The Spectral Theorem.

Math 212

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The purpose of today's lecture is to show how the Gelfand representation theorem, especially for algebras with involution, implies the spectral theorem for bounded self-adjoint (or, more generally normal) operators on a Hilbert space. An operator T is called **normal** if it commutes with T^*. More generally, an element T of a Banach algebra A with involution is called normal if $TT^* = T^*T$.

Let B be the closed commutative subalgebra generated by e, T and T^*. We know that B is isometrically isomorphic to the ring of continuous functions on $\mathcal{M}$, the space of maximal ideals of B, which is the same as the space of continuous homomorphisms of B into the complex numbers. The plan is to show that $\mathcal{M}$ can also be identified with $\text{Spec}_A(T)$, the set of all $\lambda \in \mathbf{C}$ such that $(T - \lambda e)$ is not invertible. This means that every continuous continuous function $\hat{f}$ on $\mathcal{M} = \text{Spec}_A(T)$ corresponds to an element f of B. In the case where A is the algebra of bounded operators on a Hilbert space, we will show that this homomorphism extends to the space of Borel functions on $\text{Spec}(T)$.(In general the image of the extended homomorphism will lie in A, but not necessarily in B.)

If U is a Borel subset of $\text{Spec}(T)$, let us denote the element of A corresponding to $\mathbf{1}_U$ by $P(U)$. Then

$$P(U)^2 = P(U) \quad \text{and} \quad P(U)^* = P(U)$$

so $P(U)$ is a self adjoint (i.e. "orthogonal") projection. Also, if $U \cap V = \emptyset$ then $P(U)P(V) = 0$ and

$$P(U \cup V) = P(U) + P(V).$$

Thus $U \mapsto P(U)$ is finitely additive. In fact, it is completely additive in the weak sense that for any pair of vectors x, y in our Hilbert space H the map

$$\mu_{x,y} : \quad U \mapsto (P(U)x, y)$$

is a complex valued measure on $\mathcal{M}$. We shall prove these results in partially reversed order, in that we first prove the existence of the complex valued measure $\mu_{x,y}$ using the Riesz-Markov theorem, and then deduce the existence of the **resolution of the identity** or **projection valued measure**

$$U \mapsto P(U),$$

(more precise definition below) from which we can recover T. The key tool, in addition to the Gelfand representation theorem and the Riesz-Markov theorem describing all continuous linear functions on $C(\mathcal{M})$ as being signed measures is our old friend, the Riesz representation theorem for continuous linear functions on a Hilbert space.

Contents

1 Resolutions of the identity.

In this section B denotes a closed commutative self-adjoint subalgebra of the algebra of all bounded linear operators on a Hilbert space H. (Self-adjoint means that $T \in B \Rightarrow T^* \in B$.) By the Gelfand representation theorem we know that B is isometrically isomorphic to $C(\mathcal{M})$ under a map we denote by

$$T \mapsto \hat{T}$$

and we know that

$$T^* \mapsto \overline{\hat{T}}.$$

Fix

$$x, y \in H.$$

The map

$$\hat{T} \mapsto (Tx, y)$$

is a linear function on $C(\mathcal{M})$ with

$$|(Tx, y)| \leq \|T\|\|x\|\|y\| = \|\hat{T}\|_\infty \|x\|\|y\|.$$

In particular it is a continuous linear function on $C(\mathcal{M})$. Hence, by the Riesz-Markov theorem, there exists a unique complex valued bounded measure

$$\mu_{x,y}$$

such that

$$(Tx, y) = \int_{\mathcal{M}} \hat{T} d\mu_{x,y} \quad \forall \hat{T} \in C(\mathcal{M}).$$

When $\hat{T}$ is real, $T = T^*$ so $(Tx, y) = (x, Ty) = \overline{(Ty, x)}$. The uniqueness of the measure implies that

$$\mu_{y,x} = \overline{\mu_{x,y}}.$$

Thus, for each fixed Borel set $U \subset \mathcal{M}$ its measure $\mu_{x,y}(U)$ depends linearly on x and anti-linearly on y. We have

$$\mu(\mathcal{M}) = \int_{\mathcal{M}} \mathbf{1} d\mu_{x,y} = (ex, y) = (x, y)$$

