

# Problem set 2

Math 212a

October 4, 2001, Due October 16

There are 10 problems in all, scattered through the text.

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## 1 The $L_2$ Martingale Convergence Theorem.

Let  $\mathbf{H}$  be a Hilbert space and let  $\{x_n\}$  be a sequence of elements of  $\mathbf{H}$  which satisfy the following two conditions:

- The  $\|x_n\|$  are bounded, i.e. there is a constant  $M$  such that

$$\|x_n\| \leq M \quad \text{for all } n$$

and

- For all pairs  $m$  and  $n$  with  $n > m$

$$(x_n - x_m, x_m) = 0. \tag{1}$$

1. Show that the  $x_n$  converge to an element  $x$  and that  $\|x\| \leq M$ . In fact, show that  $\|x\| = L$  where  $L$  is the greatest lower bound of the  $M$  in the first condition.

Once you have figured out this problem, you will agree that the proof is quite transparent. You may wonder about the fancy title. In fact, this result is (part of, or a version of) a key result in probability theory which unifies many so called “limit theorems”. It is also a very good entry into getting an intuitive understanding of these theorems. The appropriate language for stating these theorems is measure theory, and therefore any advanced book on probability theory either assumes that the reader has had a course on measure theory, or spends the early chapters on an exposition of measure theory. The martingale convergence theorem usually does not make its appearance until the last third of the book. I plan to get into measure theory in the next few weeks, but in this problem set I want to develop a good bit of the language and some of the results of probability theory using Hilbert space tools.

Here is an example of an “abstract” limit theorem which is in fact a consequence of the martingale convergence theorem, but is easier to prove directly:

2. Let  $y_k \in \mathbf{H}$  be such that

$$(y_i, y_j) = 0 \quad i \neq j$$

and such that there is a constant  $K$  such that

$$\|y_j\|^2 \leq K$$

for all  $j$ . Show that for any real number  $s > \frac{1}{2}$  we have

$$\lim_{n \rightarrow \infty} \frac{1}{n^s} (y_1 + \cdots + y_n) \rightarrow 0.$$

A matter of great importance is to understand what happens (under additional hypotheses) at the critical value  $s = \frac{1}{2}$ . To understand why Problem 2 is in fact a consequence of Problem 1, consider the case  $s = 1$  and take

$$x_n := \sum_1^n \frac{1}{k} y_k.$$

Then the  $\|x_n\|$  are bounded because  $\sum 1/k^2$  converges, and condition (1) clearly holds. We have

$$y_k = k(x_k - x_{k-1})$$

so

$$y_1 + \cdots + y_n = \sum_1^n k(x_k - x_{k-1}) = nx_n - \sum_1^n x_{k-1}.$$

Dividing by  $n$  gives

$$\frac{1}{n}(y_1 + \cdots + y_n) = x_n - \frac{1}{n} \sum_1^n x_{k-1}.$$

Since we know that  $x_n \rightarrow x$  we also know that the “Cesaro average”  $(1/n) \sum_1^n x_{k-1}$  approaches  $x$  so the right hand side tends to  $x - x = 0$ . The same “summation by parts” argument works for any  $s > \frac{1}{2}$  to derive Problem 2 from Problem 1 but is a little harder to write down. The direct proof is easier.

## 2 Probabilistic language in Hilbert space terms.

In what follows we will take  $\mathbf{H}$  to be  $L_2([0, 1])$ , the completion of the space of continuous functions on the unit interval relative to the metric determined by the scalar product

$$(f, g) = \int_0^1 f(t)\overline{g(t)}dt.$$

(There is, of course, no difference between  $L_2(\mathbf{T})$  and  $L_2([0, 1])$ , but since the convention is to let probabilities range between 0 and 1, we will take  $\mathbf{H} = L_2([0, 1])$ .)

The space  $L_2([0, 1])$  has certain additional structures, beyond merely being a Hilbert space. Once we have extracted the necessary additional structures we could assemble them into a collection of axioms for what we might call a “probability Hilbert space”.

### 2.1 Expectation.

The space  $\mathbf{H}$  has a preferred element  $\mathbf{1}$  which we might think of as the function which is identically one. More precisely, it is the linear function on  $\mathcal{C}([0, 1])$  which assigns to any continuous function its integral over the unit interval:

$$f \mapsto \int_0^1 f(t)dt = (f, \mathbf{1}).$$

We have

$$(\mathbf{1}, \mathbf{1}) = 1.$$

We make the definition

$$E(X) := (X, \mathbf{1}) \quad \text{for any } X \in \mathbf{H}$$

and call  $E(X)$  the **expectation** of  $X$ . We will call an arbitrary element of  $\mathbf{H}$  a **random variable**. (Strictly speaking we should call these “square integrable

random variables”, but we won’t be considering any other kind in this problem set.)

Suppose that  $A \subset [0, 1]$  is an interval, or a finite union of intervals. The linear function which assigns to any continuous function its integral over  $A$ ,

$$f \mapsto \int_A f(t) dt$$

is continuous in the  $L_2$  norm:

$$\left| \int_A f(t) dt \right| \leq \int_A |f(t)| dt \leq \int_{[0,1]} |f(t)| dt = (|f|, \mathbf{1}) \leq \|f\| \|\mathbf{1}\|.$$

So it is given by scalar product with an some element of  $\mathbf{H}$  which we shall denote by  $\mathbf{1}_A$ . The meaning of this notation is that we can “represent” this element by the indicator function of the set  $A$ , i.e. by the function which is one on  $A$  and zero on the complement of  $A$ . Indeed, we can find a sequence of continuous functions which converge to this indicator function in the  $L_2$  norm. There is little danger in thinking of  $\mathbf{1}_A$  as being an actual function, or as representing the set  $A$ , provided that we understand that modifying the set  $A$  by inserting or removing a finite number of points (or, as we shall see later, by a set of measure zero) gives the same element  $\mathbf{1}_A \in \mathbf{H}$ . Clearly

$$E(\mathbf{1}_A) = (\mathbf{1}_A, \mathbf{1}) = (\mathbf{1}_A, \mathbf{1}_A) = \|\mathbf{1}_A\|^2$$

is just the sum of the lengths of the disjoint intervals which comprise  $A$ . (It requires a little combinatorial lemma at this point to prove that this value does not depend on how we break  $A$  up into a union of disjoint intervals.) We will denote this value by  $P(A)$  and call it the **probability** of the **event**  $A$ . So

$$P(A) := E(\mathbf{1}_A) = \|\mathbf{1}_A\|^2. \tag{2}$$

## 2.2 Conditional probability is a Fourier coefficient.

Suppose that  $\mathbf{1}_A \neq 0$ , and let  $\pi_A$  denote orthogonal projection onto the one dimensional space spanned by  $\mathbf{1}_A$ . Let  $B$  be some other finite union of intervals, so that  $A \cap B$  is again a finite union of intervals.

