SOLUTION SET # 5

1. Steiner Symmetry

First we notice that the greatest difference in the value of two points will occur between points on the boundary. If to points x, y have maximal distance, and either is not on the boundary, then they must be interior points, and we can find points in A beyond x on the ray through x from y, or (similarly for y), and the distance between these points and y will be strictly greater than the alleged diameter. By the construction for Steiner symmetry, a point in A can be represented as a point (h, y), where y is a real number, and since A is compact, its intersection with the line $\{h\}\times\mathbb{R}$ is closed and bounded. We therefore have a maximum $y^+(h)$ and a minimum $y^-(h)$ for the $y-$values in A over h. Clearly, the points $(h, y^+(h)), (h, y^-(h))$ are the boundary points of A as h is allowed to vary over H. By our earlier observation the diameter will be greater than or equal to the distance between two boundary points, and clearly this is largest when we compare the top part of one fiber with the bottom part of another. We therefore have

$$\text{diam}(A) \geq \max\{\left\|(h, y^+(h)) - (k, y^-(k))\right\|, \left\|(k, y^+(k))\right\|, \left\|(h, y^-(h))\right\|\}. \tag{1}$$

Notice that the four points $(h, y^{\pm}(h)), (k, y^{\pm}(k))$ form a trapazoid with parallel sides perpendicular to H. The Steiner symmetrization can be done in two steps. First we move everything down so that the figure is symmetric about H, then rescale to "remove gaps." The first result very much changes the norms. Let $\hat{y}$ be the image of y under the first operation. We know then that $\hat{y^-} = -\hat{y^+}$. This makes our trapazoid into an isoscolese trapazoid. The max in Equation 1 is the max of the two diagonals in the trapazoid, and geometry tells us that this max is minimized when the trapazoid is isosoles. Additionally, the lengths of the parallel sides remains the same, as does the height. So we conclude that

$$\begin{aligned}\max\{\left\|(h, y^+(h)) - (k, y^-(k))\right\|, \left\|(k, y^+(k))\right\|, \left\|(h, y^-(h))\right\|\} \\ \geq \left\|(h, \hat{y^+}(h)) - (k, -\hat{y^+}(k))\right\|.\end{aligned} \tag{2}$$

This essentially gives us the problem. We know now that if $s(h)$ is the maximum value of the last coordinate in the Steiner symmetrization of A, then $s(h) \leq \frac{1}{2}(\hat{y^+}(h) - \hat{y^-}(h)) = \hat{y^+}(h)$, since the interval $[\hat{y^-}(h), \hat{y^+}(h)]$ covers the part of A over h. (We only need to look at what happens in the second part, since on the fibers the first transformation is an isometry). This means that:

$$\begin{aligned}\left\|(h, \hat{y^+}(h)) - (k, -\hat{y^+}(k))\right\| &= \sqrt{\|h-k\|^2 + (\hat{y^+}(h) + \hat{y^+}(k))^2} \\ &\geq \sqrt{\|h-k\|^2 + (s(h) + s(k))^2} = \|(h, s(h)) - (k, -s(k))\|.\end{aligned} \tag{3}$$

However, using the above argument and compactness of A, we see that for some pair of boundary points of the Steiner symmetrization, the diameter is the distance between these two points. Again, this is maximized if we compare the two different extreme values on two different fibers. If the points in question are $(h, s(h))$ and $(k, -s(k))$, then the last term in Equation 3 is the diameter of $St(A)$. Tracing through the string of inequalities, we see that $\mathrm{diam}(St(A)) \leq \mathrm{diam}(A)$.

2. Reduction Part

Let's assume that A has empty interior. If both A and B have empty interior, then the right hand side is 0, so the result is trivially true, and since there is no real distinguishable difference between A and B in the equation, this reduction makes no difference. We have therefore that

$$\mathrm{Vol}((1-\lambda)A + \lambda B)^{\frac{1}{n}} = \left(\int_{\mathbb{R}^n} 1_{(1-\lambda)A+\lambda B} dx\right)^{\frac{1}{n}}.$$

We can rescale x by λ, and this makes λB just B. The net effect is a multiplication by λ^n, so the integral becomes

$$\left(\lambda^n \int_{\mathbb{R}^n} 1_{\frac{1-\lambda}{\lambda}A+B} dx\right)^{\frac{1}{n}} \geq \lambda Vol(B).$$

The last inequality follows from the fact that $\mu A + B$ always contains an isometric copy of B, assuming A is non-empty, given by $\mu a + B$ for some fixed $a \in A$.

