

# Banach Algebras

Math 212a

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## Contents

<b>1</b>	<b>Maximal ideals.</b>	<b>2</b>
1.1	Existence. . . . .	2
1.2	The maximal spectrum of a ring. . . . .	3
1.3	Maximal ideals in a commutative algebra. . . . .	4
1.4	Maximal ideals in the ring of continuous functions. . . . .	4
<b>2</b>	<b>Normed algebras.</b>	<b>6</b>
<b>3</b>	<b>The Gelfand representation.</b>	<b>7</b>
3.1	Invertible elements in a Banach algebra form an open set. . . . .	8
3.1.1	The Gelfand-Mazur theorem. . . . .	10
3.2	The Gelfand representation for commutative Banach algebras. . . . .	11
3.3	The spectral radius. . . . .	11
3.4	The generalized Wiener theorem. . . . .	13
<b>4</b>	<b>Self-adjoint algebras.</b>	<b>14</b>
4.1	An important generalization. . . . .	17
4.2	An important application. . . . .	18
<b>5</b>	<b>The Spectral Theorem for Bounded Normal Operators, Functional Calculus Form.</b>	<b>19</b>
5.1	Statement of the theorem. . . . .	20
5.2	$\text{Spec}_B(T) = \text{Spec}_A(T)$ . . . . .	21
5.3	A direct proof of the spectral theorem. . . . .	23

In what follows, all rings will be assumed to be associative and to have an identity element, usually denoted by  $e$ . If an element  $x$  in the ring is such that  $(e - x)$  has a right inverse, then we may write this inverse as  $(e - y)$ , and the equation

$$(e - x)(e - y) = e$$

expands out to

$$x + y - xy = 0.$$

Following Loomis, we call  $y$  the right **adverse** of  $x$  and  $x$  the left adverse of  $y$ . Loomis introduces this term because he wants to consider algebras without identity elements. But it will be convenient to use it even under our assumption that all our algebras have an identity. If an element has both a right and left inverse then they must be equal by the associative law, so if  $x$  has a right and left adverse these must be equal. When we say that an element has (or does not have) an inverse, we will mean that it has (or does not have) a two sided inverse. Similarly for adverse.

All algebras will be over the complex numbers. The **spectrum** of an element  $x$  in an algebra is the set of all  $\lambda \in \mathbf{C}$  such that  $(x - \lambda e)$  has no inverse. We denote the spectrum of  $x$  by  $\text{Spec}(x)$ .

**Proposition 0.1** *If  $P$  is a polynomial then*

$$P(\text{Spec}(x)) = \text{Spec}(P(x)). \quad (1)$$

**Proof.** The product of invertible elements is invertible. For any  $\lambda \in \mathbf{C}$  write  $P(t) - \lambda$  as a product of linear factors:

$$P(t) - \lambda = c \prod (t - \mu_i).$$

Thus

$$P(x) - \lambda e = c \prod (x - \mu_i e)$$

in  $A$  and hence  $(P(x) - \lambda e)^{-1}$  fails to exist if and only if  $(x - \mu_i e)^{-1}$  fails to exist for some  $i$ , i.e.  $\mu_i \in \text{Spec}(x)$ . But these  $\mu_i$  are precisely the solutions of

$$P(\mu) = \lambda.$$

Thus  $\lambda \in \text{Spec}(P(x))$  if and only if  $\lambda = P(\mu)$  for some  $\mu \in \text{Spec}(x)$  which is precisely the assertion of the proposition. QED

## 1 Maximal ideals.

### 1.1 Existence.

**Theorem 1.1** *Every proper right ideal in a ring is contained in a maximal proper right ideal. Similarly for left ideals. Also any proper two sided ideal is contained in a maximal proper two sided ideal.*