3. Show that

$$\pi_A(\mathbf{1}_B) = \frac{P(A \cap B)}{P(A)} \mathbf{1}_A.$$

In elementary probability theory the expression  $\frac{P(A \cap B)}{P(A)}$  is known as the “conditional probability of  $B$  given  $A$ ” and is denoted by  $P(B|A)$ ,

$$P(B|A) := \frac{P(A \cap B)}{P(A)}.$$

So we can write

$$\pi_A(\mathbf{1}_B) = P(B|A)\mathbf{1}_A. \quad (3)$$

In what follows, we will enlarge our collection of “events” and take (2) and (3) as definitions. For this we need to make use of some additional structure of  $L_2([0, 1])$ .

### 2.3 Reality.

If  $\{g_n\}$  is a Cauchy sequence (in the  $L_2$  norm) of elements of  $\mathcal{C}([0, 1])$  then so is the sequence  $\{\overline{g_n}\}$ , where, of course,

$$(\overline{f})(t) := \overline{f(t)}.$$

This implies that the complex conjugation operation extends to  $\mathbf{H}$  as an anti-linear map of  $\mathbf{H}$  onto itself, and  $\|\overline{g}\|_2 = \|g\|_2$ . Thus we can define  $\mathbf{H}_{\mathbf{R}}$  to consist of those  $g \in \mathbf{H}$  which satisfy  $\overline{g} = g$ . It is a vector space over the real numbers, and it is easy to check that every such element is the limit of a Cauchy sequence of real continuous functions. The scalar product of any two elements of  $\mathbf{H}_{\mathbf{R}}$  is real. An element of  $\mathbf{H}_{\mathbf{R}}$  will be called a **real valued random variable**.

Alternatively, we could have developed the whole theory of Hilbert spaces over the real numbers, and then obtained  $\mathbf{H}_{\mathbf{R}}$  as the completion of the real valued continuous functions on  $[0, 1]$  under the  $L_2$  norm.

### 2.4 Covariance and Variance.

Let  $X$  and  $Y$  be real valued random variables. Define the **covariance** of  $X$  and  $Y$  by

$$\text{cov}(X, Y) := (X - E(X)\mathbf{1}, Y - E(Y)\mathbf{1}) = (X, Y) - E(X)E(Y).$$

Define the **variance** of  $X$  by

$$\text{var}(X) := \text{cov}(X, X) = (X, X) - E(X)^2.$$

We say that  $X$  and  $Y$  are **uncorrelated** if  $\text{cov}(X, Y) = 0$ . Notice that if  $X = \mathbf{1}_A$  and  $Y = \mathbf{1}_B$  are events, then  $X$  and  $Y$  are uncorrelated if and only if  $P(A \cap B) = P(A)P(B)$  or (if  $P(A) \neq 0$ ) if  $P(B|A) = P(B)$ . We say that the events  $A$  and  $B$  are **independent**. For more general random variables, we will introduce a notion of independence which is much more restrictive than being uncorrelated.

### 2.5 The $L_2$ law of large numbers.

Suppose  $\{Z_i\}$  is a family of pairwise uncorrelated random variables whose variances are uniformly bounded:

$$\text{var}(Z_i) \leq K$$

for some constant  $K$  for all  $i$ . We can then apply Problem 2 to  $y_i := Z_i - E(Z_i)\mathbf{1}$  to conclude that

$$\frac{1}{n}(Z_1 + \cdots + Z_n) - \frac{1}{n}(E(Z_1) + \cdots + E(Z_n))\mathbf{1} \rightarrow 0.$$

For example, if all the  $E(Z_i)$  are equal, say equal to some number  $m$ , then

$$\frac{1}{n}(Z_1 + \cdots + Z_n) \rightarrow m\mathbf{1}.$$

In this equation, convergence is in the sense of  $L_2$ . It will require more work for us to formulate and prove a corresponding result where convergence is “almost everywhere”. We will do this after developing the machinery of measure theory - we will then be able to formulate and prove an “almost everywhere” version of the martingale convergence theorem.

But all versions of the law of large numbers are meant to justify the identification of the intuitive notion of probability with that of long term behavior.

We are going to greatly enlarge the class of “events” and this is going to take a lot of technical work. The idea is this: let  $S$  be any subset of  $[0, 1]$ . We can consider the function  $1_S$  where  $1_S(x) = 1$  for  $x \in S$  and  $1_S(x) = 0$  for  $x \in S^c$ . Conversely, let  $f$  be any function on  $[0, 1]$  which takes on only the values 1 and 0, then we can let  $S$  be the set of  $x$  where  $f(x) = 1$ , and then  $f = 1_S$ . Now a function  $f$  which takes on only the values 0 and 1 can be characterized as follows:

$$\begin{aligned} 0 \leq f(x) \leq 1 \quad \forall x, \\ \max\{f(x), 1 - f(x)\} = 1, \quad \forall x, \end{aligned}$$

and

$$\min\{f(x), 1 - f(x)\} = 0 \quad \forall x.$$

Since the elements of  $\mathbf{H}_{\mathbf{R}}$  are not functions, we can not use this characterization of the indicator function of a set directly. The trouble is with the quantifier  $\forall x$ . Changing “values” at a set of measure zero does not have any effect on an element of  $\mathbf{H}_{\mathbf{R}}$ . So what we need is to show that there is a notion of partial order  $\preceq$  on which is compatible with the Hilbert space structure which is a substitute for the order relation  $f(x) \leq g(x) \forall x$ . Before embarking on this technical stuff, I thought it might be useful to develop some probabilistic intuition for the concepts we are introducing.

### 3 Simulation.

The m.file randomwalk.m in the MATLAB folder simulates a one dimensional random walk with  $N$  steps. For convenience, I reproduce it here:

1. % plots a random walk of length N

2. close all
3. clc
4. N=input('N= ');
5. Q=zeros(1,N);
6. R=rand(1,N);
7. Q=Q+(R>.5);
8. P=2\*Q-1;
9. W=[0 cumsum(P)];
10. plot(W)

Step 1 is not a command at all, and is just there to describe what the m.file does. Steps 2 and 3 clear the display and command window and Step 4 calls for the input of the number of random walk steps desired. Step 5 is just an initialization. The key steps are 6 and 7. The MATLAB command rand(M,N) produces a  $M \times N$  matrix each of whose entries is a number “chosen at random” between 0 and 1, and all the matrix entries chosen completely independently of one another.

There is obviously a deep paradox involved in how a purely mechanical device like a computer can produce numbers “chosen at random”. I will leave this issue for courses in the CS department or the philosophy department. Perhaps also the theology department. I will just take this computer capacity for granted.

In step 7, the entry  $R > .5$  produces a matrix the same size as the matrix  $R$ , whose entries are 1 or 0 according as the corresponding entry of  $R$  is or is not  $> .5$ . In other words, it has the effect of applying the function  $\mathbf{1}_{[.5,1]}$  to each of the entries of  $R$ . The net effect of step 7 is to produce a vector of length  $N$  whose entries are 0 or 1, each occurring with probability one-half. Step 8 converts this into a vector whose entries are  $\pm 1$  each with probability one-half. Step 9 produces a vector whose first entry is 0 and whose next entries are the cumulative sums of the entries of  $P$  and step 10 plots the entries of  $W$  against the index of the entry, i.e plots the points  $(i, W(i))$  and joins these points by straight line segments.