For the other part, we just do some simplification. Choose a μ in $(0, 1)$. Since the result is clearly true if $\lambda = 0, 1$ we want to show that the statement does give the right result for any value of λ between 0 and 1. The proof hinges on the fact that $\lambda \mapsto \frac{1-\lambda}{\lambda}$ is a bijection between $[0, 1]$ and $[0, \infty]$. If we know that $\mathrm{Vol}((1-\lambda)A+\lambda B) \geq 1$, when $\mathrm{Vol}(A) = \mathrm{Vol}(B) = 1$, and C, D are arbitrary convex bodies with non-empty interior, then since $\frac{1}{\mathrm{Vol}(C)^{\frac{1}{n}}}C$ has unit volume (by rescaling) and similarly for D, and noticing that if $x \geq 1$, then $x^{\frac{1}{n}} \geq 1$ for all n,

$$\mathrm{Vol}((1-\lambda)\frac{C}{\mathrm{Vol}(C)^{\frac{1}{n}}} + \lambda\frac{D}{\mathrm{Vol}(D)^{\frac{1}{n}}})^{\frac{1}{n}} \geq 1 = (1-\lambda) + \lambda.$$

Now we'll repeatedly use the fact that multiplication of a body by a scalar changes the volume by the scalar to the n^{th} power. Applying this to the scalar $\frac{\lambda}{\mathrm{Vol}(D)^{\frac{1}{n}}}$, we see that

$$\frac{\lambda}{\mathrm{Vol}(D)^{\frac{1}{n}}}\left(\mathrm{Vol}\left(\left(\frac{\mathrm{Vol}(D)}{\mathrm{Vol}(C)}\right)^{\frac{1}{n}}\frac{1-\lambda}{\lambda}C + D\right)\right)^{\frac{1}{n}} \geq 1 - \lambda + \lambda.$$

Now take λ so that

$$\frac{1-\lambda}{\lambda} = \left(\frac{\mathrm{Vol}(C)}{\mathrm{Vol}(D)}\right)^{\frac{1}{n}}\frac{1-\mu}{\mu} > 0.$$

Dividing both sides of our earlier inequality by $\lambda/\mathrm{Vol}(D)^{1/n}$ gives us:

$$\mathrm{Vol}\left(\frac{1-\mu}{\mu}C + D\right)^{\frac{1}{n}} \geq \mathrm{Vol}(D)^{\frac{1}{n}}\frac{1-\lambda}{\lambda} + \mathrm{Vol}(D)^{\frac{1}{n}} = \mathrm{Vol}(C)^{\frac{1}{n}}\frac{1-\mu}{\mu} + \mathrm{Vol}(D)^{\frac{1}{n}}.$$

Now we multiply both sides by μ to conclude that

$$\mathrm{Vol}\left((1-\mu)C + \mu D\right)^{\frac{1}{n}} \geq (1-\mu)\,\mathrm{Vol}(C)^{\frac{1}{n}} + \mu\,\mathrm{Vol}(D)^{\frac{1}{n}},$$

as required.

3. Computation

(Thanks to Tom for the hint)

We know that $x \mapsto x^{-1/p}$ is concave up (the second derivative is positive for all positive x). This means that the chord between any two points $A = (a^p, a^{-1})$, $B = (b^p, b^{-1})$ lies entirely above the curve, except at the end points. Symbolically, if (s,t) is on the chord $\overline{AB}$, then $t \geq s^{-1/p}$. However, a points (s,t) is on the chord $\overline{AB}$ iff there is a $\lambda \in [0,1]$ such that

$$(s,t) = (1-\lambda)(a^p, a^{-1}) + \lambda(b^p, b^{-1}) \Leftrightarrow (s,t) = \big((1-\lambda)a^p + \lambda b^p, (1-\lambda)a^{-1} + \lambda b^{-1}\big).$$

This implies that

$$(1-\lambda)a^{-1} + \lambda b^{-1} \geq \big((1-\lambda)a^p + \lambda b^p\big)^{-\frac{1}{p}},$$

or equivalently (since everything is positive and therefore non-zero)

$$\big((1-\lambda)a^p + \lambda b^p\big)^{\frac{1}{p}}\big((1-\lambda)a^{-1} + \lambda b^{-1}\big) \geq 1. \tag{4}$$

4. Proof of the Inequality

4.1. Inclusions of Sets.

First note that since $k_A(\tau) = A \cap H(z_A(\tau))$, $k_A(\tau) \subset A$. Similarly, $k_B(\tau) \subset B$, and therefore

$$(1-\lambda)k_A(\tau) + \lambda k_B(\tau) \subset (1-\lambda)A + \lambda B = K_\lambda.$$

(The scalar multiplication and addition applies to both sides and so does not effect inclusions).

To show the final part, note that the final coordinate of any element of $k_A(\tau)$ is $z_A(\tau)$ (since this is in $H(z_A(\tau)))$). The final coordinate of any element of $k_B(\tau)$ is $z_B(\tau)$, and by linearity, the final coordinate of any element of $(1-\lambda)k_A(\tau)+\lambda k_B(\tau)$ is just $(1-\lambda)z_A(\tau) + \lambda z_B(\tau)$. This is just $z_\lambda(\tau)$, so we conclude that

$$(1-\lambda)k_A(\tau) + \lambda k_B(\tau) \subset H(z_\lambda(\tau)).$$

Since the set in question is contained in both sets, it is contained in their intersection, as was required to show.

