G} |g(y)|)(\sum_{w\in G} |f(w)| \quad \text{i.e.} \\
\|f\star g\| &\leq \|f\|\|g\|.
\end{aligned}$$

If $\delta_x \in L^1(G)$ is defined by

$$\delta_x(t) = \begin{cases} 1 & \text{if } t = x \\ 0 & \text{otherwise} \end{cases}$$

then

$$\delta_x \star \delta_y = \delta_{xy}.$$

We know that the most general continuous linear function on $L^1(G)$ is obtained from multiplication by an element of $L^\infty(G)$ and then integrating = summing. That is it is given by

$$f \mapsto \sum_{x\in G} f(x)\rho(x)$$

where ρ is some bounded function. Under this linear function we have

$$\delta_x \mapsto \rho(x)$$

and so, if this linear function is to be multiplicative, we must have

$$\rho(xy) = \rho(x)\rho(y).$$

Since $\rho(x^n) = \rho(x)^n$ and $|\rho(x)|$ is to be bounded, we must have $|\rho(x)| \equiv 1$.

A function ρ satisfying these two conditions:

$$\rho(xy) = \rho(x)\rho(y) \quad \text{and} \quad |\rho(x)| \equiv 1$$

is called a **character** of the commutative group G. The space of characters is itself a group, denoted by $\hat{G}$.

We have shown that $\mathrm{Mspec}(L^1(G)) = \hat{G}$. In particular we have a topology on $\hat{G}$ and the Gelfand transform $f \mapsto \hat{f}$ sends every element of $L^1(G)$ to a continuous function on $\hat{G}$.

For example, if $G = \mathbf{Z}$ under addition, the condition to be a character says that

$$\rho(m+n) = \rho(m)\rho(n), \qquad |\rho| \equiv 1.$$

So

$$\rho(n) = \rho(1)^n$$

where

$$\rho(1) = e^{i\theta}$$

for some $\theta \in \mathbf{R}/(2\pi\mathbf{Z})$. Thus

$$\hat{f}(\theta) = \sum_{n\in\mathbf{Z}} f(n)e^{in\theta}$$

is just the Fourier series with coefficients $f(n)$. The image of the Gelfand transform is just the set of Fourier series which converge absolutely. We conclude from Theorem 3.5 that if F is an absolutely convergent Fourier series which vanishes nowhere, then $1/F$ has an absolutely convergent Fourier series. Before Gelfand, this was a deep theorem of Wiener.

4 Self-adjoint algebras.

Let A be a semi-simple commutative Banach algebra. Since "semi-simple" means that the radical is $\{0\}$, we know that the Gelfand transform is injective. A is called **self adjoint** if for every $x \in A$ there exists an $x^* \in A$ such that

$$(x^*)\hat{} = \overline{\hat{x}}.$$

By the injectivity of the Gelfand transform, the element x^* is uniquely specified by this equation.

In general, for any Banach algebra, a map $f \mapsto f^\dagger$ is called an **involutory anti-automorphism** if

- $(fg)^\dagger = g^\dagger f^\dagger$
- $(f+g)^\dagger = f^\dagger + g^\dagger$
- $(\lambda f)^\dagger = \overline{\lambda} f^\dagger$ and
- $(f^\dagger)^\dagger = f$.

For example, if B is the algebra of bounded operators on a Hilbert space, then the map $T \mapsto T^*$ sending every operator to its adjoint is an example of an involutory anti-automorphism.

Also, the map $x \mapsto x^*$ is an involutory anti-automorphism. It has this further property:

$$f = f^* \Rightarrow 1 + f^2 \quad \text{is invertible.}$$

indeed, if $f = f^*$ then $\hat{f}$ is real valued, so $1 + \hat{f}^2$ vanishes nowhere, and so $1 + f^2$ is invertible by Theorem 3.5. Conversely

Theorem 4.1 *Let A be a commutative semi-simple Banach algebra with an involutory anti-automorphism $f \mapsto f^\dagger$ such that $1 + f^2$ is invertible whenever $f = f^\dagger$. Then A is self-adjoint and $\dagger = *$.*

Proof. We must show that $(f^\dagger)\hat{} = \overline{\hat{f}}$. We first prove that if we set

$$g := f + f^\dagger$$

then $\hat{g}$ is real valued. Suppose the contrary, that

$$\hat{g}(M) = a + ib, \quad b \neq 0 \quad \text{for some } M \in \text{Mspec}(A).$$

Now $g^\dagger = f^\dagger + (f^\dagger)^\dagger = g$ and hence $(g^2)^\dagger = g^2$ so

$$h := \frac{ag^2 - (a^2 - b^2 g}{b(a^2 + b^2)}$$

satisfies

$$h^\dagger = h.$$

We have

$$h(M) = \frac{a(a + ib)^2 - (a^2 - b^2)(a + ib)}{b(a^2 + b^2} = i.$$

So

$$1 + h(M)^2 = 0$$

contradicting the hypothesis that $1 + h^2$ is invertible. Now let us apply this result to $\frac{1}{2}f$ and to $\frac{1}{2i}f$. We have

$$f = g + ih \quad \text{where} \quad g = \frac{1}{2}(f + f^\dagger), \quad h = \frac{1}{2i}(f - f^\dagger)$$

and we know that $\hat{g}$ and $\hat{h}$ are real. So

$$\overline{\hat{f}} = \overline{\hat{g} + i\hat{h}} = \hat{g} - i\hat{h} = (f^\dagger)\hat{}.$$