**Proof by Zorn's lemma.** The proof is the same in all three cases: Let  $I$  be the ideal in question (right left or two sided) and  $\mathcal{F}$  be the set of all proper ideals (of the appropriate type) containing  $I$  ordered by inclusion. Since  $e$  does not belong to any proper ideal, the union of any linearly ordered family of proper ideals is again proper, and so has an upper bound. Now Zorn guarantees the existence of a maximal element. QED

## 1.2 The maximal spectrum of a ring.

For any ring  $R$  we let  $\text{Mspec}(R)$  denote the set of maximal (proper) two sided ideals of  $R$ . For any two sided ideal  $I$  we let

$$\text{Supp}(I) := \{M \in \text{Mspec}(R) : I \subset M\}.$$

Notice that

$$\text{Supp}(\{0\}) = \text{Mspec}(R)$$

and

$$\text{Supp}(R) = \emptyset.$$

For any family  $I_\alpha$  of two sided ideals, a maximal ideal contains all of the  $I_\alpha$  if and only if it contains the two sided ideal  $\sum_\alpha I_\alpha$ . In symbols

$$\bigcap_\alpha \text{Supp}(I_\alpha) = \text{Supp}\left(\sum_\alpha I_\alpha\right).$$

Thus the intersection of any collection of sets of the form  $\text{Supp}(I)$  is again of this form. Notice also that if

$$A = \text{Supp}(I)$$

then

$$A = \text{Supp}(J) \quad \text{where } J = \bigcap_{M \in A} M.$$

(Here  $I \subset J$ , but  $J$  might be a strictly larger ideal.) We claim that

$$\begin{aligned} A = \text{Supp}\left(\bigcap_{M \in A} M\right) \quad \text{and} \quad B = \text{Supp}\left(\bigcap_{M \in B} M\right) \\ \Rightarrow A \cup B = \text{Supp}\left(\bigcap_{M \in A \cup B} M\right). \end{aligned} \tag{2}$$

Indeed, if  $N$  is a maximal ideal belonging to  $A \cup B$  then it contains the intersection on the right hand side of (2) so the left hand side contains the right. We must show the reverse inclusion. So suppose the contrary. This means that there is a maximal ideal  $N$  which contains the intersection on the right but does not belong to either  $A$  or  $B$ . Since  $N$  does not belong to  $A$ , the ideal  $J(A) := \bigcap_{M \in A} M$  is not contained in  $N$ , so  $J(A) + N = R$ , and hence there exist  $a \in J(A)$  and  $m \in N$  such that  $a + m = e$ . Similarly, there exist  $b \in J(B)$  and  $n \in N$  such that  $b + n = e$ . But then

$$e = e^2 = (a + m)(b + n) = ab + an + mb + mn.$$

Each of the last three terms on the right belong to  $N$  since it is a two sided ideal, and so does  $ab$  since

$$ab \in \left( \bigcap_{M \in A} M \right) \cap \left( \bigcap_{M \in B} M \right) = \left( \bigcap_{M \in A \cup B} M \right) \subset N.$$

Thus  $e \in N$  which is a contradiction.

The above facts show that the sets of the form  $\text{Supp}(I)$  give the closed sets of a topology.

If  $A \subset \text{Mspec}(R)$  is an arbitrary subset, its closure is given by

$$\overline{A} = \text{Supp} \left( \bigcap_{M \in A} M \right).$$

(For the case of commutative rings, a major advance was to replace maximal ideals by prime ideals in the preceding construction - giving rise to the notion of  $\text{Spec}(R)$  - the prime spectrum of a commutative ring. But the motivation for this development in commutative algebra came from these constructions in the theory of Banach algebras.)