Suppose we wanted to simulate an unfair coin where the probability of heads is  $p$ , not necessarily  $.5$ . We could then apply the function  $\mathbf{1}_{[1-p,1]}$  to each of entries of  $R$ . This will produce a matrix whose entries are 0 or 1, with 0 occurring with probability  $1 - p$  and 1 with probability  $p$ .

More generally, suppose we want to simulate an experiment whose outcome can be any integer from 0 to  $n$ , where the probability of getting  $i$  is  $p_i$ . Then all we have to do is construct a “step function” whose values are these integers, where the value  $i$  is taken on on an interval of length  $p_i$ . For example, the function  $f$  graphed in Figure 1 will do. Then applying  $f$  to each entry of  $R$  will

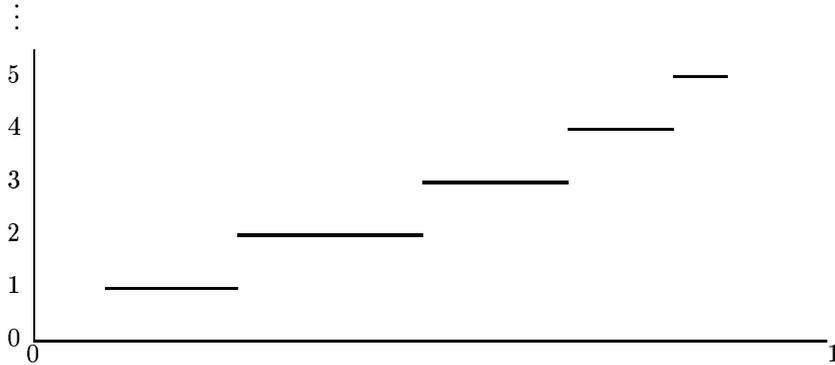


Figure 1: The graph of function  $f$  such that  $f(\text{hboxrand})$  simulates an experiment which has outcome  $i$  with probability  $p_i$ . The length of the  $i$ -th interval of constancy is  $p_i$ .

produce a simulation of independent trials of the experiment, one trial for each position of  $R$ .

In general, if we want to simulate a random variable which takes on non-negative integer values, we don't have to restrict ourselves to the function graphed in Figure 1. In fact, let  $g$  be any function defined on  $[0, 1]$  which takes on non-negative integer values, and suppose (for the sake of simplicity) that for each non-negative integer  $k$ , the set

$$g^{-1}(k)$$

is a finite union of intervals. (An interval can be open (without end points), closed (with both end points) or half open (with one end point as in Figure 1).) Suppose that

$$\mu[g^{-1}(k)] = p_k$$

Here  $\mu(S)$  denotes the "measure" of a set  $S$ . If  $S$  is a disjoint union of intervals, then  $\mu(S)$  is the sum of the lengths of constituent intervals. If  $g$  satisfies this condition, then clearly

$$X = g(\text{rand})$$

simulate an experiment with

$$\text{Prob}(X = i) = p_i, \quad i = 0, 1, 2, \dots, \quad \sum p_i = 1.$$

So the function  $f$  in Figure 1 is just one of many. It is characterized by being monotone (non-decreasing). Any function  $g$  that we choose belongs to (more precisely "represents") an element of  $\mathbf{H}_R$ , and this is why we call elements of  $\mathbf{H}_R$  random variables.

We can allow for the possibility that any non-negative integer value be achieved. The condition that a  $g$  simulating such an experiment belong to  $\mathbf{H}_R$  is that

$$\sum i^2 p_i < \infty.$$

Thus our space  $\mathbf{H}_R$  will not simulate *all* random experiments, only the square integrable ones.

### 3.1 The Poisson distribution.

This describes an experiment whose outcome can be any non-negative integer, and where the probability of having the outcome  $i$  is

$$p_i(\lambda) = \frac{\lambda^i}{i!} e^{-\lambda}.$$

Here  $\lambda$  is any non-negative real number. (In the mathematics literature,  $\lambda$  is a standard name for this parameter, but MATLAB prefers the English alphabet, so we will also use letters such as  $r$  or  $p$  for this parameter.)

4. Let  $X$  be a random variable which takes on only non-negative integer values  $i$  probability  $p_i(\lambda)$  as given above. Show that

$$E(X) = \lambda \quad \text{and} \quad \text{var}(X) = \lambda.$$

[A neat way of doing this is (for any random variable  $X$  that takes on only non-negative integer values  $n$ , with probability  $p_n$ ) is to introduce the **generating function**

$$p_X(z) := \sum_0^{\infty} p_n z^n$$

and then to evaluate  $p'(1)$  and  $p''(1)$ .]

As mentioned above, many different random variables can simulate the same experiment, but it will be convenient for us to choose the monotone step function as a standard simulator. So we let  $Y_\lambda$  denote the monotone step function with values in the non-negative integers, and where the interval where  $Y_\lambda = i$  has length  $p_i(\lambda)$ .

The m.file poisson.m calls for the input of the parameter  $r$  ( $= \lambda$ ) and an integer  $N$ , and produces a row vector  $Q$  of size  $N$  whose entries are non-negative integers chosen independently according to the Poisson law. Here is the m.file:

1. %produces a vector Q of size N with integer entries
2. %distributed according to the Poisson distribution
3. N=input('N= ')
4. r=input('r= ')
5. p=exp(-r);
6. F=p;
7. i=1;
8. Q=zeros(1,N);

9. R=rand(1,N);
10. while p > eps;
11. Q=Q+(R > F);
12. p=(r/i)\*p;
13. F=F+p;
14. i=i+1;
15. end

It works by first checking whether the entry is  $> p_0(r) = e^{-r}$ . If so, it adds 1 to that entry. Then it checks whether the entry is  $> p_0(r) + p_1(r)$ . If so, it adds another 1 etc. It computes  $p_i(r)$  as it goes along. We do not want the machine to continue once  $i$  is so large that  $p_i(r)$  is below the tolerance that MATLAB uses in its computations. This is the meaning of line 10: the command “eps” returns the value of the machine accuracy, and we want the procedure to stop when  $p_i(r)$  gets below this value.

There will be problems with this program if  $r$  is so large that  $e^{-r} < \text{eps}$  for then the program can't even get started. Also, this program is not the most efficient when the parameter  $r$  is only moderately large, since it spends a lot of time checking events of low probability, but it is fine for low values of  $r$  and useful for our theoretical discussion.

### 3.2 Comparing Poisson and Bernoulli.

We will let  $B_p$  denote the monotone step function which is equal to 0 on the interval  $[0, 1 - p)$  and equal to 1 on the interval  $[1 - p, 1]$ . The letter  $B$  stands for Bernoulli, who considered random variables which could take on only the values 0 and 1, and takes on the value 1 with probability  $p$ . Of course  $B_p$  is only one of many such random variables, but all have the same expectation and variance, namely

$$E(B_p) = p \quad \text{and} \quad \text{var}(B_p) = p - p^2.$$

We also note the following useful fact:

$$e^{-p} = 1 - p + \frac{p^2}{2!} - \frac{p^3}{3!} + \dots$$

is (for  $0 \leq p \leq 1$ ) an alternating series with terms decreasing in absolute value, and so  $e^{-p} - 1 + p > 0$  or

$$1 - p < e^{-p}.$$

This says that the interval where  $B_p = 0$  is shorter than the interval where  $Y_p = 0$ .

