Problem set 3

Math 212a

October 18, 2001 due Oct. 30

The purpose of this exercise set, which is mainly computational, is to give you some experience with the ideas associated to **generalized functions** also known as **distributions**. This theory comes in various flavors, vanilla, chocolate, and some 32 others. In each version there is a choice of "test functions" which are the "very good" functions of the theory, and then a collection of "very bad" functions which are the continuous linear functionals on the space of test functions. When we use the word "continuous" we mean that have a topology on the space of test functions, and so a complete discussion of the theory would require us to get into the discussion of topological vector spaces, especially Frechet spaces. In class, we finessed this topic by considering the Sobolev spaces on the torus, and so had a number of simplifications - we were dealing with functions on a compact space, so did not have to worry about questions of support or of growth, and we had a collection of Hilbert space norms which gave the topology. We also used a slightly different convention than is standard involving complex conjugation, as we shall explain momentarily. (The standard choice is motivated by the multiplication law for the Fourier transform.)

A thorough and highly readable text on the subject is the five volume work by Gelfand et al published in the 1950's (available in English translation). This is enjoyable but long reading. A short and readable text (which does not delve into all the technical details) is the book *A Guide to Distribution Theory and Fourier Transforms*(1994) by Robert Strichartz, CRC Press, Boca Raton.

We will start with the (chocolate) version involving the space $\mathcal{S}$ as the space of test functions, and then get to the more popular (vanilla) version using the space $\mathcal{D}$ later on.

Contents

1 The space $\mathcal{S}$ and its dual space $\mathcal{S}'$.

Recall that in our study of the Fourier transform we introduced the space $\mathcal{S}$ consisting of all functions on $\mathbf{R}$ which have derivatives of all order which vanish rapidly at infinity, and that on this space we have a countable family of norms

$$\|f\|_{m,n} := \|x^m D^n f\|_\infty .$$

Convergence in $\mathcal{S}$ means convergence with respect to all of these norms. So a linear function ℓ on this space is continuous if $|\ell(f_k)| \to 0$ whenever $\|f_k\|_{m,n} \to 0$ for all m and n. We let $\mathcal{S}'$ denote this dual space, the space of continuous linear functions on $\mathcal{S}$.

For example, consider the "Heaviside function"

$$H(x) := \begin{cases} 1 & \text{for} \quad x > 0 \\ 0 & \text{for} \quad x \leq 0 \end{cases} .$$

This function does not belong to $\mathcal{S}$, but it defines a continuous linear function on $\mathcal{S}$ by

$$\langle H, f \rangle = \int_{-\infty}^{\infty} f(x)H(x)dx = \int_0^\infty f(x)dx.$$

Of course, any element of $\mathcal{S}$ certainly defines a linear function on $\mathcal{S}$ by the same procedure, the linear function associated to $g \in \mathcal{S}$ is just the Hilbert space scalar product with the complex conjugate $\overline{g}$ of g:

$$f \mapsto (f, \overline{g}) = \int_{-\infty}^{\infty} f(x)g(x)dx.$$

This is the standard convention that we mentioned above. We also write this as

$$\langle g, f \rangle := \int_{-\infty}^{\infty} f(x)g(x)dx. \tag{1}$$

This also works for any $g \in L_2(\mathbf{R})$, but $H \notin L_2(\mathbf{R})$. From the Riesz representation theorem we thus know that the linear function given by H is not continuous with respect to the L_2 norm, but it is continuous relative to the topology given above by the countable family of norms $\|\ \|_{m,n}$.

We let $\mathcal{S}'$ denote the space of continuous linear functions on $\mathcal{S}$. For example, in order for a continuous function g to belong to $\mathcal{S}'$ via the definition (1), it is sufficient (and necessary) that g not grow at infinity faster than any polynomial, i.e. that there exists some integer N such that $(1+|x|)^{-N}|g(x) \to 0$ as $x \to \pm\infty$. Thus the function e^x does not belong to $\mathcal{S}'$. The elements of $\mathcal{S}'$ are called **tempered distributions**. The word "tempered" in this context siginifying "not too rapid growth at infinity".

