Tychonoff's theorem

Math 212

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Let I be a set (serving as an "index set"). Suppose that for each $\alpha \in I$ we are given a non-empty topological space S_α. Recall that the Cartesian product

$$\prod_{\alpha \in I} S_\alpha$$

is defined to be the collection of all functions x whose domain is I and such that $x(\alpha) \in S_\alpha$. This space is not empty by the axiom of choice. We frequently write x_α instead of $x(\alpha)$. The map

$$f_\alpha : \prod_{\alpha \in I} S_\alpha =: S \to S_\alpha, \quad x \mapsto x_\alpha$$

is called the projection of S onto S_α. We put on S the weakest topology such that each of these projections is continuous. So the open sets of S are generated by the sets of the form

$$f_\alpha^{-1}(U_\alpha) \quad \text{where } U_\alpha \subset S_\alpha \ \text{ is open.}$$

Theorem 1 [**Tychonoff.**] *If all the S_α are compact then so is $S = \prod_\alpha S_\alpha$.*

Proof. Let $\mathcal{F}$ be a family of closed subsets of S with the finite intersection property. We must show that the intersection of all the elements of $\mathcal{F}$ is non-empty. Using Zorn, extend $\mathcal{F}$ to a maximal family $\mathcal{F}_0$ of subsets of S (not necessarily all closed) with the property that the intersection of any finite collection of elements of $\mathcal{F}_0$ is non-empty. For each α, the projection $f_\alpha(\mathcal{F}_0)$ of these sets onto S_α has the property that there is a point $x_\alpha \in S_\alpha$ which is in the closure of all the sets belonging to $f_\alpha(\mathcal{F}_\prime)$. Let x be the point whose α coordinate is x_α. We will show that x is in the closure of every set of $\mathcal{F}_0$ which will complete the proof.

Let U be an open set containing x. Be the definition of the topology, there are finitely many α_i and open subsets $U_{\alpha_i} \subset S_{\alpha_i}$ such that

$$x \in \bigcap_{i=1}^{n} f_{\alpha_i}^{-1}(U_{\alpha_i}) \subset U.$$

So for each $i = 1, \ldots, n,\quad x_{\alpha_i} \in U_{\alpha_i}$ and so U_{α_i} intersects every set belonging to $f_{\alpha_i}(\mathcal{F}_0)$. So $f_{\alpha_i}^{-1}(U_{\alpha_i})$ intersects each set belonging to $\mathcal{F}_0$ and hence must belong to $\mathcal{F}_0$ by maximality. Hence

$$\bigcap_{i=1}^{n} f_{\alpha_i}^{-1}(U_{\alpha_i}) \in \mathcal{F}_0.$$

This says that U intersects every set of $\mathcal{F}_0$. This says that any neighborhood of x intersects every set belonging to $\mathcal{F}_0$, which is just another way of saying that x belongs to the closure of every set belonging to $\mathcal{F}_0$. QED