Solutions: Problem Set 1

1. We have

$$|C(f,n,x)| = \left|\frac{1}{2\pi}\int_0^{2\pi} f(x-y)K_n(y)\,dy\right| \le \|f\|_\infty \frac{1}{2\pi}\int_0^{2\pi} |K_n(y)|\,dy \tag{1}$$

$$= \|f\|_\infty \frac{1}{2\pi}\int_0^{2\pi} K_n(y)\,dy = \|f\|_\infty, \tag{2}$$

where the second to last equality follows from the fact that $K_n(y)$ is positive, since

$$K_n(y) = \frac{1}{n+1}\left(\frac{\sin(n+1)y/2}{\sin y/2}\right)^2,$$

and the last equality follows from the fact that the integral of $K_n(y)$ is equal to 2π.

2. We have

$$\frac{L^2-4\pi A}{2\pi^2} = \frac{1}{2\pi}\int_0^{2\pi}\left(2\left(\frac{L}{2\pi}\right)^2 + 4x(t)y'(t)\right)dt \tag{3}$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\left(2\left((x'(t))^2+(y'(t))^2\right)+4x(t)y'(t)\right)dt, \tag{4}$$

where the first equality follows from the fact that $A = \int_0^{2\pi} x(t)y'(t)\,dt$ and the second follows from the fact that $(x'(t))^2+(y'(t))^2 = (L/2\pi)^2$. Integration by parts shows that

$$\int_0^{2\pi} x(t)y'(t)\,dt = -\int_0^{2\pi} x'(t)y(t)\,dt,$$

and so

$$\frac{L^2-4\pi A}{2\pi^2} = \frac{1}{2\pi}\int_0^{2\pi}\left(2\left((x'(t))^2+(y'(t))^2\right)+2x(t)y'(t)-2x'(t)y(t)\right)dt \tag{5}$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\left(|x'(t)+y(t)|^2+|y'(t)-x(t)|^2-|x(t)|^2-|y(t)|^2\right)dt \tag{6}$$

$$= \sum_{n\in\mathcal{Z}}\left(|ina_n+b_n|^2+|inb_n-a_n|^2+|na_n|^2+|nb_n|^2-|a_n|^2-|b_n|^2\right)$$

$$= \sum_{n\neq 0}\left(|na_n-ib_n|^2+|nb_n+ia_n|^2+(n^2-1)\left(|a_n|^2+|b_n|^2\right)\right), \tag{7}$$

where the second to last equality follows from Parseval's identity. Therefore we conclude that $\frac{L^2-4\pi A}{2\pi^2} \ge 0$, and so $A \le \frac{L^2}{4\pi}$, as desired.

3. The nth Fourier coefficient of e^{ax} is

$$a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ax}e^{-inx}\,dx = \frac{1}{2\pi}\frac{1}{a-in}\left(e^{(a-in)\pi}-e^{-(a-in)\pi}\right).$$

Thus the Fourier series of e^{ax} is

$$\frac{1}{2\pi}\sum_{n\in\mathcal{Z}}\frac{1}{a-in}\left(e^{(a-in)\pi}-e^{-(a-in)\pi}\right)e^{inx},$$

which can be rewritten as

$$\frac{e^{a\pi}-e^{-a\pi}}{2\pi a}+\frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^n\frac{a(e^{a\pi}-e^{-a\pi})}{a^2+n^2}\cos(nx)-(-1)^n\frac{n(e^{a\pi}-e^{-a\pi})}{a^2+n^2}\sin(nx).$$

In order to calculate the Fourier series of $\cos(ax)$, we can use the above computations to find the Fourier series for e^{iax} and e^{-iax} and then average them together. This results in

$$\cos(ax)=\frac{1}{\pi a}\sin(a\pi)+\frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^n\frac{2a\sin(a\pi)}{a^2-n^2}\cos(nx).$$

If we evaluate the Fourier series for $\cos(ax)$ that we just obtained at $x=\pi$ and divide by $\sin(a\pi)/\pi$, we get

$$\pi\cot(a\pi)=\frac{1}{a}+2a\sum_{n=1}^{\infty}\frac{1}{a^2-n^2},$$

which is the desired formula.

