The Stone-Weierstrass theorem.

Math 212

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This is an important generalization of Weierstrass's theorem which asserted that the polynomials are dense in the space of continuous functions or any compact interval, when we use the uniform topology.

An algebra A of (real valued) functions on a set S is said to *separate points* if for any $p, q \in S,\ p \neq q$ there is an $f \in A$ with $f(p) \neq f(q)$.

Theorem 1 **[Stone-Weierstrass.]** *Let S be a compact space and A an algebra of continuous real valued functions on S which separates points. Then the closure of A in the uniform topology is either the algebra of all continuous functions on S, or is the algebra of all continuous functions on S which all vanish at a single point, call it x_∞.*

We first state and prove some preliminary lemmas:

Lemma 1 *An algebra A of bounded real valued functions on a set S which is closed in the uniform topology is also closed under the lattice operations $\vee$ and $\wedge$.*

Proof. Since $f \vee g = \frac{1}{2}(f + g + |f - g|)$ and $f \wedge g = \frac{1}{2}(f + g - |f - g|)$ we must show that

$$f \in A \quad \Rightarrow |f| \in A.$$

Replacing f by $f/\|f\|_\infty$ we may assume that

$$|f| \leq 1.$$

The Taylor series for the function $t \mapsto (t + \epsilon^2)^{\frac{1}{2}}$ converges uniformly on $[0, 1]$. So there exists, for any $\epsilon > 0$ there is a polynomial P such that

$$|P(x^2) - (x^2 + \epsilon^2)^{\frac{1}{2}}| < \epsilon \quad \text{on} \quad [-1, 1].$$

Let

$$Q := P - P(0).$$

We have $|P(0) - \epsilon| < \epsilon$ so

$$|P(0)| < 2\epsilon.$$

So $Q(0) = 0$ and

$$|Q(x^2) - (x^2 + \epsilon^2)^{\frac{1}{2}}| < 3\epsilon.$$

But

$$(x^2+\epsilon^2)^{\frac{1}{2}} - |x| \le \epsilon$$

for small ϵ. So

$$|Q(x^2) - |x|\,| < 4\epsilon \quad \text{on} \quad [0,1].$$

As Q does not contain a constant term, and A is an algebra, $Q(f^2) \in A$ for any $f \in A$. Since we are assuming that $|f| \le 1$ we have

$$Q(f^2) \in A, \quad \text{and} \quad \|Q(f^2) - |f|\,\|_\infty < 4\epsilon.$$

Since we are assuming that A is closed under $\|\cdot\|_\infty$ we conclude that $|f| \in A$ completing the proof of the lemma.

Lemma 2 *Let A be a set of real valued continuous functions on a compact space S such that*

$$f, g \in A \Rightarrow \quad f \wedge g \in A \quad \text{and} \quad f \vee g \in A.$$

Then the closure of A in the uniform topology contains every continuous function on S which can be approximated at every pair of points by a function belonging to A.

Proof. Suppose that f is a continuous function on S which can be approximated at any pair of points by elements of A. So let $p, q \in S$ and $\epsilon > 0$, and let $f_{p,q,\epsilon} \in A$ be such that

$$|f(p) - f_{p,q,\epsilon}(p)| < \epsilon, \quad |f(q) - f_{p,q,\epsilon}(q)| < \epsilon.$$

Let

$$U_{p,q,\epsilon} := \{x | f_{p,q,\epsilon}(x) < f(x) + \epsilon\}, \quad V_{p,q,\epsilon} := \{x | f_{p,q,\epsilon}(x) > f(x) - \epsilon\}.$$

Fix q and ϵ. The sets $U_{p,q,\epsilon}$ cover S as p varies. Hence a finite number cover S since we are assuming that S is compact. We may take the minimum $f_{q,\epsilon}$ of the corresponding finite collection of $f_{p,q,\epsilon}$. The function $f_{q,\epsilon}$ has the property that

$$f_{q,\epsilon}(x) < f(x) + \epsilon$$

and

$$f_{q,\epsilon}(x) > f(x) - \epsilon$$

for

$$x \in \bigcap_p V_{p,q,\epsilon}$$

where the intersection is again over the same finite set of p's. We have now found a collection of functions $f_{q,\epsilon}$ such that

$$f_{q,\epsilon} < f + \epsilon$$

and $f_{q,\epsilon} > f - \epsilon$ on some neighborhood $V_{q,\epsilon}$ of q. We may choose a finite number of q so that the $V_{q,\epsilon}$ cover all of S. Taking the maximum of the corresponding $f_{q,\epsilon}$ gives a function $f_\epsilon \in A$ with $f - \epsilon < f < f + \epsilon$, i.e.

$$\|f - f_\epsilon\|_\infty < \epsilon.$$

Since we are assuming that A is closed in the uniform topology we conclude that $f \in A$, completing the proof of the lemma.

Proof of the Stone-Weierstrass theorem. Suppose first that for every $x \in S$ there is a $g \in A$ with $g(x) \neq 0$. Let $x \neq y$ and $h \in A$ with $h(y) \neq 0$. Then we may choose real numbers c and d so that $f = cg + dh$ is such that

$$0 \neq f(x) \neq f(y) \neq 0.$$

Then for any real numbers a and b we may find constants A and B such that

$$Af(x) + Bf^2(x) = a \quad \text{and } Af(y) + Bf^2(y) = b.$$

We can therefore approximate (in fact hit exactly on the nose) any function at any two distinct points. We know that the closure of A is closed under $\vee$ and $\wedge$ by the first lemma. By the second lemma we conclude that the closure of A is the algebra of all real valued continuous functions.

The second alternative is that there is a point, call it p_∞ at which all $f \in A$ vanish. We wish to show that the closure of A contains all continuous functions vanishing at p_∞. Let B be the algebra obtained from A by adding the constants. Then B satisfies the hypotheses of the Stone-Weierstrass theorem and contains functions which do not vanish at p_∞. so we can apply the preceding result. If g is a continuous function vanishing at p_∞ we may, for any $\epsilon > 0$ find an $f \in A$ and a constant c so that

$$\|g - (f + c)\|_\infty < \frac{\epsilon}{2}.$$

Evaluating at p_∞ gives $|c| < \epsilon/2$. So

$$\|g - f\|_\infty < \epsilon.$$

QED

The reason for the apparently strange notation p_∞ has to do with the notion of the one point compactification of a locally compact space. A topological space S is called **locally compact** if every point has a closed compact neighborhood. We can make S compact by adding a single point. Indeed, let p_∞ be a point not belonging to S and set

$$S_\infty := S \cup p_\infty.$$

We put a topology on S_∞ by taking as the open sets all the open sets of S together with all sets of the form

$$O \cup p_\infty$$

where O is an open set of S whose complement is compact. The space S_∞ is compact, for if we have an open cover of S_∞, at least one of the open sets in this cover must be of the second type, hence its complement is compact, hence covered by finitely many of the remaining sets. If S itself is compact, then the empty set has compact complement, hence $p\infty$ has an open neighborhood disjoint from S, and all we have done is add a disconnected point to S. The space S_∞ is called the **one-point compactification** of S. In application of the Stone-Weierstrass theorem, we shall frequently have to do with an algebra of functions on a locally compact space consisting of functions which "vanish at infinity" in the sense that for any $\epsilon > 0$ there is a compact set C such that $|f| < \epsilon$ on the complement of C. We can think of these functions as being defined on S_∞ and all vanishing at p_∞.