Note that a corollary to this is that

$$\mathrm{Vol}_{n-1}((1-\lambda)k_A(\tau) + \lambda k_B(\tau)) \leq \mathrm{Vol}_{n-1}(K_\lambda \cap H(z_\lambda(\tau))).$$

4.2. Final Proof.

Now note that if A has slices with non-trivial volume on the interval $[z_A(0), z_A(1)]$, then μA has slices with non-trivial volume on $[\mu z_A(0), \mu z_A(1)]$ $(\mu \neq 0)$. This is just because scaling by μ effects all parts of A equally. Additionally, we can conclude that any vertical translation (in the direction of the last coordinate) will just translate this interval by the same amount. This, together with the idea that $\mu A + \lambda B$ is just a translate of A by a number of different lengths, all parameterized by B, gives us the result that K_λ has slices with non-trivial volume on the interval $[z_\lambda(0), z_\lambda(1)]$. By Fubini's theorem:

$$\mathrm{Vol}(K_\lambda) = \int_{z_\lambda(0)}^{z_\lambda(1)} \mathrm{Vol}(K_\lambda \cap H(t))dt = \int_0^1 \mathrm{Vol}(K_\lambda \cap H(z_\lambda(\tau)))z'_\lambda(\tau)d\tau, \tag{5}$$

by a change of variables.

By the remark at the end of the last subsection, and by the monotonicity of the integral (using the value of the derivative of z_λ computed from the derivatives of z_A and z_B by linearity), Equation 5 is greater than or equal to

$$\int_0^1 \mathrm{Vol}_{n-1}\left((1-\lambda)k_A(\tau)+\lambda k_B(\tau)\right)\left(\frac{1-\lambda}{\nu_A(z_A(\tau))}+\frac{\lambda}{\nu_B(z_B(\tau))}\right)d\tau. \tag{6}$$

By the $(n-1)-$dimensional Brunn-Minkowski theorem,

$$\begin{aligned} \mathrm{Vol}_{n-1}\left((1-\lambda)k_A(\tau)+\lambda k_B(\lambda)\right) \\ \geq \left((1-\lambda)\,\mathrm{Vol}_{n-1}(k_A(\tau))^{\frac{1}{n-1}}+\lambda\,\mathrm{Vol}_{n-1}(k_B(\tau))^{\frac{1}{n-1}}\right)^{n-1}. \end{aligned} \tag{7}$$

We know also that $k_A(\tau)=A\cap H(z_A(\tau))$, so $\mathrm{Vol}_{n-1}(k_A(\tau))=\nu_A(z_A(\tau))$. Writing ν_A for $\nu_A(z_A(\tau))$, and combining this result with Inequalities 6 and 7, we deduce that

$$\mathrm{Vol}(K_\lambda)\geq\int_0^1\left((1-\lambda)\nu_A^{\frac{1}{n-1}}+\lambda\nu_B^{\frac{1}{n-1}}\right)^{n-1}\left(\frac{1-\lambda}{\nu_A}+\frac{\lambda}{\nu_B}\right)d\tau \tag{8}$$

By Inequality 4 from problem 6, we know that the integrand is greater than or equal to 1, (take $p=\frac{1}{n-1}$), so we conclude that the entire integral is greater than or equal to 1, and the theorem is proved.

5. Mixed Volumes

By the chain rule and the definition of f,

$$f'(0)=\frac{1}{n}\frac{\tilde{K}_0'}{\tilde{K}_0^{\frac{n-1}{n}}}+\mathrm{Vol}(A)^{\frac{1}{n}}-\mathrm{Vol}(B)^{\frac{1}{n}}\geq 0.$$

$\tilde{K}_\lambda$ is defined as

$$\sum\binom{n}{i}(1-\lambda)^{n-i}\lambda^i V_i.$$

If $\lambda=0$, then this sum reduces to $V_0=\mathrm{Vol}(A)$ (we don't have to have the other $A's$ since the mixed volume here is just the volume, as shown by the vanishing of f at 0). For the derivative, we have both the $i=0$ term and the $i=1$ term. This makes $K_0'=nV_1-nV_0$. The expression for $f'(0)$ therefore becomes

$$\frac{V_1-\mathrm{Vol}(A)}{\mathrm{Vol}(A)^{\frac{n-1}{n}}}+\mathrm{Vol}(A)^{\frac{1}{n}}\geq\mathrm{Vol}(B)^{\frac{1}{n}}.$$

Raising everything to the n^{th} power and simplifying, we immediately see that

$$V(A,\ldots,A,B)^n\geq(\mathrm{Vol}(A))^{n-1}\,\mathrm{Vol}(B).$$