### 1.3 Maximal ideals in a commutative algebra.

**Proposition 1.1** *An ideal  $M$  in a commutative algebra is maximal if and only if  $R/M$  is a field.*

**Proof.** If  $J$  is an ideal in  $R/M$ , its inverse image under the projection  $R \rightarrow R/M$  is an ideal in  $R$ . If  $J$  is proper, so is this inverse image. Thus  $M$  is maximal if and only if  $F := R/M$  has no ideals other than 0 and  $F$ . Thus if  $0 \neq X \in F$ , the set of all multiples of  $X$  must be all of  $F$  if  $M$  is maximal. In particular every non-zero element has an inverse. Conversely, if every non-zero element of  $F$  has an inverse, then  $F$  has no proper ideals. QED

### 1.4 Maximal ideals in the ring of continuous functions.

Let  $S$  be a compact Hausdorff space, and let  $\mathcal{C}(S)$  denote the ring of continuous complex valued functions on  $S$ . For each  $p \in S$ , the map of  $\mathcal{C}(S) \rightarrow \mathbf{C}$  given by

$$f \mapsto f(p)$$

is a surjective homomorphism. The kernel of this map consists of all  $f$  which vanish at  $p$ . By the preceding proposition, this is then a maximal ideal, which we shall denote by  $M_p$ .

**Theorem 1.2** *If  $I$  is a proper ideal of  $\mathcal{C}(S)$ , then there is a point  $p \in S$  such that*

$$I \subset M_p.$$

*In particular every maximal ideal in  $\mathcal{C}(S)$  is of the form  $M_p$  so we may identify  $\text{Mspec}(\mathcal{C}(S))$  with  $S$  as a set. This identification is a homeomorphism between the original topology of  $S$  and the topology given above on  $\text{Mspec}(\mathcal{C}(S))$ .*

**Proof.** Suppose that for every  $p \in S$  there is an  $f \in I$  such that  $f(p) \neq 0$ . Then  $|f|^2 = f\bar{f} \in I$  and  $|f(p)|^2 > 0$  and  $|f|^2 \geq 0$  everywhere. Thus each point of  $S$  is contained in a neighborhood  $U$  for which there exists a  $g \in I$  with  $g \geq 0$  everywhere, and  $g > 0$  on  $U$ . Since  $S$  is compact, we can cover  $S$  with finitely many such neighborhoods. If we take  $h$  to be the sum of the corresponding  $g$ 's, then  $h \in I$  and  $h > 0$  everywhere. So  $h^{-1} \in \mathcal{C}(S)$  and  $e = 1 = hh^{-1} \in I$  so  $I = \mathcal{C}(S)$ , a contradiction. This proves the first part of the the theorem.

To prove the last statement, we must show that the closure of any subset  $A \subset S$  in the original topology coincides with its closure in the topology derived from the maximal ideal structure. That is, we must show that

$$\text{closure of } A \text{ in the topology of } S = \text{Supp} \left( \bigcap_{M \in A} M \right).$$

Now

$$\bigcap_{M \in A} M$$

consists exactly of all continuous functions which vanish at all points of  $A$ . Any such function must vanish on the closure of  $A$  in the topology of  $S$ . So the left hand side of the above equation is contained in the right hand side. We must show the reverse inclusion. Suppose  $p \in S$  does not belong to the closure of  $A$  in the topology of  $S$ . Then Urysohn's Lemma asserts that there is an  $f \in \mathcal{C}(S)$  which vanishes on  $A$  and  $f(p) \neq 0$ . Thus  $p \notin \text{Supp}(\bigcap_{M \in A} M)$ . QED

**Theorem 1.3** *Let  $I$  be an ideal in  $\mathcal{C}(S)$  which is closed in the uniform topology on  $\mathcal{C}(S)$ . Then*

