

# The Uniform Boundedness Principle.

Math 212

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**Theorem 1** *Let  $\mathbf{B}$  be a Banach space and  $\{F_n\}$  be a sequence of elements in  $B^*$  such that for every fixed  $x \in \mathbf{B}$  the sequence of numbers  $\{|F_n(x)|\}$  is bounded. Then the sequence of norms  $\{\|F_n\|\}$  is bounded.*

**Proof.** The proof will be by a Baire category style argument. We will prove

**Proposition 1** *There exists some ball  $B = B(y, r)$ ,  $r > 0$  and a constant  $K$  such that  $|F_n(z)| \leq K$  for all  $z \in B$ .*

**Proof that the proposition implies the theorem.** For any  $z$  with  $\|z\| < r$  we have

$$|F_n(z)| \leq |F_n(z - y)| + |F_n(y)| \leq 2K.$$

So

$$\|F_n\| \leq \frac{2K}{r}$$

for all  $n$  proving the theorem from the proposition.

**Proof of the proposition.** If the proposition is false, we can find  $n_1$  such that  $|F_{n_1}(x)| > 1$  at some  $x \in B(0, 1)$  and hence in some ball of radius  $\epsilon < \frac{1}{2}$  about  $x$ . Then we can find an  $n_2$  with  $|F_{n_2}(z)| > 2$  in some non-empty closed ball of radius  $< \frac{1}{3}$  lying inside the first ball. Continuing inductively, we choose a subsequence  $n_m$  and a family of nested non-empty balls  $B_m$  with  $|F_{n_m}(z)| > m$  throughout  $B_m$  and the radii of the balls tending to zero. Since  $B$  is complete, there is a point  $x$  common to all these balls, and  $\{|F_n(x)|\}$  is unbounded, contrary to hypothesis. QED

We will have occasion to use this theorem in a “reversed form”. Recall that we have the norm preserving injection  $B \rightarrow B^{**}$  sending  $x \mapsto x^{**}$  where  $x^{**}(F) = F(x)$ . Since  $B^*$  is a Banach space (even if  $B$  is incomplete) we have

**Corollary 1** *If  $\{x_n\}$  is a sequence of elements in a normed linear space such that the numerical sequence  $\{|F(x_n)|\}$  is bounded for each fixed  $F \in B^*$  then the sequence of norms  $\{\|x_n\|\}$  is bounded.*