2 The transpose method of extending operators.

Suppose that

$$T : \mathcal{S} \to \mathcal{S}$$

is a continuous linear operator. Suppose that there is another continuous operator T^* mapping $\mathcal{S} \to \mathcal{S}$ such that

$$\langle Tg, f\rangle = \langle g, T^* f\rangle \tag{2}$$

for all $f, g \in \mathcal{S}$. Now the right hand side of this equation makes sense when $f \in S$ and g is an arbitrary element of $\mathcal{S}'$. This right hand side is a continuous linear functional on $\mathcal{S}$. So we can use (2) as the *definition* of Tg. Here are some examples.

2.1 Differentiation of generalized functions.

The operation of differentiation

$$\frac{d}{dx} : \mathcal{S} \to \mathcal{S}, \quad f \mapsto f' = \frac{d}{dx} f$$

is a continuous linear operator. It has a well defined transpose

$$\left(\frac{d}{dx}\right)^* : \ \mathcal{S} \to \mathcal{S} \qquad \left\langle \left(\frac{d}{dx}\right) \ell, f \right\rangle := \left\langle \ell, \left(\frac{d}{dx}\right)^* f \right\rangle$$

given by

$$\left(\frac{d}{dx}\right)^* := -\left(\frac{d}{dx}\right).$$

Indeed, for $f, g \in \mathcal{S}$ we have

$$\int_{-\infty}^{\infty} f'(x)g(x)dx = -\int_{-\infty}^{\infty} f(x)g'(x)dx$$

by the integration by parts formula: The "boundary terms" in the integration by parts formula disappear at $\pm\infty$ because the functions f and g vanish rapidly at infinity.

Our transpose method then tells us to *define* the operator of differentiation on elements of $\mathcal{S}$ by

$$\left\langle \left(\frac{d}{dx}\right)\ell, f\right\rangle := -\langle \ell, f'\rangle, \quad \ell \in \mathcal{S}', \quad f \in \mathcal{S}.$$

For example, if we take $\ell = H$ to be the Heaviside function defined above, we have

$$\left\langle \left(\frac{d}{dx}\right)H, f\right\rangle = -\langle H, f'\rangle = -\int_0^\infty f'(x)dx = f(0).$$

So if we define the **Dirac delta "function"** $\delta = \delta_0 \in \mathcal{S}'$ by

$$\langle \delta, f\rangle := f(0)$$

We have

$$\frac{d}{dx}H = \delta. \tag{3}$$

We have learned how to differentiate discontinuous functions! When Heaviside wrote down formulas like this at the beginning of the twentieth century he was derided by mathematicians. When Dirac did the same, mathematicians were puzzled, but a bit more respectful because of Dirac's fantastic achievements in physics. The advent of theorems like the Riesz representation theorem gave rise to the idea that we might want to think of a function as a functional, i.e. as a linear function on a suitable space of functions. The choices of the appropriate spaces of "test functions" such as $\mathcal{S}$ (there are others which are convenient for different purposes) with their topologies and the whole subject of "generalized functions" or "distributions" was developed by Laurent Schwartz around 1945, after earlier work by Sobolev in the 1930's, work that we studied in class.

2.2 The Fourier transform.

Recall the multiplication formula:

$$\int_{\mathbf{R}} \hat{f}(x)g(x)dx = \int_{\mathbf{R}} f(x)\hat{g}(x)dx$$

for any $f, g \in \mathcal{S}$. This says that the Fourier transform $\mathcal{F}$ is its own adjoint. Our general method tells us to define

$$\mathcal{F} : \mathcal{S}' \to \mathcal{S}', \quad \ell \mapsto \hat{\ell}$$

by

$$\langle \hat{\ell}, f\rangle := \langle \ell, \hat{f}\rangle. \tag{4}$$

1. What is the Fourier transform of the Dirac delta function?

2. What is the k-th derivative of the Dirac delta function?

We may generalize the Heaviside function by considering the function

$$x_+^\lambda := \begin{cases} x^\lambda & \text{for} \quad x > 0 \\ 0 & \text{for} \quad x \leq 0 \end{cases} . \tag{5}$$