so

$$|\mu_{x,y}(\mathcal{M})| \leq \|x\| \|y\|.$$

So if f is any bounded Borel function on $C(\mathcal{M})$, the integral

$$\int_{\mathcal{M}} f d\mu_{x,y}$$

is well defined, and is bounded in absolute value by $\|f\|_\infty \|x\| \|y\|$. If we hold f and x fixed, this integral is a bounded anti-linear function of y, and hence by the Riesz representation theorem there exists a $w \in H$ such that this integral is given by (w, y). The w in question depends linearly on f and on x because the integral does, and so we have defined a linear map O from bounded Borel functions on $\mathcal{M}$ to bounded operators on H such that

$$(O(f)x, y) = \int_{\mathcal{M}} f d\mu_{x,y}$$

and

$$\|O(f)\| \leq \|f\|_\infty.$$

On continuous functions we have

$$O(\hat{T}) = T$$

so O is an extension of the inverse of the Gelfand transform from continuous functions to bounded Borel functions. So we know that O is multiplicative and takes complex conjugation into adjoint when restricted to continuous functions. Let us prove these facts for all Borel functions. If f is real we know that $(O(f)y, x)$ is the complex conjugate of $(O(f)x, y)$ since $\mu_{y,x} = \overline{\mu_{x,y}}$. Hence $O(f)$ is self-adjoint if f is real from which we deduce that

$$O(\overline{f}) = O(f)^*.$$

Now to the multiplicativity: For $S, T \in B$ we have

$$\int_{\mathcal{M}} \hat{S}\hat{T} d\mu_{x,y} = (STx, y) = \int_{\mathcal{M}} \hat{S} d\mu_{Tx,y}.$$

Since this holds for all $\hat{S} \in C(\mathcal{M})$ (for fixed T, x, y) we conclude by the uniqueness of the measure that

$$\mu_{Tx,y} = \hat{T}\mu_{x,y}.$$

Therefore, for any bounded Borel function f we have

$$(Tx, O(f)^* y) = (O(f)Tx, y) = \int_{\mathcal{M}} f d\mu_{Tx,y} = \int_{\mathcal{M}} \hat{T} f d\mu_{x,y}.$$

This holds for all $T \in C(\mathcal{M})$ and so by the uniqueness of the measure again, we conclude that

$$\mu_{x,O(f)^*y} = f\mu_{x,y}$$

and hence

$$(O(fg)x, y) = \int_{\mathcal{M}} gf d\mu_{x,y} = \int_{\mathcal{M}} g d\mu_{x,O(f)^*y} = (O(g)x, O(f)^*y) = (O(f)O(g)x, y)$$

or

$$O(fg) = O(f)O(g)$$

as desired.

We have now extended the homomorphism from $C(\mathcal{M})$ to A to a homomorphism from the bounded Borel functions on $\mathcal{M}$ to bounded operators on H.

Now define:

$$P(U) := O(\mathbf{1}_U)$$

for any Borel set U. The following facts are immediate:

1. $P(\emptyset) = 0$
2. $P(\mathcal{M}) = e$ the identity
3. $P(U \cap V) = P(U)P(V)$ and $P(U)^* = P(U)$. In particular, $P(U)$ is a self-adjoint projection operator.
4. If $U \cap V = \emptyset$ then $P(U \cup V) = P(U) + P(V)$.
5. For each fixed $x, y \in H$ the set function $P_{x,y} : U \mapsto (P(U)x, y)$ is a complex valued measure.

Such a P is called a **resolution of the identity**. It follows from the last item that for any fixed $x \in H$, the map $U \mapsto P(U)x$ is an H valued measure.

We have shown that any commutative closed self-adjoint subalgebra B of the algebra of bounded operators on a Hilbert space H gives rise to a unique resolution of the identity on $\mathcal{M} = \mathrm{Mspec}(B)$ such that

$$T = \int_{\mathcal{M}} \hat{T} dP \tag{1}$$

in the "weak" sense that

$$(Tx, y) = \int_{\mathcal{M}} \hat{T} d\mu_{x,y} \quad \mu_{x,y}(U) = (P(U)x, y).$$