5. Show that the set  $A := \{x \in [0, 1] \mid B_p(x) \neq Y_p(x)\}$  consists of two intervals, and

$$P(A) \leq p^2.$$

We can write this result in the more convenient shorter notation

$$P(B_p \neq Y_p) \leq p^2.$$

### 3.3 Distance in law.

We continue to discuss the situation where our random variables take on only integer values, and the inverse image of each integer is a finite union of intervals. Neither of these hypotheses is essential, but I want to develop some intuition and these assumptions provide a useful crutch.

Each such random variable  $X$  assigns a non-negative weight or probability  $p_k$  to each integer  $k$  where the  $p_k$  are non-negative and  $\sum_k p_k = 1$ . Suppose the  $Y$  is a second non-negative integer valued random variable which has the probability assignments  $P(Y = k) = q_k$ . We say that  $X$  and  $Y$  “have the same law” or are “equal in law” if  $p_k = q_k$  for all  $k$ . More generally we might want to measure how far the overall distribution of the  $p$ ’s is from that of the  $q$ ’s by defining

$$d_{\text{law}}(X, Y) \stackrel{\text{def}}{=} \sup_A |\text{Prob}(X \in A) - \text{Prob}(Y \in A)| \quad (4)$$

where the supremum is taken over all subsets  $A$  of the integers.

Notice that if  $d_{\text{law}}(X, Y) = 0$  then  $X$  and  $Y$  are equal in law, and if  $X, Y, Z$  are three random variables then for any (measurable) set  $A$  we have

$$\begin{aligned} |\text{Prob}(X \in A) - \text{Prob}(Z \in A)| &= |\text{Prob}(X \in A) - \text{Prob}(Y \in A) \\ &\quad + \text{Prob}(Y \in A) - \text{Prob}(Z \in A)| \\ &\leq |\text{Prob}(X \in A) - \text{Prob}(Y \in A)| \\ &\quad + |\text{Prob}(Y \in A) - \text{Prob}(Z \in A)| \\ &\leq d_{\text{law}}(X, Y) + d_{\text{law}}(Y, Z). \end{aligned}$$

Taking the supremum over all  $A$  gives the triangle inequality

$$d_{\text{law}}(X, Z) \leq d_{\text{law}}(X, Y) + d_{\text{law}}(Y, Z).$$

More generally, suppose that  $X$  and  $Y$  are discrete random variables with possible values  $\{x_k\}$  and  $\{y_k\}$

$$\text{Prob}(X = x_k) = r_k, \quad \text{Prob}(Y = y_k) = s_k$$

so that

$$\sum_k r_k = \sum_k s_k = 1$$

when the sums are taken over all  $k$ . By throwing in 0's for  $p$ 's or  $q$ 's corresponding to values which don't occur in  $X$  or  $Y$ , we can assume that the sets  $\{x_k\}$  and  $\{y_k\}$  are the same. Now, when we measure distance in law, we may let  $A$  range over all subsets of these possible values.

Let  $B_+$  denote the set of all  $x_k$  for which  $r_k \geq s_k$  and  $B_-$  the complement, i.e. the set where  $r_k < s_k$ . In words,  $B_-$  is the set of values where  $Y$  has a larger probability of occurring than  $X$ . For any set  $A$  we have the disjoint union

$$A = (A \cap B_+) \cup (A \cap B_-)$$

and so

$$\begin{aligned} \text{Prob}(X \in A) - \text{Prob}(Y \in A) &= \text{Prob}(X \in A \cap B_+) - \text{Prob}(Y \in A \cap B_+) \\ &\quad + \text{Prob}(X \in A \cap B_-) - \text{Prob}(Y \in A \cap B_-) \\ &= \sum_{x_k \in (A \cap B_+)} (r_k - s_k) + \sum_{x_k \in (A \cap B_-)} (r_k - s_k). \end{aligned}$$

The summands in the first sum are all non-negative, and in the second sum are all negative. So there will be cancellation between these two sums unless one or the other vanishes. So in taking the supremum in the definition (4) we may assume that  $A \subset B_+$  or  $A \subset B_-$ , and that, in fact  $A = B_+$  or  $A = B_-$ . But since

$$\text{Prob}(X \in B_+) + \text{Prob}(X \in B_-) = 1$$

and similarly for  $Y$ , we have

$$\text{Prob}(X \in B_+) - \text{Prob}(Y \in B_+) = -[\text{Prob}(X \in B_-) - \text{Prob}(Y \in B_-)]$$

so the supremum in (4) is achieved by taking  $A = B_+$  or  $A = B_-$ . Since these two sums are equal, their common value is equal to one half of their sum and so we get

$$d_{\text{law}}(X, Y) = \frac{1}{2} \sum_k |r_k - s_k| \tag{5}$$

where the sum is over all  $k$ . so this is an alternative definition for the distance in law between two discrete random variables.

Let  $f$  be any function satisfying

$$|f(x)| \leq 1$$

for all  $x$ . Then

$$|E(f(X)) - E(f(Y))| = \left| \sum_{x_k \in B_+} f(x_k)(r_k - s_k) + \sum_{x_k \in B_-} f(x_k)(r_k - s_k) \right|.$$

The right hand side of this expression will be maximized (subject to the constraint  $|f(x)| \leq 1$ ) if we choose  $f \equiv 1$  on  $B_+$  and  $f \equiv -1$  on  $B_-$  in which case the expression on the right becomes

$$\sum_k |r_k - s_k| = 2d_{\text{law}}(X, Y).$$

So if we define

$$d_f(X, Y) = \frac{1}{2} |E(f(X)) - E(f(Y))|, \quad (6)$$

we have

$$d_{\text{law}}(X, Y) = \sup_{|f| \leq 1} d_f(X, Y). \quad (7)$$

Let us assume that the set  $X \neq Y$  is an event; for example that this subset is a finite union of intervals as in Exercise 5. (Later we shall see that we have enough events so that  $X \neq Y$  is always an event. )

**Lemma 1** *Let  $X$  and  $Y$  be any two random variables. Then*

$$d_{\text{law}}(X, Y) \leq \text{Prob}(X \neq Y). \quad (8)$$

**Proof.** By definition, for any set  $A$ ,

$$\text{Prob}(X \in A) = \mu[X^{-1}(A)]$$

where  $X^{-1}(A)$  is the subset of the unit interval where the function  $X$  takes values in  $A$ , and  $\mu[X^{-1}(A)]$  is the measure (total length) of this subset. So

$$\text{Prob}(X \in A) - \text{Prob}(Y \in A) = \mu[X^{-1}(A)] - \mu[Y^{-1}(A)].$$