As a *function* of x, this is defined for all complex values of λ by the rule

$$x^\lambda = e^{\lambda \log x}.$$

For $\lambda = 0$ we get $H(x)$. For Re$\lambda > -1$ the integral

$$\int_0^\infty f(x) x_+^\lambda dx$$

converges for all $f \in \mathcal{S}$ and so x_+^λ defines a continuous linear function on $\mathcal{S}$. So for these values of λ, we can consider x_+^λ as defined by (5) as an element of $\mathcal{S}'$. We never have any trouble with the convergence of the above integral at infinity. But we have convergence problems at 0 when Re $\lambda \leq -1$. We will therefore have to modify the definition (5) for these values of λ.

Notice that for Re $\lambda > 0$ we have

$$\frac{d}{dx}(x_+^\lambda) = \lambda x_+^{\lambda - 1} \tag{6}$$

both as functions and as elements of $\mathcal{S}'$.

3. Compute the derivative $\frac{d}{dx}(x_+^\lambda)$ for $-1 < \text{Re } \lambda < 0$. In fact, show that

$$\langle (x_+^\lambda)', f \rangle = \int_0^\infty \lambda x^{\lambda-1} [f(x) - f(0)] dx.$$

[Hint: Write $\int_0^\infty x^\lambda f'(x) dx$ as $\lim_{\epsilon \to 0} \int_\epsilon^\infty x^\lambda f'(x) dx$ and do an integration by parts with $f'(x)dx = du$, $u = f(x) + C$, and $v = x^\lambda$. Choose C appropriately so as to be able to evaluate the limit.]

Notice that the function $f(x) - f(0)$ will not belong to the space $\mathcal{S}$ if $f(0) \neq 0$. What makes the above integral converge at infinity is that $x^{\lambda - 1}$ vanishes rapidly enough at infinity if Re $\lambda < 0$.

This suggests the following strategy: For Re $\lambda > -1$ we have

$$\int_0^1 x^\lambda dx = \frac{1}{\lambda + 1}.$$

So we can write

$$\langle x_+^\lambda, f \rangle = \int_0^1 x_+^\lambda [f(x) - f(0)] dx + \int_1^\infty x^\lambda f(x) dx + \frac{f(0)}{\lambda + 1}. \tag{7}$$

Notice that the right hand side of this equation makes sense for all λ such that Re $\lambda > -2$ with the exception of the single point $\lambda = -1$. (In the language of complex variable theory we would say that the left hand side is a meromorphic function of λ with a pole at -1 with residue δ.) We will therefore take the right hand side of (7) as a new definition of x_+^λ valid for all $\lambda \neq -1$ such that Re $\lambda > -2$.

4. Show that in the range $-2 <$ Re $\lambda < -1$ we can write the above expression as

$$\int_0^\infty x^\lambda [f(x) - f(0)]dx.$$

Conclude that we can write the result of Problem 2 as asserting the validity of (6) on the range Re $\lambda > -2, \lambda \neq -1$.

5. Obtain a formula for x_+^λ which extends its range into the region

$$\text{Re } \lambda > -n-1, \quad \lambda \neq -1, -2, \cdots, -n$$

for any positive integer n, and obtain a simpler formula valid on the range $-n-1 <$ Re $\lambda < -n$ to conclude the validity of (6) on the strip $-n <$ Re $\lambda < -n-1$.

3 The Poisson summation formula.

For any real number y, let

$$\delta_y$$

sometimes denoted by

$$\delta(\cdot - y)$$

be the element of $\mathcal{S}'$ given by

$$\langle \delta_y, f \rangle := f(y), \quad \forall \quad f \in \mathcal{S}.$$

Let a be a positive number. Define the **Dirac comb with grid** a, by

$$\diamondsuit_a := \sum_{n=-\infty}^{\infty} \delta_{na}.$$

6. Show that $\diamondsuit_a \in \mathcal{S}'$.

7. What is the Fourier transform of $\diamondsuit_a$? For which value of a does $\diamondsuit_a$ equal its own Fourier transform?

