# Wiener measure.

Math 212

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We begin by constructing Wiener measure following a paper by Nelson *HOurnal of Mathematical Physics* **5** (1964) 332-343.

## 1 Path space.

Let $\dot{\mathbf{R}}^{\mathbf{n}}$ denote the one point compactification of $\mathbf{R}^n$. Let

$$\Omega := \prod_{0 \leq t < \infty} \dot{\mathbf{R}}^{\mathbf{n}}$$

be the product of copies of $\dot{\mathbf{R}}^{\mathbf{n}}$, one for each non-negative $t$. This is an uncountable product, and so a huge space, but by Tychonoff's theorem, it is compact and Hausdorff. We can think of a point $\omega$ of $\Omega$ as being a function from $\mathbf{R}_+$ to $\dot{\mathbf{R}}^{\mathbf{n}}$, i.e. as as a "curve" with no restrictions whatsoever.

Let $F$ be a continuous function on the $m$-fold product:

$$F : \prod_{i=1}^{m} \dot{\mathbf{R}}^{\mathbf{n}} \to \mathbf{R},$$

and let $t_1 \leq t_2 \leq \cdots \leq t_m$ be fixed "times". Define

$$\phi = \phi_{F;t_1,\ldots,t_m} : \ \Omega \to \mathbf{R}$$

by

$$\phi(\omega) := F(\omega(t_1), \ldots, \omega(t_m)).$$

We can call such a function a **finite** function since its value at $\omega$ depends only on the values of $\omega$ at finitely many points. The set of such functions satisfies our abstract axioms for a space on which we can define integration. Furthermore, the set of such functions is an algebra containing 1 and which separates points, so is dense in $C(\Omega)$ by the Stone-Weierstrass theorem. Let us call the space of such functions $C_{fin}(\Omega)$.

If we define an integral $L$ on $C_{fin}(\Omega)$ then, by the Stone-Weierstrass theorem it extends to $C(\Omega)$ and therefore gives us a regular Borel measure on $\Omega$.

For each $x \in \mathbf{R}^n$ we are going to define such an integral, $L_x$ by

$$L_x(\phi) =$$

$$\int \cdots \int F(x_1, x_2, \ldots, x_m) p(x, x_1; t_1) p(x_1, x_2; t_2 - t_1) \cdots p(x_{m-1}, x_m, t_m - t_{m-1}) dx_1 \ldots dx_m$$

when $\phi = \phi_{F, t_1}, \ldots, t_m$ where

$$p(x, y; t) = \frac{1}{(4\pi t)^{n/2}} e^{-(x-y)^2/4t}$$

(with $p(x, \infty) = 0$) and all integrations are over $\dot{\mathbf{R}}^{\mathbf{n}}$. In order to check that this is well defined, we have to verify that if $F$ does not depend on a given $x_i$ then we get the same answer if we define $\phi$ in terms of the corresponding function of the remaining $m - 1$ variables. This amounts to the computation

$$\int p(x, y; s) p(y, z, t) dy = p(x, z; s + t)$$

which is just the expression of the semi-group property of the heat kernel. We denote the measure corresponding to $L_x$ by $\mathrm{pr}_x$. It is a probablity measure in the sense that $\mathrm{pr}_x(\Omega) = 1$.

The intuitive idea behind the definition of $L_x$ is that it assigns probability

$$pr_x(E) := \int_{E_1} \cdots \int_{E_m} p(x, x_1; t_1) p(x_1, x_2; t_2 - t_1) \cdots p(x_{m-1}, x_m, t_m - t_{m-1}) dx_1 \ldots dx_m$$

to the set of all paths $\omega$ which start at $x$ and pass through the set $E_1$ at time $t_1$, the set $E_2$ at time $t_2$ etc. and we have denoted this set of paths by $E$.