Let  $\{X = Y\}$  denote the subset of the unit interval where the functions  $X$  and  $Y$  take on the same value:

$$\{X = Y\} = \{x | X(x) = Y(x)\}.$$

Its complement is the set  $\{X \neq Y\}$  where the functions  $X$  and  $Y$  take on unequal values, and the right hand side of (8) is  $\mu[\{X \neq Y\}]$ . For any set  $A$  we have the disjoint union

$$X^{-1}(A) = (X^{-1}(A) \cap \{X = Y\}) \cup (X^{-1}(A) \cap \{X \neq Y\})$$

and hence

$$\text{Prob}(X \in A) = \mu[X^{-1}(A) \cap \{X = Y\}] + \mu[X^{-1}(A) \cap \{X \neq Y\}]$$

with a similar expression for  $Y$ . But clearly we have the equality of the two sets

$$X^{-1}(A) \cap \{X = Y\} = Y^{-1} \cap \{X = Y\}$$

and so their measures are the same. So

$$\text{Prob}(X \in A) - \text{Prob}(Y \in A) = \mu[X^{-1}(A) \cap \{X \neq Y\}] - \mu[Y^{-1}(A) \cap \{X \neq Y\}].$$

The expression on the right lies between  $-\mu[\{X \neq Y\}]$  and  $+\mu[\{X \neq Y\}]$  so

$$|\text{Prob}(X \in A) - \text{Prob}(Y \in A)| \leq \mu[\{X \neq Y\}].$$

Taking the supremum over all  $A$  proves (8).

As an example of the use of this lemma we conclude from Problem 5 that

$$d_{\text{law}}(B_p, Y_p) \leq p^2.$$

Notice that the proof of this inequality depended on a specific choice of random variables  $B_p$  and  $Y_p$ . But distance in law does not depend on this choice: the triangle inequality allow us to replace  $B_p$  by any Bernoulli random variable with parameter  $p$  and  $Y_p$  by any Poisson random variable with parameter  $p$  and to conclude

**Lemma 2** *Let  $X_p$  be a Bernoulli random variable with parameter  $p$  and let  $W_p$  be a Poisson random variable with the same parameter,  $p$ . Then*

$$d_{\text{law}}(X_p, W_p) \leq p^2. \tag{9}$$

For discrete random variables we have a simple definition of “independence”. We say that  $X$  and  $Y$  are independent if for any pair of possible values  $x$  of  $X$  and  $y$  of  $Y$  we have

$$P((X = x) \cap (Y = y)) = P(X = x)P(Y = y),$$

in other words, the events  $X = x$  and  $Y = y$  are independent. It is instructive to write this as

$$P[(X, Y) = (x, y)] = P((X = x))P((Y = y)).$$

If  $X$  and  $Y$  are (discrete) random variables, then  $X + Y$  is again a (discrete) random variable and we can consider the expectation of  $f(X + Y)$  where  $f$  is a function. If  $X$  and  $Y$  are independent,

$$E(f(X + Y)) = \sum_{x,y} f(x + y)p_x q_y$$

in the obvious notation. We can write this double sum as an iterated sum

$$E(f(X + Y)) = \sum_x p_x \sum_y f(x + y)q_y = \sum_x p_x E_Y(f(x + Y)).$$

Here, for fixed  $x$  we are thinking of  $f(x + y)$  as a function of  $y$  and taking the expectation with respect to  $Y$ . Here is even better notation, we form the function  $E_Y(f(\cdot + Y))$  which sends  $x \mapsto E_Y(f(x + Y))$  and then take the expectation of this with respect to  $X$ :

$$E(f(X + Y)) = E_X[E_Y(f(X + Y))].$$

Here is a lemma which says that adding an independent random variable “fuzzes out” differences in law:

6. Let  $X, Y, Z$  be three random variables with  $Z$  independent of  $X$  and of  $Y$ . Then

$$d_{\text{law}}(X + Z, Y + Z) \leq d_{\text{law}}(X, Y). \quad (10)$$

[Hint: Use the definition of distance in law which involves minimizing over function  $f$ .]

7. Show that if  $Z$  and  $W$  are independent Poisson random variables with parameters  $\lambda$  and  $\mu$  respectively, then  $Z + W$  is a Poisson random variable with parameter  $\lambda + \mu$ . [Hint: A neat way of doing this is to show that if  $X$  and  $Y$  are independent non-negative integer valued random variables then we have the relation  $p_{X+Y}(z) = p_X(z)p_Y(z)$  for their generating functions.]

### 3.4 Poisson's law of small numbers.

Here is one version:

**Theorem 1** *Let  $X_1, \dots, X_n$  be independent Bernoullis with parameters  $p_i$ . Let*

$$\lambda = p_1 + \dots + p_n,$$

*and let  $Z_\lambda$  be Poisson with parameter  $\lambda$ . Then*

$$d_{\text{law}}(X_1 + \dots + X_n, Z_\lambda) \leq p_1^2 + \dots + p_n^2. \quad (11)$$

For example, suppose that all the  $p_i$  are equal, and so equal  $\lambda/n$ . Then the right hand side of (11) is also equal to  $\lambda/n$ . So for  $\lambda$  fixed and  $n$  large, this says that a sum of many identical Bernoullis whose expectations add up to finite number  $\lambda$ , can be approximated (in law) by a Poisson. The theorem is a bit more general, in that we don't need to have all the  $p_i$  equal to estimate the right hand side of (11), it is enough that they all be roughly of size  $\lambda/n$ .

**Proof of the theorem.** Choose Poisson random variables  $Y_i$  with parameters  $p_i$  which are independent of all the  $X_i$  and of each other. Then

$$\begin{aligned} d_{\text{law}}(X_1 + \dots + X_n, X_1 + \dots + X_{n-1} + Y_n) &\leq d_{\text{law}}(X_n, Y_n) \\ &\leq p_n^2. \end{aligned}$$

The first inequality is just (10) with  $Z = X_1 + \dots + X_{n-1}$  and the second inequality is (9). Next we replace  $X_{n-1}$  by  $Y_{n-1}$  via

$$\begin{aligned} d_{\text{law}}(X_1 + \dots + X_{n-1} + Y_n, X_1 + \dots + X_{n-2} + Y_{n-1} + Y_n) &\leq d_{\text{law}}(X_{n-1}, Y_{n-1}) \\ &\leq p_{n-1}^2 \end{aligned}$$

where the first inequality is (10) with  $Z = X_1 + \dots + X_{n-2} + Y_n$  and the second inequality is (9). Proceeding in this way we end up, via the triangle inequality with

$$d_{\text{law}}(X_1 + \dots + X_n, Y_1 + \dots + Y_n) \leq p_1^2 + \dots + p_n^2.$$

But the sum of the  $Y_i$ 's is Poisson with parameter  $\lambda = p_1 + \dots + p_n$  by the proposition. QED

There are several important improvements on this theorem, whose proofs are a bit harder, and so we will content ourselves with the statements. The first has to do with the estimate in the theorem. If  $\lambda$  is a large number, we can do better. We can replace (11) by

$$d_{\text{law}}(X_1 + \dots + X_n, Z_\lambda) \leq 2 \frac{1 - e^{-\lambda}}{\lambda} \sum_i p_i^2. \quad (12)$$