4 Differentiating under the limit.

Let $\{\ell_n\}$ be a weakly convergent sequence of elements of $\mathcal{S}'$. By definition, this means that there is an $\ell \in \mathcal{S}'$ such that for every $f \in \mathcal{S}$

$$\langle \ell_n, f \rangle \to \langle \ell, f \rangle.$$

But then

$$\ell_n' \to \ell'$$

as well, since by definition

$$\langle \ell_n', f \rangle = -\langle \ell_n, f' \rangle \to -\langle \ell, f' \rangle = \langle \ell', f \rangle.$$

Similarly, if we have a series $\ell_1 + \ell_2 + \cdots$ of generalized functions which converges in this weak sense, we may differentiate term by term to obtain the derivative of the sum.

For example, let h be a piecewise smooth function which is periodic of period 2π. Its Fourier series converges at all points, and hence (by say the dominated convergence theorem) it converges as a series in the weak topology of $\mathcal{S}'$. We know (from Gibbs!) that

$$\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \cdots = \frac{1}{2}(\pi - x) \quad 0 < x < 2\pi$$

and periodic of period 2π on $\mathbf{R}$.

8. Differentiate this formula and conclude that

$$\cdots + e^{-2ix} + e^{-ix} + 1 + e^{ix} + e^{2ix} + \cdots = 2\pi \sum_{-\infty}^{\infty} \delta(x - 2\pi n).$$

and from this conclude the Poisson summation formula.

Let

$$v(x,t) := \frac{1}{2\sqrt{\pi t}} \exp\left(-\frac{x^2}{4t}\right), \quad t > 0$$

9. Show that

$$\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right) v \equiv 0$$

and

$$\lim_{t \to 0} v(x,t) = \delta$$

in the sense of generalized functions.

5 Extending the domain of definition.

We may define the **support** of a generalized function $\ell \in \mathcal{S}'$ as follows: We say that ℓ vanishes on an open set U if $\langle \ell, f \rangle = 0$ for all f with $\mathrm{supp}(f) \subset U$. It is not hard to verify that there is a maximal open set with this property, and the support of ℓ is defined to be the complement of this set. Put contrapositively, what this says if that $x \in \mathrm{supp}\ \ell$ if for every neighborhood U of x there is an $f \in \mathcal{S}$ with supp $(f) \subset U$ such that $\langle \ell, f \rangle \neq 0$.

For example, the generalized functions x_+^λ all have their support on the set of non-negative real numbers.

Suppose that we have a function f which does not necessarily belong to $\mathcal{S}$, but which has the property that there is a function ϕ which is identically one in a neighborhood of $\mathrm{supp}(\ell)$ and such that

$$\phi f \in \mathcal{S}.$$

We could then define

$$\langle \ell, f \rangle := \langle \ell, \phi f \rangle$$

and this definition will not depend on ϕ. For example, the function e^{-x} does not belong to $\mathcal{S}$ because of the blow up at $-\infty$. But nevertheless we may apply the x_+^λ to it. In fact Euler's Gamma function defined by

$$\Gamma(\lambda) = \int_0^\infty x^{\lambda-1} e^{-x} dx$$

can be thought of as applying the generalized function $x_+^{\lambda-1}$ to the function e^{-x} which is valid even though $e^{-x} \notin \mathcal{S}$. The results of Problem 5 can be thought of as a generalization of the "analytic continuation of the Gamma function".

6 The spaces $\mathcal{D}$ and $\mathcal{D}'$.

So far, we defined a generalized function to be a continuous linear function on the Schwartz space $\mathcal{S}$, and denoted the space of generalized functions by $\mathcal{S}'$. There is no difficulty in passing from the one dimensional case, in which we have been working, to the n-dimenionsl case.

For various reasons (especially when we want to extend the theory to manifolds) it is convenient to study other spaces of "test functions" and "generalized functions". This is the most common flavor of the theory.

Let $\mathcal{D}$ denote the space of infinitely differentiable functions each of which vanishes outside a compact set K (which may depend on the function). A sequence $f_n \in \mathcal{D}$ is said to converge to an $f \in \mathcal{D}$ if there is a fixed compact set K such that $\mathrm{supp}\, f_n \subset K$ for all n, and then such that the f_n converge to f uniformly, together with uniform convergence of all the derivative. Then $\mathcal{D}'$ consists of all linear functions which are continuous relative to this notion of convergence.