4. Let a_n be an enumeration of the rationals, and let I_{ij} be an interval of length $1/2^{i+j}$ centered at a_i. Let $O_j=\cup_{i=1}^{\infty}I_{ij}$. Then

$$O_1\supset O_2\supset\cdots,$$

and $A=\bigcap O_i$ is the set we want. Also, each O_j is open and dense, since each O_j is a union of open sets and contains $\mathcal{Q}$, which is dense. So A is a countable intersection of open dense sets.

Moreover, A has measure zero. This follows since the measure of O_j is less than or equal to

$$\sum_{i=1}^{\infty}|I_{ij}|=\sum_{i=1}^{\infty}\frac{1}{2^{i+j}}=\frac{1}{2^j},$$

and so, since the O_j's satisfy

$$O_1\supset O_2\supset\cdots,$$

we see that the measure of their intersection is just the limit as $j\to\infty$ of the measure of O_j, which is bounded by $\lim_{j\to\infty}\frac{1}{2^j}$, and so is just 0.

Consider, for example, the proposition that a real number is in A. This holds quasi-surely, since it holds on A. However, its negation holds almost everywhere, namely on all the elements not in A, since A has measure zero (by the last paragraph), and so anything which is true about their complement is true almost everywhere.

5. We have

$$S_n(f,t) = \sum_{j=-n}^{n} a_j e^{ijt} = \sum_{j=-n}^{n} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-ijx}\, dx e^{ijt} \tag{8}$$

$$= \sum_{j=-n}^{n} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{ij(t-x)}\, dx$$

$$= \sum_{j=-n}^{n} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t-x) e^{ijx}\, dx \tag{9}$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t-x) \sum_{j=-n}^{n} e^{ijx}\, dx. \tag{10}$$

Now

$$\sum_{j=-n}^{n} e^{ijx} = e^{-inx} \sum_{j=0}^{2n} e^{ijx} = e^{-inx} \frac{e^{i(2n+1)x} - 1}{e^{ix} - 1} = \frac{\sin\left(n + \frac{1}{2}\right) t}{\sin \frac{t}{2}},$$

as desired.

6. About $t = 0$, we have

$$\frac{1}{\sin t} - \frac{1}{t} = \frac{t - \sin t}{t \sin t}.$$

This last quantity has a removable singularity at 0. We can see, for example, by using L'Hopital's rule and differentiating the numerator and denominator twice that the limit at $t = 0$ is the same as the value of

$$\frac{\sin t}{2\cos t - t \sin t},$$

which is just 0. Thus, in a neighborhood of $t = 0$, $g(t) = \frac{1}{\sin t} - \frac{1}{t}$ is a continuous function if we set $g(0) = 0$. In particular, it is continuous in the region $-\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$. (Note however that g is certainly not continuous at $t = k\pi$ for $k \neq 0$.)

We have

$$S_n(f,t) - S_n^*(f,t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t-x)(D_n(x) - F_n(x))\, dx \tag{11}$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t-x) g(x/2) \sin\left(\left(n + \frac{1}{2}\right) x\right)\, dx. \tag{12}$$

By making the change of variables $x \mapsto x + \frac{\pi}{n+\frac{1}{2}}$, we see also that

$$S_n(f,t) - S_n^*(f,t) = -\frac{1}{2\pi} \int_{-\pi}^{\pi} f\left(t - x - \frac{\pi}{n + \frac{1}{2}}\right) g\left(\frac{x}{2} + \frac{\pi}{2n+1}\right) \sin\left(\left(n + \frac{1}{2}\right) x\right)\, dx.$$

Taking the average of these two expressions for $S_n(f,t) - S_n^*(f,t)$, we see that

$$S_n(f,t) - S_n^*(f,t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(f(t-x) g(x/2) - f\left(t - x - \frac{\pi}{n + \frac{1}{2}}\right) g\left(\frac{x}{2} + \frac{\pi}{2n+1}\right)\right) \sin\left(\left(n + \frac{1}{2}\right) x\right)\, dx,$$

and so

$$|S_n(f,t) - S_n^*(f,t)| \leq \frac{1}{2} \sup_{x \in [-\pi,\pi]} \left| f(t-x) g(x/2) - f\left(t - x - \frac{\pi}{n + \frac{1}{2}}\right) g\left(\frac{x}{2} + \frac{\pi}{2n+1}\right)\right|.$$

As $n \to \infty$, this last expression goes to 0 uniformly in t, since both f and g are continuous.