Actually, given any resolution of the identity we can give a meaning to the integral

$$\int_{\mathcal{M}} f dP$$

for any bounded Borel function f in the strong sense as follows: if

$$s = \sum \alpha_i \mathbf{1}_{U_i}$$

is a simple function where

$$\mathcal{M} = U_1 \cup \cdots \cup U_n, \quad U_i \cap U_j = \emptyset, \quad i \neq j$$

and $\alpha_1, \ldots, \alpha_n \in \mathbf{C}$, define

$$O(s) := \sum \alpha_i P(U_i) =: \int_{\mathcal{M}} s dP.$$

This is well defined on simple functions (is independent of the expression) and is multiplicative

$$O(st) = O(s)O(t).$$

Also, since the $P(U)$ are self adjoint,

$$O(\overline{s}) = O(s)^*.$$

It is also clear that O is linear and

$$(O(s)x, y) = \int_{\mathcal{M}} s dP_{x,y}.$$

As a consequence, we get

$$\|O(s)x\|^2 = (O(s)^* O(s)x, x) = \int_{\mathcal{M}} |s|^2 dP_{x,x}$$

so

$$\|O(s)x)\|^2 \leq |s\|_\infty \|x\|^2.$$

If we choose i such that $|\alpha_i| = \|s\|_\infty$ and take $x = P(U_i)y \neq 0$, then we see that

$$\|O(s)\| = \|s\|_\infty$$

provided we now take $\|f\|_\infty$ to denote the **essential supremum** which means the following:

It follows from the properties of a resolution of the identity that if U_n is a sequence of Borel sets such that $P(U_n) = 0$, then $P(U) = 0$ if $U = \bigcup U_n$. So if f is any complex valued Borel function on $\mathcal{M}$, there will exist a largest open subset $V \subset \mathbf{C}$ such that $P(f^{-1}(V)) = 0$. We define the **essential range** of f to be the complement of V, say that f is **essentially bounded** if its essential range is compact, and then define its essential supremum $\|f\|_\infty$ to be the supremum of $|\lambda|$ for λ in the essential range of f. Furthermore we identify two essentially bounded functions f and g if $\|f - g\|_\infty = 0$ and call the corresponding space $L^\infty(P)$.

Every element of $L^\infty(P)$ can be approximated in the $\| \|_\infty$ norm by simple functions, and hence the integral

$$O(f) = \int_{\mathcal{M}} f dp$$

is defined as the strong limit of the integrals of the corresponding simple functions. The map $f \mapsto O(f)$ is linear, multiplicative, and satisfies

$$O(\overline{f}) = O(f)^*$$

and

$$\|O(f)\| = \|f\|_\infty$$

as before.

If S is a bounded operator on H which commutes with all the $O(f)$ then it commutes with all the $P(U) = O(\mathbf{1}_U)$. Conversely, if S commutes with all the $P(U)$ it commutes with all the $O(s)$ for s simple and hence with all the $O(f)$.

Putting it all together we have:

Theorem 1 *Let B be a commutative closed self adjoint subalgebra of the algebra of all bounded operators on a Hilbert space H. Then there exists a resolution of the identity P defined on $\mathcal{M} = \mathrm{Mspec}(B)$ such that (1) holds. The map $\hat{T} \mapsto T$ of $C(\mathcal{M}) \to B$ given by the inverse of the Gelfand transform extends to a map O from $L^\infty(P)$ to the space of bounded operators on H*

$$O(f) = \int_{\mathcal{M}} f dP.$$

Furthermore, $P(U) \neq 0$ for any non-empty open set U and an operator S commutes with every element of B if and only if it commutes with all the $P(U)$ in which case it commutes with all the $O(f)$.

We must prove the last two statements. If U is open, we may choose $T \neq 0$ such that $\hat{T}$ is supported in U (by Urysohn's lemma). But then (1) implies that $T = 0$, a contradiction.

For any bounded operator S and any $x, y \in H$ and $T \in B$ we have

$$(STx, y) = (Tx, S^*y) = \int \hat{T} dP_{x,S^*y}$$

while

$$(TSx, y) = \int \hat{T} dP_{Sx,y}.$$

If $ST = TS$ for all $T \in B$ this means that the measures $P_{Sx,y}$ and P_{x,S^*y} are the same, which means that

$$(P(U)Sx, y) = (P(U)x, S^*y) = (SP(U)x, y)$$

for all x and y which means that

$$SP(U) = P(U)S$$

for all U. We already know that $SP(U) = P(U)S$ for all S implies that $SO(f) = O(f)S$ for all $f \in L^\infty(P)$. QED