We should really write $\mathcal{D}(\mathbf{R})$ if we are thinking of functions of one real variable, but we could equally well consider functions of n variables, in which case we would write $\mathcal{D}(\mathbf{R}^n)$ or write $\mathcal{D}(V)$ where V is any finite dimensional vector space.

For example, if Δ denotes the Laplacian on $\mathbf{R}^3$ (or $\mathbf{R}^n$) then one way of writing Green's theorem is

$$\int_G f\Delta\phi dx = \int_G (\Delta f)\phi dx + \int_{\partial G}\left(f\frac{\partial\phi}{\partial n} - \frac{\partial f}{\partial n}\phi\right) dS \tag{8}$$

where f and ϕ are smooth functions, G is a region with smooth (or piecewise smooth) boundary ∂G, where dS denotes the surface measure on the boundary and $\partial/\partial n$ denotes normal derivative. If we take both f and ϕ to be in $\mathcal{D}$, and take G so large that both $\operatorname{supp} f$ and $\operatorname{supp}\phi$ lie in the interior of G, so that there are no boundary terms,then (8) becomes

$$\langle\Delta f, \phi\rangle = \langle f, \Delta\phi\rangle. \tag{9}$$

For a generalized function f we take this as the definition of Δf, i.e. Δf is defined to be that generalized function given by

$$\langle\Delta f, \phi\rangle := \langle f, \Delta\phi\rangle \quad \forall\phi \in \mathcal{D}.$$

For example, suppose that g is a smooth function, that G is as in (8), and we define f to be equal to g inside G and to be zero outside G, i.e. $f = \mathbf{1}_G g$. Then (8) says that Δf is the sum of the function $(\Delta g)\mathbf{1}_G$ plus two other terms supported on the boundary.

In three dimensions, consider the function $1/r$ (where r is the distance from the origin) which is smooth away from the origin, and (by direct computation) satisfies $\Delta(1/r) = 0$ there. Since the function $1/r$ is integrable in three dimensions at the origin, it defines an element of $\mathcal{S}'$ and of $\mathcal{D}'$ and so it makes sense to compute $\Delta(1/r)$ according to the above definition.

10. Show that

$$\Delta\left(\frac{1}{r}\right) = -4\pi\delta.$$

[Hint: Apply (8) to the region consisting of $\epsilon \le r \le R$ where R is chosen so large that $\operatorname{supp}\phi$ is contained in the open ball of radius R so that there are no terms coming from the outer boundary $r = R$. Compute the terms coming from the inner boundary and let $\epsilon \to 0$.]

7 Cauchy principle value.

Back to one dimension temporarily. The function $x \mapsto 1/x$ is not locally integrable in one dimension, but we do have $\log(|x|)' = 1/x$ at all $x \neq 0$ and $\log|x|$ is locally integrable and so defines a generalized function. We may therefore try to define a generalized function by taking the derivative of $\log |x|$ in the sense of generalized functions.

11. Show that $(\log |x|)' = \text{pv}\left(\frac{1}{x}\right)$ where

$$\langle \text{pv}\left(\frac{1}{x}\right), \phi\rangle = \int_0^\infty \frac{\phi(x) - \phi(-x)}{x} dx.$$

The symbol "pv" stands for principal value, a notion that was introduced by Cauchy. Also compute the second derivative of $\log |x|$.

12. Show that if $U \in \mathcal{D}'$ has derivative zero, then U is a constant, i.e. $\langle U, \phi\rangle = C \int_{\mathbf{R}} \phi dx$. [Hint: let $\mathcal{D}_0$ denote the set of all $\psi \in \mathcal{D}$ such that $\int_{\mathbf{R}} \psi dx = 0$. First show that U vanishes on $\mathcal{D}_0$. Then choose an element $\theta \in \mathcal{D}$ with $\int_{\mathbf{R}} \theta dx = 1$. Write every $\phi \in \mathcal{D}$ as $\phi = \psi + a\theta$ where $a = \int_{\mathbf{R}} \phi dx$.]