Another improvement has to do with relaxing the condition of independence. To illustrate, consider the *birthday problem*: Suppose that we have  $n$  people and we want to compute the probability that at least one pair of the group have the same birthday. Let us make the approximation that there are  $c = 365$  days in the year and that birthdays are independently distributed with probability  $p = 1/c$ . Then the probability of no two people having the same birthday is clearly

$$\prod_{k=1}^{n-1} \left( \frac{c-k}{c} \right) = \prod_{k=1}^{n-1} \left( 1 - \frac{k}{c} \right).$$

If we make the approximations

$$\ln\left(1 - \frac{k}{c}\right) \doteq -\frac{k}{c}, \quad 1 \leq k \leq n$$

the product becomes approximated by

$$\exp\left(-\frac{(n-1)(n-2)}{2c}\right).$$

Assuming that  $n$  is small with respect to  $c$  this can be approximated by

$$e^{-\frac{n^2}{2c}}. \quad (13)$$

This suggests that the probability of getting  $k$  matching pairs of birthdays should be Poisson with parameter

$$\lambda \doteq \frac{n^2}{2c} \doteq \frac{n(n-1)}{2c}.$$

This is indeed the case for  $n$  moderately large but relatively small in comparison to  $c$ . To relate this to our theorem, let  $\mathcal{E}$  denote the set of pairs of individuals, so an element  $e \in \mathcal{E}$  is a pair,  $e = \{x, y\}$  of individuals. For each such  $e \in \mathcal{E}$ , let  $X_e$  denote the random variable which is one or zero according to whether  $x$  and  $y$  do or do not have the same birthday. Clearly  $X_e$  is a Bernoulli random variable, and our assumptions are that

$$\text{Prob}(X_e = 1) = \frac{1}{c}.$$

The sum

$$\sum_{e \in \mathcal{E}} X_e$$

gives the total number of matching pairs, and there are  $\frac{n(n-1)}{2}$  pairs, so we would *like* to be able to say that this sum is Poisson distributed with parameter  $\lambda$  as given above. But the theorem does not quite apply, since the random variables  $X_e$  are not independent: If  $x$  and  $y$  have the same birthday, and  $y$  and  $z$  have the same birthday, then so do  $x$  and  $z$ . On the other hand, we might expect that this dependency is a second order effect which does not affect the validity of the conclusion of the theorem. We will not go into these more refined version, referring them to a course in probability.

We now interrupt this discussion of simulation and return to  $\mathbf{H}_R$ .

## 4 Order.

### 4.1 Non-negativity.

It makes sense to say that a real valued function is non-negative:

$$\phi \succeq 0 \quad \Leftrightarrow \quad \phi(t) \geq 0 \quad \text{for all } t.$$

We have two natural candidates for how to extend this notion to  $\mathbf{H}_R$  and we want to check that these two extended definitions are in fact the same. We could define

1.  $g \succeq 0$  if  $(g, \phi) \geq 0$  for all  $\phi \succeq 0 \in \mathcal{C}([0, 1])$  or we could define
2.  $g \succeq 0$  if there is a sequence of  $g_n \in \mathcal{C}([0, 1])$  with  $g_n \succeq 0$  and  $g_n \rightarrow g$ .

Clearly  $2 \Rightarrow 1$ , since  $(g, \phi) = \lim(g_n, \phi) \geq 0$ . We must show the reverse implication.

For any real number  $a$  define

$$a^+ = \frac{1}{2}(a + |a|) = \begin{cases} a & \text{if } a \geq 0 \\ 0 & \text{if } a \leq 0 \end{cases}.$$

By examining the four possibilities for the signs of  $a$  and  $b$  we check that

$$|a^+ - b^+| \leq |a - b|.$$

For a function  $\phi$  we define  $\phi^+$  by

$$\phi^+(t) = (\phi(t))^+.$$

It follows from the above that if  $\{g_n\}$  is a Cauchy sequence of elements of  $\mathcal{C}([0, 1])$  then so is  $g_n^+$ . So if  $g_n \rightarrow g$  in  $\mathbf{H}_R$ , then  $g_n^+$  converges to some element, call it  $g^+$ . We want to show that  $g^+ = g$ . Since  $g_n^+ - g_n \geq 0$ , we have

$$(g, g_n^+ - g_n) \geq 0$$

by condition 2, and passing to the limit we get

$$\|g\|_2^2 = (g, g) \leq (g, g^+).$$

By the Cauchy-Schwartz inequality the right hand side is  $\leq \|g\|_2 \|g^+\|_2$  so we get

$$\|g\|_2 \leq \|g^+\|_2.$$

On the other hand, from its very definition we know that

$$\|g_n^+\|_2^2 \leq \|g_n\|_2^2$$

since  $|g_n^+(t)| \leq |g_n(t)|$  for all  $t$ . Passing to the limit gives  $\|g^+\|_2 \leq \|g\|_2$  so

$$\|g\|_2 = \|g^+\|_2.$$

But then

$$\|g - g^+\|_2^2 = \|g\|_2^2 + \|g^+\|_2^2 - 2(g, g^+) \leq \|g^+\|_2^2 + \|g\|_2^2 - 2\|g\|_2^2 = \|g^+\|_2^2 - \|g\|_2^2 = 0$$

so  $g = g^+$ .

In short, it makes sense to ask whether or not  $g \succeq 0$  for any  $g \in \mathbf{H}_{\mathbf{R}}$ . That is, it makes sense to talk of non-negative random variables. If  $g_n \in \mathbf{H}_{\mathbf{R}}$  are non-negative random variables, and  $g_n \rightarrow g$  then it follows from the first definition above that  $g \succeq 0$ . In other words, the set of non-negative random variables is a closed subset of  $\mathbf{H}_{\mathbf{R}}$ . The sum of two non-negative random variables is again non-negative, and the product of a non-negative random variable by a non-negative real number is again non-negative. Applying the first definition to  $f$  and the second definition to  $g$  we see that if  $f \succeq 0$  and  $g \succeq 0$  then  $(f, g) \geq 0$ .

In fact, the above proof shows that for *any*  $g \in \mathbf{H}_{\mathbf{R}}$  we can define its “non-negative part”  $g^+$  as the limit of  $g_n^+$  where  $g_n \in \mathcal{C}([0, 1]) \rightarrow g$  and define its non-positive part  $g^-$  as  $(-g)^+$  and its absolute value  $|g|$  as  $|g| := g^+ + g^-$ . Equally well we could define  $|g|$  to be the limit of  $|g_n|$  if  $g_n$  is a sequence of elements in  $\mathcal{C}[0, 1]$  converging to  $g$ . We have  $\|g^+\|_2 \leq \|g\|_2$  and  $\|g^-\|_2 \leq \|g\|_2$  while  $\| |g| \|_2 = \|g\|_2$ . We can also check that the map  $g \mapsto g^+$  is continuous, and hence so is the map  $g \mapsto |g|$ .

We can now write

$$f \succeq g \quad \text{iff} \quad f - g \succeq 0$$

and so

$$f \succeq g \Rightarrow f + h \succeq g + h$$

for any  $h \in \mathbf{H}_{\mathbf{R}}$ .

