The Hahn-Banach theorem

Math 212

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This says:

Theorem 1 **[Hahn-Banach].** *Let M be a subspace of a normed linear space B, and let F be a bounded linear function on on M. Then F can be extended so as to be defined on all of B without increasing its norm.*

Proof by Zorn. Suppose that we can prove

Proposition 1 *Let M be a subspace of a normed linear space B, and let F be a bounded linear function on on M. Let $y \in B, y \notin M$. Then F can be extended to $M + [y]$ without changing its norm.*

Then we could order the extensions of F by inclusion, one extension being $\succeq$ than another if it is defined on a larger space. The extension defined on the union of any family of subspaces ordered by inclusion is again an extension, and so is an upper bound. The proposition implies that a maximal extension must be defined on the whole space, oherwise we can extend it further. So we must prove the propostion.

I was careful in the statement not to specify whether our spaces are over the real or complex numbers. Let us first assume that we are dealing with a real vector space, and then deduce the complex case.

We want to choose a value

$$\alpha = F(y)$$

so that if we then define

$$F(x + \lambda y) := F(x) + \lambda F(y) = F(x) + \lambda\alpha, \quad \forall x \in M, \lambda \in \mathbf{R}$$

we do not increase the norm of F. If $F = 0$ we take $\alpha = 0$. If $F \neq 0$, we may replace F by $F/\|F\|$, extend and then multiply by $\|F\|$ so without loss of generality we may assume that $\|F\| = 1$. We want to choose the extension to have norm 1, which means that we want

$$|F(x) + \lambda\alpha| \leq \|x + \lambda y\| \quad \forall x \in M, \lambda \in \mathbf{R}.$$

If $\lambda = 0$ this is true by hypothesis. If $\lambda \neq 0$ divide this inequality by λ and replace $(1/\lambda)x$ by x. We want

$$|F(x) + \alpha| \leq \|x + y\| \quad \forall\ x \in M.$$

We can write this as two separate conditions:

$$F(x_2) + \alpha \le |x_2 + y\| \ \ \forall\, x_2 \in M \ \text{ and } \quad -F(x_1) - \alpha \le \|x_1 + y\| \ \ \forall x_1 \in M.$$

Rewriting the second inequality this becomes

$$-F(x_1) - \|x_1 + y\| \le \alpha \le -F(x_2) + |x_2 + y\|.$$

The question is whether such a choice is possible In other words, is the supremum of the left hand side (over all $x_1 \in M$) less than or equal to the infimium of the right hand side (over all $x_2 \in M$)? If the answer to this question is yes, we may choose α to be any value between the sup of the left and the inf of the right hande sides of the preceding inequality. So our question is: Is

$$F(x_2) - F(x_1) \le \|x_2 + y\| + \|x_1 + y\| \quad \forall\, x_1, x_2 \in M?$$

But $x_1 - x_2 = (x_1 + y) - (x_2 + y)$ and so using the fact that $\|F\| = 1$ and the triangle inequality gives

$$|F(x_2) - F(x_1) \le \|x_2 - x_1\| \le \|x_2 + y\| + \|x_1 + y\|.$$

This completes the proof of the proposition, and hence of the Hahn-Banach theorem over the real numbers.

We now deal with the complex case. If B is a complex normed vector space, then it is also a real vector space, and the real and imaginary parts of a complex linear function are real linear functions. In other words, we can write any complex linear function F as

$$F(x) = G(x) + iH(x)$$

where G and H are real linear functions. The fact that F is complex linear says that $F(ix) = iF(x)$ or

$$G(ix) = -H(x)$$

or

$$H(x) = -G(ix)$$

or

$$F(x) = G(x) - iG(ix).$$

The fact that $\|F\| = 1$ implies that $\|G\| \le 1$. So we can adjoin the real one dimensional space spanned by y to M and extend the real linear function to it, keeping the norm ≤ 1. Next adjoin the real one dimensional space spanned by iy and extend G to it. We now have G extended to $M \oplus \mathbf{C}y$ with no increase in norm. Try to define

$$F(z) := G(z) - iG(iz)$$

on $M \oplus \mathbf{C}y$. This map of $M \oplus \mathbf{C}y \to \mathbf{C}$ is $\mathbf{R}$-linear, and coincides with F on M. We must check that it is complex linear and that its norm is ≤ 1: To check that it is complex linear it is enough to observe that

$$F(iz) = G(iz) - iG(-z) = i[G(z) - iG(iz)] = iF(z).$$

To check the norm, we may, for any z choose θ so that $e^{i\theta}F(z)$ is real and is non-negative. Then

$$|F(z)| = |e^{i\theta}F(z)| = |F(e^{i\theta}z)| = G(e^{i\theta}z) \le \|e^{i\theta}z\| = \|z\|$$

so $\|F\| \le 1$. QED

Suppose that M is a closed subspace of B and that $y \notin M$. Let d denote the distance of y to M, so that

$$d := \inf_{x \in M} \|y - x\|.$$

Suppose we start with the zero function on M, and extend it first to $M \oplus y$ by

$$F(\lambda y - x) = \lambda d.$$

This is a linear function on $M + \{y\}$ and its norm is ≤ 1. Indeed

$$\|F\| = \sup_{\lambda, x} \frac{|\lambda d|}{\|\lambda y - x\|} = \sup_{x' \in M} \frac{d}{y - x'} = \frac{d}{d} = 1.$$

Let M^0 be the set of all continuous linear functions on B which vanish on M. Then, using the Hahn-Banach theorem we get

Proposition 2 *If $y \in B$ and $y \notin M$ where M is a closed linear subspace of B, then there is an element $F \in M^0$ with $\|F\| \le 1$ and $F(y) \ne 0$. In fact we can arrange that $F(y) = d$ where d is the distance from y to M.*

We have an embedding

$$B \to B^{**} \quad x \mapsto x^{**} \quad \text{where } x^{**}(F) := F(x).$$

The first part of the preceding propostion can be formulated as

$$(M^0)^0 = M$$

if M is a closed suspace of B.

The map $x \mapsto x^{**}$ is clearly linear and

$$|x^{**}(F)| = |F(x)| \le \|F\|\|x\|.$$

Taking the sup of $|x^{**}(F)|/\|F\|$ shows that

$$\|x^{**}\| \le \|x\|$$

where the norm on the left is the norm on the space B^{**}. On the other hand, if we take $M = \{0\}$ in the preceding proposition, we can find an $F \in B^*$ with $\|F\| = 1$ and $F(x) = \|x\|$. For this F we have $|x^{**}(F)| = \|x\|$. So

$$\|x^{**}\| \ge \|x\|.$$

We have proved

Theorem 2 *The map $B \to B^{**}$ given above is a norm preserving injection.*