$$I = \bigcap_{M \in \text{Supp}(I)} M.$$

**Proof.**  $\text{Supp}(I)$  consists of all points  $p$  such that  $f(p) = 0$  for all  $f \in I$ . Since  $f$  is continuous, the set of zeros of  $f$  is closed, and hence  $\text{Supp}(I)$  being the intersection of such sets is closed. Let  $O$  be the complement of  $\text{Supp}(I)$  in  $S$ . Then  $O$  is a locally compact space, and the elements of  $\bigcap_{M \in \text{Supp}(I)} M$  when restricted to  $O$  consist of all functions which vanish at infinity.  $I$ , when restricted to  $O$  is a uniformly closed subalgebra of this algebra. If we could show that the elements of  $I$  separate points in  $O$  then the Stone-Weierstrass theorem would tell us that  $I$  consists of all continuous functions on  $O$  which "vanish at infinity", i.e. all continuous functions which vanish on  $\text{Supp}(I)$ , which is the assertion of the theorem. So let  $p$  and  $q$  be distinct points of  $O$ , and let  $f \in \mathcal{C}(S)$  vanish on  $\text{Supp}(I)$  and at  $q$  with  $f(p) = 1$ . Such a function exists by Urysohn's Lemma, again. Let  $g \in I$  be such that  $g(p) \neq 0$ . Such a  $g$  exists by the definition of  $\text{Supp}(I)$ . Then  $gf \in I$ ,  $(gf)(q) = 0$ , and  $(gf)(p) \neq 0$ . QED

## 2 Normed algebras.

A **normed algebra** is an algebra (over the complex numbers) which has a norm as a vector space which satisfies

$$\|xy\| \leq \|x\|\|y\|. \quad (3)$$

Since  $e = ee$  this implies that

$$\|e\| \leq \|e\|^2$$

so

$$\|e\| \geq 1.$$

Consider the new norm

$$\|y\|_N := \text{lub}_{\|x\| \neq 0} \|yx\|/\|x\|.$$

This still satisfies (3) and from (3) we have

$$\|y\|_N \leq \|y\|.$$

Under this norm we have

$$\|e\|_N = 1.$$

On the other hand, from its definition

$$\|y\|/\|e\| \leq \|y\|_N.$$

Combining this with the previous inequality gives

$$\|y\|/\|e\| \leq \|y\|_N \leq \|y\|.$$

In other words the norms  $\| \cdot \|$  and  $\| \cdot \|_N$  are equivalent. So with no loss of generality we can add the requirement

$$\|e\| = 1 \quad (4)$$

to our axioms for a normed algebra.

Suppose we weaken our condition and allow  $\| \cdot \|$  to be only a pseudo-norm. this means that we allow the possible existence of non-zero elements  $x$  with  $\|x\| = 0$ . Then (3) implies that the set of all such elements is an ideal, call it  $I$ . Then  $\| \cdot \|$  descends to  $A/I$ . Furthermore, any continuous (i.e. bounded) linear function must vanish on  $I$  so also descends to  $A/I$  with no change in norm. In other words,  $A^*$  can be identified with  $(A/I)^*$ .

If  $A$  is a normed algebra which is complete (i.e.  $A$  is a Banach space as a normed space) then we say that  $A$  is a **Banach algebra**.

### 3 The Gelfand representation.

Let  $A$  be a normed vector space. The space  $A^*$  of continuous linear functions on  $A$  becomes a normed vector space under the norm

$$\|\ell\| := \sup_{\|x\| \neq 0} |\ell(x)|/\|x\|.$$

Each  $x \in A$  defines a linear function on  $A^*$  by

$$x(\ell) := \ell(x)$$

and

$$|x(\ell)| \leq \|\ell\|\|x\|$$

so  $x$  is a continuous function of  $\ell$  (relative to the norm introduced above on  $A^*$ ).

Let  $B = B_1(A^*)$  denote the unit ball in  $A^*$ . In other words  $B = \{\ell : \|\ell\| \leq 1\}$ . The functions  $x(\cdot)$  on  $B$  induce a topology on  $B$  called the **weak topology**.