We now turn to some technical issues. We recall that the statement that a measure $\mu$ is regular means that for any Borel set $A$

$$\mu(A) = \inf\{\mu(G) : \ A \subset G, \ G \text{ open }\}$$

and

$$\mu(A) = \sup\{\mu(K) : \ K \subset A, \ K \text{ compact}\}.$$

This second condition has the following consequence: Suppose that $\Gamma$ is any collection of open sets which is closed under finite union. If

$$O = \bigcup_{G \in \Gamma} G$$

then

$$\mu(O) = \sup_{G \in \Gamma} \mu(G)$$

since any compact subset of $O$ is covered by finitely many sets belonging to $\Gamma$. The importance of this stems from the fact that we can allow $\Gamma$ to consist of uncountably many open sets, and we will need to impose uncountably many conditions in singling out the space of continuous paths, for example. Indeed, our first task will be to show that the measure $\mathrm{pr}_x$ is concentrated on the space of continuous path in $\mathbf{R}^n$ which do not go to infinity too fast.

We begin with the following computation in one dimension:

$$\mathrm{pr}_0(\{|\omega(t)| > r\}) = 2 \cdot \left(\frac{1}{4\pi t}\right)^{1/2} \int_r^\infty e^{-x^2/4t} dx \le \left(\frac{1}{\pi t}\right)^{1/2} \int_r^\infty \frac{x}{r} e^{-x^2/4t} dx =$$

$$\left(\frac{1}{\pi t}\right)^{1/2} \frac{2t}{r} \int_r^\infty \frac{x}{2t} e^{-x^2/4t} dx = \left(\frac{4t}{\pi}\right)^{1/2} \frac{e^{-r^2/4t}}{r}.$$

For fixed $r$ this tends to zero (very fast) as $t \to 0$. Taking the product and introducing some constants like $1/\sqrt{n}$ having to do with the size of a cube inscribed in a ball in $n$-dimensions we find that

$$\mathrm{pr}_x(\{|\omega(t) - x| > \epsilon\}) \le c\epsilon^{n/2} e^{-\epsilon^2/4t}.$$

In particular, if we let $\rho(\epsilon, \delta)$ denote the supremum of the above probability over all $0 < t \le \delta$ then

$$\rho(\epsilon, \delta) = o(\delta). \tag{1}$$

**Lemma 1** *Let $0 \le t_1 \le \cdots \le t_m$ with $t_m - t_1 \le \delta$. Let*

$$A := \{\omega|\ |\omega(t_j) - \omega(t_1)| > \epsilon \quad \textit{for some } j = 1, \ldots m\}.$$

*Then*

$$\mathrm{pr}_x(A) \le 2\rho(\frac{1}{2}\epsilon, \delta) \tag{2}$$

*independently of the number $m$ of steps.*

**Proof.** Let

$$B := \{\omega|\ |\omega(t_1) - \omega(t_n)| > \frac{1}{2}\epsilon\}$$

let

$$C_i := \{\omega|\ |\omega(t_i) - \omega(t_n)| > \frac{1}{2}\epsilon\}$$

and let

$$D_i = \{\omega|\ |\omega(t_1) - \omega(t_i)| > \frac{1}{2}\epsilon \text{ and } |\omega(t_1) - \omega(t_k)| \le \epsilon\ k = 1, \ldots i-1\}.$$

If $\omega \in A$, then $\omega \in D_i$ for some $i$ by the definition of $A$, by taking $i$ to be the first $j$ that works in the defnition of $A$. If $\omega \notin B$ and $\omega \in D_i$ then $\omega \in C_i$ since it has to move a distance of at least $\frac{1}{2}\epsilon$ to get back from outside the ball of radius $\epsilon$ to inside the ball of radius $\frac{1}{2}\epsilon$. So we have

$$A \subset B \cup \bigcup_{i=1}^n (C_i \cap D_i)$$

and hence

$$\mathrm{pr}_x(A) \le \mathrm{pr}_x(B) + \sum_{i=1}^{n} \mathrm{pr}_x(C_i \cap D_i). \tag{3}$$