For any real numbers  $a$  and  $b$  we have

$$\max\{a, b\} = \frac{1}{2}(a + b + |a - b|) \quad \text{and} \quad \min\{a, b\} = \frac{1}{2}(a + b - |a - b|).$$

So if  $f$  and  $g$  are functions, and we define

$$f \vee g := \frac{1}{2}(f + g + |f - g|), \quad f \wedge g := \frac{1}{2}(f + g - |f - g|),$$

then  $(f \vee g)(t) = \max\{f(t), g(t)\}$  and  $(f \wedge g)(t) = \min\{f(t), g(t)\}$ . But the above definitions make sense in  $\mathbf{H}_{\mathbf{R}}$  and we shall use them.

For example, if  $f$  and  $g$  are elements of  $\mathbf{H}_{\mathbf{R}}$  and  $f \succeq 0$  and  $g \succeq 0$ , then we can find  $f_n \succeq 0$ ,  $g_n \succeq 0$  in  $\mathcal{C}([0, 1])$  with  $f_n \rightarrow f$  and  $g_n \rightarrow g$ . We know that  $(f_n \wedge g_n)(t) \geq 0$  for all  $t$ , and so  $f_n \wedge g_n \succeq 0$ . From the continuity of the map  $u \mapsto |u|$  we conclude that  $f \wedge g \succeq 0$ .

Therefore, if  $f \succeq 0$  and  $g \succeq 0$  then

$$(f, g) \geq (f, f \wedge g) \geq (f \wedge g, f \wedge g) > 0 \text{ unless } f \wedge g = 0.$$

Also,  $f \vee g$  for any  $f, g \in \mathbf{H}_{\mathbf{R}}$  can be characterized as being the “least upper bound” of  $f$  and  $g$  in the sense that  $f \vee g \succeq f$ ,  $f \vee g \succeq g$ , and if  $h \succeq f$  and  $h \succeq g$  then  $h \succeq (f \vee g)$ . Similarly  $f \wedge g$  can be characterized as being the “greatest lower bound” of  $f$  and  $g$ .

If  $f, g, h \in \mathbf{H}_{\mathbf{R}}$  then

$$\begin{aligned} f - (g \vee h) &= f - \frac{1}{2}(g + h + |g - h|) \\ &= \frac{1}{2}(f - g + f - h - |f - g - (f - h)|) \\ &= (f - g) \wedge (f - h). \end{aligned} \tag{14}$$

Similarly, it follows directly from the definition that the “distributive laws”

$$f \wedge (g \vee h) = (f \wedge g) \vee (f \wedge h) \tag{15}$$

$$f \vee (g \wedge h) = (f \vee g) \wedge (f \vee h) \tag{16}$$

hold.

Finally, from the “triangle inequality”  $|u + v| \preceq |u| + |v|$  one can conclude that the operations  $f \vee g$  and  $f \wedge g$  are continuous; e.g.  $f_n \rightarrow f$  and  $g_n \rightarrow g$  implies that  $f_n \vee g_n \rightarrow f \vee g$  and similarly of  $\wedge$ . We could keep going on, so I will just mention that from the definitions we have

$$f + g = f \wedge g + f \vee g. \tag{17}$$

## 4.2 The Boolean distributive laws.

In set theory we have the following “distributive law” for unions and intersections of sets:

$$S \cap \left( \bigcup_{\alpha} S_{\alpha} \right) = \bigcup_{\alpha} (S \cap S_{\alpha})$$

with a similar distributive law with  $\cap$  and  $\cup$  interchanged.

We wish to show that similar formulas hold for  $\vee$  and  $\wedge$ . We have characterized  $f \wedge g$  as the least upper bound of  $f$  and  $g$ . More generally, suppose we are given a collection of elements  $x_{\alpha} \in \mathbf{H}_{\mathbf{R}}$ . We will say that  $a$  is the least upper bound of this set and write

$$a = \bigvee_{\alpha} x_{\alpha}$$

if

- $a \succeq x_\alpha$  for all  $\alpha$  and
- If  $b \succeq x_\alpha$  for all  $\alpha$  then  $b \succeq a$ .

If there are finitely many  $x_\alpha$  then we know that  $a$  exists. We will need to deal with the case of infinitely many  $x_\alpha$  where such an  $a$  may or may not exist. Let us assume that  $a$  exists. Then for any  $x \in \mathbf{H}_R$ ,  $x_\alpha \preceq a$  implies that  $x \wedge x_\alpha \preceq x \wedge a$ , so  $x \wedge a$  is an upper bound for all the  $x \wedge x_\alpha$ . We want to show that it is a (and hence the) least upper bound. For this we will use the fact (17):

$$x + x_\alpha = x \wedge x_\alpha + x \vee x_\alpha$$

in the form

$$x \wedge x_\alpha = x + x_\alpha - (x \vee x_\alpha).$$

Thus if  $y \succeq x \wedge x_\alpha$  then

$$x + x_\alpha - x \wedge x_\alpha \preceq y$$

or

$$x_\alpha \leq y - x + (x \wedge x_\alpha).$$

Hence

$$a \preceq y - x + x \vee a.$$

**8.** Proceeding to argue this way, show that  $x \wedge a$  is indeed the least upper bound of the  $x_\alpha$ .

In symbols, we can write this as

$$x \wedge \left( \bigvee_{\alpha} x_\alpha \right) = \bigvee_{\alpha} (x \wedge x_\alpha) \quad (18)$$

in the sense that the right hand side exists and is equal to the left hand side under the assumption that the left hand side exists.

### 4.3 More Events.

We shall call a random variable  $\mathbf{a}$  an **event** if We call  $e \in \mathbf{H}_R$  an **event** if  $0 \preceq e \preceq \mathbf{1}$  and

$$e \wedge (\mathbf{1} - e) = 0.$$

Then

$$\begin{aligned} e \vee (\mathbf{1} - e) &= -(-e \wedge (-\mathbf{1} + e)) = \mathbf{1} - \mathbf{1} - (-e \wedge (-\mathbf{1} + e)) = \\ &= \mathbf{1} - (\mathbf{1} - (-e \wedge (-\mathbf{1} + e))) = \mathbf{1} - ((\mathbf{1} - e) \wedge e) = \mathbf{1}. \end{aligned}$$

Conversely, if

$$e \vee (\mathbf{1} - e) = \mathbf{1}$$

then

$$e \wedge (\mathbf{1} - e) = 0$$

by reversing the argument.

Suppose that  $\{e_\alpha\}$  is a family of events, and suppose that this family has a least upper bound, i.e. suppose there is an  $e \in \mathbf{H}_R$  with the property that

$$e \succeq v_\alpha, \forall \alpha$$

and

$$y \succeq v_\alpha \forall \alpha \Rightarrow y \succeq e.$$

The usual notation is to write

$$e = \bigvee_{\alpha} e_{\alpha}$$

if such a least upper bound exists. We claim that  $e$ , if it exists, is an event. Indeed, since all the  $e_\alpha \succeq 0$  we see that  $e \succeq 0$ , and since all the  $e_\alpha \preceq \mathbf{1}$  we see that  $e \preceq \mathbf{1}$ . So

$$0 \preceq \mathbf{1} - e \preceq \mathbf{1} - e_\alpha$$

so from

$$e_\alpha \wedge (\mathbf{1} - e_\alpha) = 0$$

we conclude that

$$e_\alpha \wedge (\mathbf{1} - e) = 0$$

for all  $\alpha$ . Therefore by the distributive law,

$$e \wedge (\mathbf{1} - v) = \left( \bigvee_{\alpha} e_{\alpha} \right) \wedge (\mathbf{1} - e) = \bigvee_{\alpha} (e_{\alpha} \wedge (\mathbf{1} - e)) = 0.$$

All this was quite general, and left open the issue of whether the least upper bound exists. We know it exists for a finite family. We will now use the Hilbert space structure to show that it exists for a countable family of events.