13. Use the preceding problem to show the following: Suppose that f is a continuous function whose derivative in the sense of generalized functions is a continuous function. I.e. assume that $-\int_{\mathbf{R}} f\phi' dx =: \langle f', \phi\rangle = \int_{\mathbf{R}} g\phi dx =: \langle g, \phi$. Show that f is in fact continuously differentiable as a function and that $f' = g$ in the usual sense.

8 The tensor product of two generalized functions.

Let X and Y be two finite dimensional vector spaces over the real numbers, so that $X \times Y = X \oplus Y$ is again a finite dimensional vector space over the real number. We will write the typical point of $X \times Y$ as (x, y). We can consider the space $\mathcal{D}(X \times Y)$ which consists of all infinitely differentiable functions of compact of compact support on $X \times Y$. We can also consider the space $\mathcal{D}(X) \otimes \mathcal{D}(Y)$ which consists of all finite linear combinations of expressions of the form $\phi\psi$ where $\phi \in \mathcal{D}(X)$ and $\psi \in \mathcal{D}(Y)$. Any such expression defines a function on $X \times Y$ by the rule

$$(\phi\psi)(x, y) = \phi(x)\psi(y),$$

so we have an injection

$$\mathcal{D}(X) \otimes \mathcal{D}(Y) \mapsto \mathcal{D}(X \times Y)$$

which allows us to think of $\mathcal{D}(X) \otimes \mathcal{D}(Y)$ as a subspace of $\mathcal{D}(X \times Y)$. The Stone-Weierstrass theorem, or the original Weierstrass approximation theorem guarantees that $\mathcal{D}(X) \otimes \mathcal{D}(Y)$ is dense in $\mathcal{D}(X \times Y)$.

Now let $f \in \mathcal{D}'(X)$ and $g \in \mathcal{D}'(Y)$ be generalized functions. Let $\phi \in \mathcal{D}(X \times Y)$. For each fixed x, the function $\phi(x, \cdot) : \ y \mapsto \phi(x, y)$ belongs to $\mathcal{D}(Y)$ and so we can apply g to it to obtain the function $x \mapsto \langle g, \phi(x, \cdot)\rangle$. This is a function of x, and the continuity properties of g imply that this function belongs to $\mathcal{D}(X)$. We then may apply f to the function $x \mapsto \langle g, \phi(x, \cdot)\rangle$ to obtain a number. The notation is getting cumbersome, so we will shorten it and write the final result as

$$\langle f, \langle g, \phi\rangle\rangle.$$

We then define $f \otimes g$ to be this generalized function. In other words we define $f \otimes g \in \mathcal{D}'(X \times Y)$ by

$$\langle f \otimes g, \phi\rangle = \langle f, \langle g, \phi\rangle\rangle. \tag{10}$$

If $\phi = \tau \otimes \eta$ where $\tau \in \mathcal{D}(X)$ and $\eta \in \mathcal{D}(Y)$ then it is clear from the definition that

$$\langle f \otimes g, \tau\eta\rangle = \langle f, \tau\rangle\langle g, \eta\rangle.$$

This shows that on function of the form $\tau\eta$ it would not have made any difference in the definition of $f \otimes g$ had we done things in the reverse order, i.e. first apply f to the function $x \mapsto \phi(x, y)$ and then apply g to the resulting function of y. But since $\mathcal{D}(X) \otimes \mathcal{D}(Y)$ is dense in $\mathcal{D}(X \times Y)$ it follows that doing things in the reverse order yields the same answer on all of $\mathcal{D}(X \times Y)$. This is a sort of "generalized function version" of Fubini's theorem.

Similarly, if we have three vector spaces X, Y and Z and $f \in \mathcal{D}(X)$, $g \in \mathcal{D}(Y)$, $h \in \mathcal{D}(Z)$ we can form $f \otimes (g \otimes h)$ and $(f \otimes g) \otimes h$ and verify that they give the same element of $\mathcal{D}(X \times Y \times Z)$.

It is easy to check directly from the definition that

$$\operatorname{supp}(f \otimes g) = \operatorname{supp}(f) \times \operatorname{supp}(g)$$

as a subset of $X \times Y$.