2 The spectral theorem for bounded normal operators.

Let T be a bounded operator on a Hilbert space H satisfying

$$TT^* = T^*T.$$

Recall that such an operator is called normal. Let B be the closure of the algebra generated by e, T and T^*. We can apply the theorem of the preceding section to this algebra. The one useful additional fact is that *we may identify* $\mathcal{M}$ *with* $\mathrm{Spec}(T)$. Indeed, define the map

$$\mathcal{M} \to \mathrm{Spec}(T)$$

by

$$h \mapsto h(T).$$

We know that $h(T - h(T)e) = 0$ so $T - h(T)e$ lies in the maximal ideal corresponding to h and so is not invertible, consequently $h(T) \in \mathrm{Spec}(T)$. So the map is indeed into $\mathrm{Spec}(T)$. If $\lambda \in \mathrm{Spec}(T)$ then by definition $T - \lambda e$ is not invertible, hence lie in some maximal ideal, hence $\lambda = h(T)$ for some T so this map is surjective. If $h_1(T) = h_2(T)$ then $h_1(T^*) = \overline{h_1(T)} = h_2(T^*)$. Since h_1 agrees with h_2 on T and T^* they agree on all of B, hence $h_1 = h_2$. In other words the map $h \mapsto h(T)$ is injective. From the definition of the topology on $\mathcal{M}$ it is continuous. Since $\mathcal{M}$ is compact, this implies that it is a homeomorphism. QED

Thus in the theorem of the preceding section, we may replace $\mathcal{M}$ by $\mathrm{Spec}\, T$ when B is the closed algebra generated by T and T^* where T is a normal operator.

In the case that T is a self-adjoint operator, we know that $\mathrm{Spec}\, T \subset \mathbf{R}$, so our resolution of the identity P is defined as a projection valued measure on $\mathbf{R}$ and (1) gives a bounded selfadjoint operator as

$$A = \int \lambda dP$$

relative to a resolution of the identity defined on $\mathbf{R}$.

3 Stone's formula.

Let T be a bounded self-adjoint operator. We know that

$$T = \int_{\mathbf{R}} \lambda dP(\lambda)$$

for some projection valued measure P on $\mathbf{R}$. We also know that every bounded Borel function on $\mathbf{R}$ gives rise to an operator. In particular, if z is a complex number which is not real, the function

$$\lambda \mapsto \frac{1}{\lambda - z}$$

is bounded, and hence corresponds to a bounded operator

$$R(z,T) = \int_{\mathbf{R}} (z-\lambda)^{-1} dP(\lambda).$$

Since

$$(ze - T) = \int_{\mathbf{R}} (z-\lambda) dP(\lambda)$$

and our homomorphism is multiplicative, we have

$$R(z,T) = (z-T)^{-1}.$$

A conclusion of the above argument is that this inverse does indeed exist for all non-real z. The operator (valued function) $R(z,T)$ is called the **resolvent** of T. **Stone's formula** gives an expression for the projection valued measure in terms of the resolvent. It says that for any real numbers $a < b$ we have

$$\text{s-}\lim_{\epsilon \to 0} \frac{1}{2\pi i} \int_a^b \left[R(\lambda - i\epsilon, T) - R(\lambda + i\epsilon, T)\right] d\lambda = \frac{1}{2}\left(P((a,b)) + P([a,b])\right). \quad (2)$$

Although this formula cries out for a "complex variables" proof, and I plan to give one later, we can give a direct "real variables" proof in terms of what we already know. Indeed, let

$$f_\epsilon(x) := \frac{1}{2\pi i} \int_a^b \left(\frac{1}{x - \lambda - i\epsilon} - \frac{1}{x - \lambda + i\epsilon} \right) d\lambda.$$

We have

$$f_\epsilon(x) = \frac{1}{\pi} \int_a^b \frac{\epsilon}{(x-\lambda)^2 + \epsilon^2} d\lambda = \frac{1}{\pi}\left(\arctan\left[\frac{x-a}{\epsilon}\right] - \arctan\left[\frac{x-b}{\epsilon}\right]\right).$$

The expression on the right is uniformly bounded, and approaches zero if $x \notin [a,b]$, approaches $\frac{1}{2}$ if $x = a$ or $x = b$, and approaches 1 if $x \in (a,b)$. In short,

$$\lim_{\epsilon \to 0} f_\epsilon(x) = \frac{1}{2}(\mathbf{1}_{(a,b)} + \mathbf{1}_{[a,b]}).$$

We may apply the dominated convergence theorem to conclude Stone's formula. QED