QED

Theorem 4.2 *Let A be a commutative Banach algebra with an involutory anti automorphism $\dagger$ which satisfies the condition*

$$\|ff^\dagger\| = \|f\|^2 \quad \forall\ f \in A. \tag{10}$$

Then the Gelfand transform $f \mapsto \hat{f}$ is a norm preserving isomorphism which satisfies

$$(f^\dagger)\hat{} = \overline{\hat{f}}.$$

In particular, A is semi-simple and self-adjoint.

Proof. $\|f\|^2 = \|ff^\dagger\| \leq \|f\|\|f^\dagger\|$ so $\|f\| \leq \|f^\dagger\|$. Replacing f by $f^\dagger$ gives $\|f^\dagger\| \leq \|f\|$. so

$$\|f\| = \|f^\dagger\|$$

and

$$\|ff^\dagger\| = \|f\|^2 = \|f\|\|f^\dagger\|.$$

Hence

$$\|f^2\|\|(f^\dagger)^2\| = \|ff^\dagger(ff^\dagger)^\dagger\| = \|ff^\dagger\|\|ff^\dagger\| = \|f\|^2\|f^\dagger\|^2$$

or

$$\|f^2\|^2 = \|f\|^4.$$

Thus

$$\|f^2\| = \|f\|^2,$$

and therefore

$$\|f^4\| = \|f^2\|^2 = \|f\|^4$$

and by induction

$$\|f^{2^k}\| = \|f\|^{2^k}$$

for all non-negative integers k.

Hence letting $n = 2^k$ in the right hand side of (9) we see that $\|\hat{f}\|_\infty = \|f\|$ so the Gelfand transform is norm preserving, and hence injective. To show that $\dagger = *$ it is enough to show that if $f = f^\dagger$ then $\hat{f}$ is real valued, as in the proof of the preceding theorem. Suppose not, so $\hat{f}(M) = a + ib,\ b \neq 0$. For any real number c we have

$$(f + ice)\hat{}(M) = a + i(b + c)$$

so

$$|(f + ice)\hat{}(M)|^2 = a^2 + (b + c)^2 \leq \|f + ice\|^2 = \|(f + ice)(f - ice)\|$$

$$= \|f^2 + c^2 e\| \leq \|f\|^2 + c^2.$$

This says that

$$a^2 + b^2 + 2bc + c^2 \leq \|f\|^2 + c^2$$

which is impossible if we choose c so that $2bc > \|f\|^2$.

So we have proved that $\dagger = *$. Now by definition, if $f(M) = f(N)$ for all $f \in A$, the maximal ideals M and N coincide. So the image of elements of A under the Gelfand transform separate points of Mspec(A). But every $f \in A$ can be written as

$$f = \frac{1}{2}(f + f^*) + i\frac{1}{2i}(f - f^*)$$

i.e. as a sum $g+ih$ where $\hat{g}$ and $\hat{f}$ are real valued. Hence the real valued functions of the form $\hat{g}$ separate points of Mspec(A). Hence by the *Stone Weierstrass theorem* we know that the image of the Gelfand transform is dense in $\mathcal{C}(\text{Mspec}(A))$. Since A is complete and the Gelfand transform is norm preserving, we conclude that the Gelfand transform is surjective. QED

4.1 Important application.

Proposition 4.1 *If T is a bounded linear operator on a Hilbert space, then*

$$||TT^*|| = ||T||^2. \tag{11}$$

Proof.

$$\begin{aligned} ||TT^*|| &= \sup_{||\phi||=1} ||TT^*\phi|| \\ &= \sup_{||\phi||=1, ||\psi||=1} |(TT^*\phi, \psi)| \\ &= \sup_{||\phi||=1, ||\psi||=1} |(T^*\phi, T^*\psi)| \\ &\geq \sup_{||\phi||=1} (T^*\phi, T^*\phi) \\ &= ||T^*||^2 \end{aligned}$$

so

$$||T^*||^2 \leq ||TT^*|| \leq ||T|| ||T^*||$$

so

$$||T^*|| \leq ||T||.$$

Reversing the role of T and T^* give the reverse inequality so $||T|| = ||T^*||$. Inserting into the preceding inequality gives

$$||T^2|| \leq ||TT^*| \leq ||T||^2$$

so we have the equality (11).

Putting the previous facts together gives

Theorem 4.3 *Let A be any commutative subalgebra of the algebra of bounded operators on a Hilbert space which is closed in the strong topology and with the property that $T \in A \Rightarrow T^* \in A$. Then the Gelfand transform $T \mapsto \hat{T}$ gives a norm preserving isomorphism of A with $\mathcal{C}(\mathcal{M})$ where $\mathcal{M} =$ Mspec(A). Furthermore, $(T^*)\hat{} = \overline{\hat{T}}$ for all $T \in A$. In particular, if T is self-adjoint, then $\hat{T}$ is real valued.*