Now we can estimate $\mathrm{pr}_x(C_i \cap D_i)$ as follows. For $\omega$ to belong to this intersection, we must have $\omega \in D_i$ and then the path moves a distance at least $\frac{\epsilon}{2}$ in time $t_n - t_i$ and these two events are independent, so $\mathrm{pr}_x(C_i \cap D_i) \le \rho(\frac{\epsilon}{2}, \delta)\, \mathrm{pr}_x(D_i)$. Here is this argument in more detail: Let

$$F = \mathbf{1}_{\{(y,z)|\ |y-z|>\frac{1}{2}\epsilon\}}$$

so that

$$\mathbf{1}_{C_i} = \phi_{F,t_i,t_n}.$$

Similarly, let $G$ be the indicator function of the subset of $\dot{\mathbf{R}}^n \times \dot{\mathbf{R}}^n \times \cdots \times \dot{\mathbf{R}}^n$ ($i$ copies) consisting of all points with

$$|x_k - x_1| \le \epsilon, \quad k = 1, \dots, i-1, \quad |x_1 - x_i| > \epsilon$$

so that

$$\mathbf{1}_{D_i} = \phi_{G,t_1,\dots,t_j}.$$

Then

$$\mathrm{pr}_x(C_i \cap D_i) =$$

$$\int \cdots \int p(x, x_1; t_1) \cdots p(x_{i-1}, x_i; t_i - t_{i-1}) D(x_1, \dots, x_i) C(x_i, x_n) p(x_i, x_n) dx_1 \cdots dx_n.$$

The last integral (with respect to $x_n$) is $\le \rho(\frac{1}{2}\epsilon, \delta)$. Thus

$$\mathrm{pr}_x(C_i \cap D_i) \le \rho(\frac{\epsilon}{2}, \delta)\, \mathrm{pr}_x(D_i).$$

The $D_i$ are disjoint by definition, so

$$\sum \mathrm{pr}_x(D_i) \le \mathrm{pr}_x(\bigcup D_i) \le 1.$$

So

$$\mathrm{pr}_x(A) \le \mathrm{pr}_x(B) + \rho(\frac{1}{2}\epsilon, \delta) \le 2\rho(\frac{1}{2}\epsilon, \delta).$$

QED

Let

$$E : \{\omega | |\omega(t_i) - \omega(t_j)| > 2\epsilon \text{ for some } 1 \le j < k \le n\}.$$

Then $E \subset A$ since if $|\omega(t_j) - \omega(t_k)| > 2\epsilon$ then either $|\omega(t_1) - \omega(t_j)| > \epsilon$ or $|\omega(t_1) - \omega(t_k)| > \epsilon$ (or both). So

$$\mathrm{pr}_x(E) \le 2\rho(\frac{1}{2}\epsilon, \delta). \tag{4}$$

**Lemma 2** *Let $0 \leq a < b$ with $b - a \leq \delta$. Let*

$$E(a,b,\epsilon) := \{\omega | \ |\omega(s) - \omega(t)| > 2\epsilon \ \ \textit{for some} \ \ s,t \in [a,b]\}.$$

*Then*

$$\mathrm{pr}_x(E(a,b.\epsilon)) \leq 2\rho(\frac{1}{2}\epsilon, \delta).$$

**Proof.** Here is where we are going to use the regularity of the measure. Let $S$ denote a finite subset of $[a,b]$ and and let

$$E(a,b,\epsilon,S) := \{\omega | \ |\omega(s) - \omega(t)| > 2\epsilon \ \ \text{for some} \ \ s,t \in S\}.$$

Then $E(a,b,\epsilon,S)$ is an open set and $\mathrm{pr}_x(E(a,b,\epsilon,S)) < 2\rho(\frac{1}{2}\epsilon,\delta)$ for any $S$. The union over all $S$ of the $E(a,b,\epsilon,S)$ is $E(a,b,\epsilon)$. The regularity of the measure now implies the lemma. QED

Let $k$ and $n$ be integers, and set

$$\delta := \frac{1}{n}.$$

Let

$$F(k,\epsilon,\delta) := \{\omega | \ |\omega(t) - \omega(s)| > 4\epsilon \ \ \text{for some } t,s \in [0,k], \ \text{with } |t-s| < \delta\}.$$

Then we claim that

$$\mathrm{pr}_x(F(k,\epsilon,\delta)) < 2k\frac{\rho(\frac{1}{2}\epsilon,\delta)}{\delta}. \tag{5}$$