Suppose that $X = Y = Z$, and to fix the ideas (and hopefully to get the powers of 2π right, although I am not all that optimistic) that they all equal $\mathbf{R}$. If f and g were elements of L_1 we defined their convolution $f \star g$ as

$$(f \star g)(u) = \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} f(x) g(u - x) ds.$$

If we think of this as a generalized function and apply it to ϕ we get

$$\frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} \int_{\mathbf{R}} f(x) g(u - x) dx \phi(u) du = \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} \int_{\mathbf{R}} f(x) g(y) \phi(x + y) dx dy.$$

Therefore we would like to define the convolution $f \star g$ of any two elements of $\mathcal{D}'(\mathbf{R})$ by

$$\langle f \star g, \phi\rangle = \frac{1}{\sqrt{2\pi}} \langle f \otimes g, \phi(x + y)\rangle \tag{11}$$

Here $\phi \in \mathcal{D}(\mathbf{R})$ and $\phi(x + y)$ denotes the function of two variables given by $(x, y) \mapsto \phi(x + y)$. The trouble is that $\phi(x + y)$ does not have compact support as a function of two variables. Indeed it is constant on any "anti-diagonal" line $x + y = a$. So the supp $\phi(x+y)$ is the anti-diagonal strip consisting of all (x, y) such that $x + y \in \operatorname{supp}(\phi)$. So we can not use (11) in general. But by the remarks of the last section of the preceding exercise set, we can use this definition if we know that $\operatorname{supp}(f) \times \operatorname{supp}(g)$ intersects every anti-diagonal strip (of bounded width) in a compact set.

This will happen, for example, if

- Either f of g has compact support. For then $\operatorname{supp}(f \otimes g)$ will be a horizontal or a vertical strip, which then meets any anti-diagonal strip of bounded width in a compact set. Or
- $\operatorname{supp}(f) \subset [a, \infty)$ and $\operatorname{supp}(g) \subset [b, \infty)$ for then $\operatorname{supp}(f \otimes g)$ is contain in the (infinite) rectangle $[a, \infty) \times [b, \infty)$ which also intersects any bounded anti-diagonal strip in a compact set.

As an illustration of the first case, consider what happens if we take $f = \delta$. Then

$$\langle \delta \star g, \phi \rangle = \frac{1}{\sqrt{2\pi}} \langle \delta \otimes g, \phi(x+y) \rangle = \frac{1}{\sqrt{2\pi}} \langle g, \langle \delta, \phi(x+y) \rangle \rangle = \frac{1}{\sqrt{2\pi}} \langle g, \phi \rangle.$$

In other words,

$$(\sqrt{2\pi}\delta) \star g = g$$

for any $g \in \mathcal{D}'(\mathbf{R})$. Convolution with the element $\sqrt{2\pi}\delta$ is the identity operator.

Similarly, we can define the convolution of two generalized function in n dimensions, the unfortunate factor $\frac{1}{\sqrt{2\pi}}$ (which was determined by our conventions for the Fourier transform) being replaced by the factor $\frac{1}{(2\pi)^{n/2}}$. The second sufficient condition for the definition of the convolution is replaced by the condition that $\operatorname{supp}(f)$ and $\operatorname{supp}(g)$ are both contained in the same type of "orthant", for example in the "first quadrant" (or any translate thereof) in two variables.

Let D be any differential operator with constant coefficients. Define D^* to be the formal adjoint of D obtained by integration by parts and ignoring the boundary terms. So, for example, if D is a homogeneous differential operator of order k, then $D^* = (-1)^k D$. For instance $\Delta^* = \Delta$ for the Laplacian. The operator D is then defined on generalized functions by

$$\langle Df, \phi \rangle := \langle f, D^* \phi \rangle.$$

14. Show that

$$D(f \star g) = (Df) \star g = f \star Dg$$

whenever $f \star g$ is defined. In particular, in three dimensions, if f is any generalized function then

$$u = \left(\frac{c}{r}\right) \star f$$

is a solution to

$$\Delta u = f$$

where (I hope) $c = -\sqrt{\frac{\pi}{2}}$.

