Machado's Theorem

Math 212a

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Let $\mathcal{M}$ be a compact space and let $C_{\mathbf{R}}(\mathcal{M})$ denote the algebra of continuous real valued functions on $\mathcal{M}$. We let $\|\cdot\| = \|\cdot\|_\infty$ denote the uniform norm on $C_{\mathbf{R}}(\mathcal{M})$. More generally, for any closed set $F \subset \mathcal{M}$, we let

$$\|f\|_F = \sup_{x \in F} |f(x)|$$

so $\|\cdot\| = \|\cdot\|_{\mathcal{M}}$.

If $A \subset C_{\mathbf{R}}(\mathcal{M})$ is a collection of functions, we will say that a subset $E \subset \mathcal{M}$ is a **level set** (for A) if all the elements of A are constant on the set E. Also, for any $f \in C_{\mathbf{R}}(\mathcal{M})$ and any closed set $F \subset \mathcal{M}$, we let

$$d_f(F) := \inf_{g \in A} \|f - g\|_F.$$

So $d_f(F)$ measures how far f is from the elements of A on the set F. (I have suppressed the dependence on A in this notation.) We can look for "small" closed subsets which measure how far f is from A on all of $\mathcal{M}$; that is we look for closed sets with the property that

$$d_f(E) = d_f(\mathcal{M}). \tag{1}$$

Let $\mathcal{F}$ denote the collection of all non-empty closed subsets of $\mathcal{M}$ with this property. Clearly $\mathcal{M} \subset \mathcal{F}$ so this collection is not empty. We order $\mathcal{F}$ by the reverse of inclusion: $F_1 \prec F_2$ if $F_1 \supset F_2$. Let $\mathcal{C}$ be a totally ordered subset of $\mathcal{F}$. Since $\mathcal{M}$ is compact, the intersection of any nested family of non-empty closed sets is again non-empty. We claim that the intersection of all the sets in $\mathcal{C}$ belongs to $\mathcal{F}$, i.e. satisfies (1). Indeed, since $d_f(F) = d_F(\mathcal{M})$ for any $F \in \mathcal{C}$ this means that for any $g \in A$, the sets

$$\{x \in F \| f(x) - g(x)| \geq d_f(K)\}$$

are non-empty. They are also closed and nested, and hence have a non-empty intersection. So on the set

$$E = \bigcap_{F \in \mathcal{C}} F$$

we have

$$\|f - g\|_E \geq d_f(K).$$

So every chain has an upper bound, and hence by Zorn's lemma, there exists a maximum, i.e. there exists a non-empty closed subset E satisfying (1) which has the property that no proper subset of E satisfies (1). We shall call such a subset f-**minimal**.

Theorem 0.1 [**Machado.**] *Suppose that $A \subset C_{\mathbf{R}}(\mathcal{M})$ is a subalgebra which contains the constants and which is closed in the uniform topology. Then for every $f \in C_{\mathbf{R}}(\mathcal{M})$ there exists an A level set satisfying(1). In fact, every f-minimal set is an A level set.*

Proof. Let E be an f-minimal set. Suppose it is not an A level set. This means that there is some $h \in A$ which is not constant on A. Replacing h by $ah + c$ where a and c are constant, we may arrange that

$$\min_{x\in E} h = 0 \quad \text{and} \ \max_{x\in E} h = 1.$$

Let

$$E_0 := \{x \in E | 0 \leq h(x) \leq]frac23\} \ \text{ and } \ E_1 : \{x \in E | \frac{1}{3} \leq x \leq 1\}.$$

These are non-empty closed proper subsets of E, and hence the minimality of E implies that there exist $g_0, g_1 \in A$ such that

$$\|f - g_0\|_{E_0} < d_f(K) \quad \text{and} \quad \|f - g_1\|_{E_1} < d_f(K).$$

Define

$$h_n := (1 - h^n)^{2^n} \quad \text{and} \quad k_n := h_n g_0 + (1 - h_n) g_1.$$

Both h_n and k_n belong to A and $0 \leq h_n \leq 1$ on E, with strict inequality on $E_0 \cap E_1$. On this intersection we have

$$\begin{aligned} \|f - k_n\|_{E_0\cap E_1} &= \|h_n f - h_n g_0 + (1-h_n) f - (1-h_n) g_1|_{E_0\cap E_1} \\ &\leq h_n\|f - g_0|_{E_0\cap E_1} + (1-h_n)\|f - g_1|_{E_0\cap E_1} \\ &\leq h_n\|f - g_)\|_{E_0} + (1-h_n)\|f - g_1\|_{E_1} < d_f(K). \end{aligned}$$

We will now show that $h_n \to 1$ on $E_0 \setminus E_1$ and $h_n \to 0$ on $E_1 \setminus E_0$. Indeed, on $E_0 \setminus E_1$ we have

$$h^n < \left(\frac{1}{3}\right)^n$$

so

$$h_n = (1 - h^n)^{2^n} \geq 1 - 2^n h^n \geq 1 - \left(\frac{2}{3}\right)^n \to 1$$

since the binomial formula gives an alternating sum with decreasing terms. On the other hand,

$$h_n (1 + h^n)^{2^n} = 1 - h^{2\cdot 2^n} \leq 1$$

or

$$h_n \leq \frac{1}{(1 + h^n)^{2^n}}.$$

Now the binomial formula implies that for any integer k and any positive number a we have $ka \le (1+a)^k$ or $(1+a)^-k \le 1/(ka)$. So we have

$$h_n \le \frac{1}{2^n h^n}.$$

On $E_0 \setminus E_1$ we have $h^n \ge \left(\frac{2}{3}\right)^n$ so there we have

$$h_n \le \left(\frac{3}{4}\right)^n \to 0.$$

Thus $k_n \to g_0$ uniformly on $E_0 \setminus E_1$ and $k_n \to g_1$ uniformly on $E_1 \setminus E_0$. We conclude that for n large enough

$$\|f - k_n\|_E < d_f(K)$$

contradicting our assumption that $d_f(K) = d_f(K)$. QED

Corollary 0.1 [**The Stone-Weierstrass Theorem.**] *If A is a uniformly closed subalgebra of $C_{\mathbf{R}}(\mathcal{M})$ which contains the constants and separates points, then $A = C_{\mathbf{R}}(\mathcal{M})$.*

Proof. The only A-level sets are points. But since $\|f - f(a)\|_{\{a\}} = 0$, we conclude that $d_f(\Updownarrow = 0$, i.e. $f \in A$ for any $f \in C_{\mathbf{r}}(\mathcal{M})$. QED