Indeed, $[0,k]$ is the union of the $nk = k/\delta$ subintervals $[0,\delta], [\delta, 2\delta], \ldots, [k-\delta, \delta]$. If $\omega \in F(k,\epsilon,\delta)$ then $|\omega(s) - \omega(t) > 4\epsilon$ for some $s$ and $t$ which lie in either the same or in adjacent subintervals. So $\omega$ must lie in $E(a,b,\epsilon)$ for one of these subintervals, and ther are $kn$ of them. QED

Let $\omega \in \Omega$ be a continuous path in $\mathbf{R}^n$. Restricted to any interval $[0,k]$ it is uniformly continuous. This means that for any $\epsilon > 0$ it belongs to the complement of the set $F(k,\epsilon,\delta)$ for some $\delta$. We can let $\epsilon = 1/p$ for some integer $p$. Let $\mathcal{C}$ denote the set of continuous paths from $[0,\infty)$ to $\mathbf{R}^n$. Then

$$\mathcal{C} \subset \bigcap_k \bigcap_\epsilon \bigcup_\delta F(k,\epsilon,\delta)^c$$

so the complement $\mathcal{C}^c$ of the set of continuous paths is contained in

$$\bigcup_k \bigcup_\epsilon \bigcap_\delta F(k,\epsilon,\delta),$$

a countable union of sets of measure zero since

$$\mathrm{pr}_x\left(\bigcap_\delta F(k,\epsilon,\delta)\right) \leq \lim_{\delta\to 0} 2k\rho(\frac{1}{2}\epsilon,\delta)/\delta = 0.$$

We have thus proved a famous theorem of Wiener:

**Theorem 1 Weiner.]** *The measure* $\mathrm{pr}_x$ *is concentrated on the space of continuous paths, i.e.* $\mathrm{pr}_x(\mathcal{C}) = 1$. *In particular, there is a probability measure on the space of continuous paths starting at the origin with*

$$\mathrm{pr}_0(E) = \int_{E_1} \cdots \int_{E_m} p(0, x_1; t)_1 p(x_1, x_2; t_2 - t_1) \cdots p(x_{m-1}, x_m, t_m - t_{m-1}) dx_1 \ldots dx_m$$

*to the set of all paths* $\omega$ *which start at* 0 *and pass through the set* $E_1$ *at time* $t_1$, *the set* $E_2$ *at time* $t_2$ *etc. and we have denoted this set of paths by* $E$.

For convenience in notation let me now specialize to the case $n = 1$. Let

$$\mathcal{W} \subset \mathcal{C}$$

consist of those paths $\omega$ with $\omega(0) = 0$ and

$$\int_0^\infty (1+t)^{-2} w(t) dt < \infty.$$

**Proposition 1 [Stroock]** *The Wiener measure* $\mathrm{pr}_0$ *is concentrated on* $\mathcal{W}$.

Indeed, we let $E(|\omega(t)|)$ denote the expectation of the function $|\omega(t)|$ of $\omega$ with respect to Wiener measure, so

$$E(|\omega(t)|) = \frac{1}{\sqrt{4\pi t}} \int_{\mathbf{R}} |x| e^{-x^2/4t} dx = \frac{1}{\sqrt{4\pi t}} \cdot 2t \int_0^\infty \frac{x}{2t} e^{-x^2/4t} dx = Ct^{-1/2}.$$

Thus, by Fubini,

$$E\left(\int_0^\infty (1+t)^{-2} |w(t)| dt\right) = \int_0^\infty (1+t)^{-2} E(|w(t)|) < \infty.$$

Hence the set of $\omega$ with $\int_0^\infty (1+t)^{-2} |w(t)| dt = \infty$ must have measure zero. QED

Now each element of $\mathcal{W}$ defines a tempered distribution, i.e. an element of $\mathcal{S}'$ according to the rule

$$\langle \omega, \phi \rangle = \int_0^\infty \omega(t) \phi(t) dt.$$

This map from $\mathcal{W}$ to $\mathcal{S}'$ is continuous, hence the Wiener measure pushes forward to give a measure on $\mathcal{S}'$. Our next step will be to examine this push forward measure due to Dan Stroock.