9 The wave equation in one dimension.

The convolution, where it is defined, is a continuous function of its variable. For example if f_t is a family of generalized functions (say of compact support) which depend differentiably on a parameter t, then $f_t \star g$ depends differentiably on t and we have

$$\frac{d}{dt}(f_t \star g) = \frac{d}{dt} f_t \star g.$$

For example, consider the function $F = \frac{1}{2}\mathbf{1}_{\mathbf{C}}$ where $\mathbf{C}$ is the "forward cone $|x| \le t$ in the (x,t) plane. Let $F_t(x) = F(x,t)$. So

$$F_t(x) = \frac{1}{2} \quad \text{if } |x| \le t$$

and equals zero otherwise. For each fixed t this is a function of x which we can consider as a generalized function of x, and hence apply $\frac{d}{dx}$ to it. By abuse of language we will denote this operation by $\frac{\partial}{\partial x}F$. On the other hand, we can differentiate F_t with respect to t to get a generalized function again depending on t. By abuse of language we will denote this operation by $\frac{\partial}{\partial t}$.

15. Verify that

$$\begin{aligned}
\frac{\partial F}{\partial x} &= \frac{1}{2}\delta(x+t) - \frac{1}{2}\delta(x-t) \\
\frac{\partial^2 F}{\partial x^2} &= \frac{1}{2}\delta'(x+t) - \frac{1}{2}\delta'(x-t) \\
\frac{\partial F}{\partial t} &= \frac{1}{2}\delta(x+t) + \frac{1}{2}\delta(x-t) \\
\frac{\partial^2 F}{\partial t^2} &= \frac{1}{2}\delta'(x+t) - \frac{1}{2}\delta'(x-t).
\end{aligned}$$

From the second and fourth equation we get

$$\frac{\partial^2 F}{\partial t^2} = \frac{\partial F}{\partial x^2}$$

which says that F is a solution of the wave equation. From its definition we have $F_t \to 0$ as $t \to 0$ in the sense of generalized functions. From the third equation we get

$$\lim_{t \to 0} \frac{\partial F}{\partial t} = \delta(x).$$

So if we set $E = \sqrt{2\pi}F$ so as to cancel the stupid factor of $1/\sqrt{2\pi}$ which is our convention for convolution, we see that

$$w = E \star v$$

is the solution to the wave equation

$$\frac{\partial^2 w}{\partial t^2} = \frac{\partial^2 w}{\partial x^2}$$

with the initial conditions

$$w(x,0)=0, \qquad \frac{\partial w}{\partial t}(x,0)=v.$$

This is true for any generalized function v. If v is actually a locally integrable function then this reads

$$w=\int_{-\infty}^{\infty}F(s,t)v(x-s)ds=\frac{1}{2}\int_{-t}^{t}v(x-s)ds$$

or

$$w(x,t)=\frac{1}{2}\int_{x-t}^{x+t}v(r)dr.$$

For the sake of completeness we should record here how D'Alembert used this formula to find the solution of the wave equation with general initial conditions

$$u(x,0)=u_0(x), \qquad \frac{\partial u}{\partial t}(x,0)=u_1(x).$$

First solve the wave equation with the initial conditions $w(x,0)=0$ and $\frac{\partial w}{\partial}(x,0)=u_0(x)$ so

$$w(x,t)=\frac{1}{2}\int_{x-t}^{x+t}u_0(r)dr.$$

The derivative with respect to t of any solution of the wave equation is again a solution of the wave equation. So

$$f(x.t):=\frac{\partial w}{\partial t}(x,t)=\frac{1}{2}[u_0(x+t)+u_0(x-t)]$$

is a solution of the wave equation with initial conditions

$$f(x,0)=u_0(x), \qquad \frac{\partial f}{\partial t}(x,0)=\frac{1}{2}[u_0'(x)-u_0'(x)]=0.$$

So subtracting f and applying the preceding result gives D'Alembert's formula

$$u(x,t)=\frac{1}{2}[u_0(x+t)+u_0(x-t)]+\frac{1}{2}\int_{x-t}^{x+t}u_1(r)dr.$$