Urysohn's Lemma

Math 212

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A topological space S is called **normal** if it is Hausdorff, and if for any pair F_1, F_2 of closed sets with $F_1 \cap F_2 = \emptyset$ there are disjoint open sets U_1, U_2 with $F_1 \subset U_1$ and $F_2 \subset U_2$. For example, suppose that S is Hausdorff and compact. For each $p \in F_1$ and $q \in F_2$ there are neighborhoods O_q of p and W_q of q with $O_p \cap W_q = \emptyset$. This is the Hausdorff axiom. A finite number of the W_q cover F_2 since it is compact. Let the intersection of the corresponding O_q be called U_p and the union of the corresonding W_q be called V_p. Thus for each $p \in F_1$ we have found a neighborhood U_p of p and an open set V_p containing F_2 with $U_p \cap V_p = \emptyset$. Once again, finitely many of the U_p cover F_1. So the union U of these and the intersection V of the corresponding V_p give disjoint open sets$U$ containing F_1 and V containing F_2. So any compact Hausdorff space is normal.

Theorem 1 [**Urysohn's lemma.**] *If F_0 and F_1 are disjoint closed sets in a normal space S then there is a continuous real valued function $f : S \to \mathbf{R}$ such that $0 \leq f \leq 1$, $f = 0$ on F_0 and $f = 1$ on F_1.*

Proof. Let

$$V_1 := F_1^c.$$

We can find an open set $V_{\frac{1}{2}}$ containing F_0 and whose closure is contained in V_1, since we can choose $V_{\frac{1}{2}}$ disjoint from an open set containing F_1. So we have

$$F_0 \subset V_{\frac{1}{2}}, \quad \overline{V_{\frac{1}{2}}} \subset V_1.$$

Applying our normality assumption to the sets F_0 and $V_{\frac{1}{2}}^c$ we can find an open set $V_{\frac{1}{4}}$ with $F_0 \subset V_{\frac{1}{4}}$ and $\overline{V_{\frac{1}{4}}} \subset V_{\frac{1}{2}}$. Similarly, we can find an open set $V_{\frac{3}{4}}$ with $\overline{V_{\frac{1}{2}}} \subset V_{\frac{3}{4}}$ and $\overline{V_{\frac{3}{4}}} \subset V_1$. So we have

$$F_0 \subset V_{\frac{1}{4}},\ \overline{V_{\frac{1}{4}}} \subset V_{\frac{1}{2}},\ \overline{V_{\frac{1}{2}}} \subset V_{\frac{3}{4}},\ \overline{V_{\frac{3}{4}}} \subset V_1 = F_1^c.$$

Continuing in this way, for each $0 < r < 1$ where r is a dyadic rational, $r = m/2^k$ we produce an open set V_r with $F_0 \subset V_r$ and $\overline{V_r} \subset V_s$ if $r < s$, including $\overline{V_r} \subset V_1 = F_1^c$. So $f(x) = 1$ for $x \in F_1$. Otherwise, define

$$f(x) = \inf\{r | x \in V_r\}.$$

So $f = 0$ on F_0.

If $0 < b \leq 1$, then $f(x) < b$ means that $x \in V_r$ for some $r < b$. Thus

$$f^{-1}([0, b)) = \bigcap_{r<b} V_r.$$

This is a union of open sets, hence open. Similarly, $f(x) > a$ means that there is some $r > a$ such that $x \notin \overline{V_r}$. Thus

$$f^{-1}((a, 1]) = \bigcap_{r>a} (\overline{V_r})^c,$$

also a union of open sets, hence open. So we have shown that

$$f^{-1}([0, b)) \quad \text{and} \quad f^{-1}((a, 1])$$

are open. Hence $f^{-1}((a, b))$ is open. Since the intervals $[0, b), (a, 1]$ and (a, b) form a basis for the open sets on the interval $[0, 1]$, we see that the inverse image of any open set under f is open, which says that f is continuous.