**This is the replacement (via Hilbert space) of the notion countable additivity of measures that we will introduce in class when we get to measure theory - soon, I hope.**

So suppose that  $e_i, i = 1, 2, 3 \dots$  is a sequence of events.

9. Show that the

$$f_n := \bigvee_1^n e_n$$

form a Cauchy sequence, and then that their limit is the least upper bound of the  $\{e_i\}$  and so is an event. [Hint: Begin by showing that  $0 \preceq f_n - f_m \preceq \mathbf{1}$  for  $n > m$  and then comparing  $\|f_n - f_m\|^2$  with  $(f_n, \mathbf{1}) - (f_m, \mathbf{1})$ .]

Here is another illustration of the interplay between the Hilbert space structure and the order structure.

Suppose that  $\mathbf{C}$  is a closed subset of  $\mathbf{H}_{\mathbf{R}}$  which has an upper bound  $x$ . In other words,

$$x \succeq u \quad \forall u \in \mathbf{C}.$$

Suppose also that  $\mathbf{C}$  is closed under the  $\vee$  operation. In other words, assume that

$$y \in \mathbf{C}, z \in \mathbf{C} \Rightarrow y \vee z \in \mathbf{C}.$$

Then,

**10.** Under the above hypotheses, show that the least upper bound  $\bigvee \mathbf{C}$  exists and belongs to  $\mathbf{C}$ . [Hint: Since  $\mathbf{C}$  is closed, we can find a  $z \in \mathbf{C}$  which is closest to  $x$  in the sense that

$$\|x - z\| \leq \|x - y\| \quad \forall y \in \mathbf{C}.$$

Show that  $z$  is the desired  $\bigvee \mathbf{C}$  by showing that  $z \vee y = z$  for any  $y \in \mathbf{C}$ . Show that  $0 \preceq x - (y \vee z) \preceq x - z$ . Use this to get  $\|x - (z \vee y)\| \leq \|x - z\|$ , and then expand out  $\|x - z\|^2 = \|x - (y \vee z) + (y \vee z) - z\|^2$  and use the non-negativity of the cross terms.]

Consider any element  $X \in \mathbf{H}_{\mathbf{R}}$ . Let  $\mathcal{C}_X$  denote the set of all events  $e$  such that

$$e \preceq X^+.$$

This satisfies the conditions above, and hence there exists a least upper bound, which is an event. We shall denote this event by

$$\mathbf{1}(X \geq 1).$$

Similarly, for any positive number  $a$  we can consider the event  $\mathbf{1}_{\frac{1}{a}X \geq 1}$  which we can write as

$$\mathbf{1}_{X \geq a}$$

and in a similar fashion we can construct events such as

$$\mathbf{1}_{a \leq X \leq b}$$

for any random variable  $X$ .

## 5 Conditional expectation is orthogonal projection.

This leads to the following idea: Let us call a subspace  $S \subset \mathbf{H}_{\mathbf{R}}$  a **probabilistically closed subspace** if

- $\mathbf{1} \in S$  and
- If  $X$  and  $Y \in S$  implies that  $X \wedge Y$  and  $X \vee Y$  are in  $S$ .

The simplest example of a probabilistically closed subspace is the one dimensional space spanned by  $\mathbf{1}$ . If  $Z \in \mathbf{H}_{\mathbf{R}}$  is any random variable, orthogonal projection of  $Z$  onto this subspace is given by

$$Z \mapsto E(Z)\mathbf{1}.$$

Here is a slightly less simple but still elementary example: Let  $\mathbf{1}_{A_1}, \dots, \mathbf{1}_{A_n}$  a collection of events such that

$$\sum_1^n \mathbf{1}_{A_i} = \mathbf{1}.$$

In other words we have a finite set of mutually exclusive and exhaustive events. Then the space  $S$  that they span is the set of “step functions”

$$\sum a_i \mathbf{1}_{A_i}.$$

Notice that if

$$X = \sum a_i \mathbf{1}_{A_i} \quad \text{and} \quad Y = \sum b_i \mathbf{1}_{A_i}$$

then

$$X \vee Y = \sum c_i \mathbf{1}_{A_i} \quad \text{where} \quad c_i = \max(a_i, b_i)$$

and so belongs to  $S$ , similarly for  $X \wedge Y$ . Orthogonal projection onto the space  $S$  is given by

$$\pi_S : Z \mapsto \sum_{i=1}^n \frac{(Z, \mathbf{1}_{A_i})}{P(A_i)} \mathbf{1}_{A_i}.$$

The image  $\pi(Z)$  is “measurable” with respect to the (finite)  $\sigma$ -field of events generated by the events  $\mathbf{1}_{A_i}$ . The finite valued element  $\pi(Z)$  is called the conditional expectation of  $Z$  relative to this finite  $\sigma$ -field.

So in general, given any probabilistically closed subspace  $S$ , we call orthogonal projection onto  $S$  the **conditional expectation** relative to  $S$ . It is sometimes written as

$$\pi_S(Z) = E(Z||S).$$

In general, suppose we are given a family of probabilistically closed subspaces. Then their intersection is again a probabilistically closed subspace. So given any subset  $B$  of  $\mathbf{H}_{\mathbf{R}}$ , the intersection of all probabilistically closed subspaces containing it will be called the **probabilistically closed subspace generated** by  $B$ . For instance, if  $X$  is any element of  $\mathbf{H}_{\mathbf{R}}$ , the probabilistically closed subspace generated by  $X$  will contain all the events  $\mathbf{1}_{a \leq X \leq b}$ . This is a rather subtle notion.

For instance, suppose we believe in MATLAB’s capability of producing  $(X, Y) = \text{rand}(1,2)$ . (It is easy enough to see that we can theoretically produce such a pair of uniformly distributed random variables over the unit square in the plane.) Then the events in the probabilistically closed subspace generated

by  $X$ , model all subsets of the unit square of the form  $A \times I$  where  $A$  is a subset of the unit interval; in other words sets of the form  $\{(x, y) | x \in A\}$ .

We can now also write down the general notion of **independence** of two random variable in terms of these concepts: Let  $S_X$  be the probabilistically closed subspace generated by  $X$ , and let  $S_Y$  be the probabilistically closed subspace generated by  $Y$ . Then  $X$  and  $Y$  are independent if and only if

$$\text{cov}(f, g) = 0 \quad \forall f \in S_X, \quad g \in S_Y.$$

Notice that this independence a much more stringent condition than being uncorrelated which merely says that  $\text{cov}(X, Y) = 0$ .