**Proposition 3.1**  *$B$  is compact under the weak topology.*

**Proof.** For each  $x \in A$ , the values assumed by the set of  $\ell \in B$  at  $x$  lie in the closed disk  $D_{\|x\|}$  of radius  $\|x\|$  in  $\mathbf{C}$ . Thus

$$B \subset \prod_{x \in A} D_{\|x\|}$$

which is compact by Tychonoff's theorem - being the product of compact spaces. To prove that  $B$  is compact, it is sufficient to show that  $B$  is a closed subset of this product space. Suppose that  $f$  is in the closure of  $B$ . For any  $x$  and  $y$  in  $A$  and any  $\epsilon > 0$ , we can find an  $\ell \in B$  such that

$$|f(x) - \ell(x)| < \epsilon, \quad |f(y) - \ell(y)| < \epsilon, \quad \text{and} \quad |f(x+y) - \ell(x+y)| < \epsilon.$$

Since  $\ell(x+y) = \ell(x) + \ell(y)$  this implies that

$$|f(x+y) - f(x) - f(y)| < 3\epsilon.$$

Since  $\epsilon$  is arbitrary, we conclude that

$$f(x+y) = f(x) + f(y).$$

Similarly,  $f(\lambda x) = \lambda f(x)$ . In other words,  $f \in B$ . QED

Let  $\Delta \subset A^*$  denote the set of all continuous homomorphisms of  $A$  onto the complex numbers. In other words, in addition to being linear, we demand of  $h \in \Delta$  that

$$h(xy) = h(x)h(y) \quad \text{and} \quad h(e) = 1.$$

Let  $E := h^{-1}(1)$ . Then  $E$  is closed under multiplication. In particular, if  $x \in E$  we can not have  $\|x\| < 1$  for otherwise  $x^n$  is a sequence of elements in  $E$  tending to 0, and so by the continuity of  $h$  we would have  $h(0) = 1$  which

is impossible. So  $\|x\| \geq 1$  for all  $x \in E$ . If  $y$  is such that  $h(y) = \lambda \neq 0$ , then  $x := y/\lambda \in E$  so

$$\|h(y)\| \leq \|y\|,$$

and this clearly also holds if  $h(y) = 0$ . In other words,

$$\Delta \subset B.$$

Since the conditions for being a homomorphism will hold for any weak limit of homomorphisms (the same proof as given above for the compactness of  $B$ ), we conclude that  $\Delta$  is compact.

Once again we can turn the tables and think of  $y \in A$  as a function  $\hat{y}$  on  $\Delta$  via

$$\hat{y}(h) := h(y).$$

This map from  $A$  into an algebra of functions on  $\Delta$  is called the **Gelfand representation**.

The inequality  $h(y) \leq \|y\|$  for all  $h$  translates into

$$\|\hat{y}\|_\infty \leq \|y\|. \tag{5}$$

Putting it all together we get

**Theorem 3.1**  *$\Delta$  is a compact subset of  $A^*$  and the Gelfand representation  $y \mapsto \hat{y}$  is a norm decreasing homomorphism of  $A$  onto a subalgebra  $\hat{A}$  of  $\mathcal{C}(\Delta)$ .*

The above theorem is true for any normed algebra - we have not used any completeness condition. For Banach algebras, i.e. complete normed algebras, we can proceed further and relate  $\Delta$  to  $\text{Mspec}(A)$ . Recall that an element of  $\text{Mspec}(A)$  corresponds to a homomorphism of  $A$  onto some field. In the commutative Banach algebra case we will show that this field is  $\mathbf{C}$  and that any such homomorphism is automatically continuous. So for commutative Banach algebras we can identify  $\Delta$  with  $\text{Mspec}(A)$ .

### 3.1 Invertible elements in a Banach algebra form an open set.

In this section  $A$  will be a Banach algebra.

**Proposition 3.2** *If  $\|x\| < 1$  then  $x$  has an adverse  $x'$  given by*

$$x' = - \sum_{n=1}^{\infty} x^n$$

*so that  $e - x$  has an inverse given by*

$$e - x' = e + \sum_{n=1}^{\infty} x^n.$$

*Both are continuous functions of  $x$*

**Proof.** Let

$$s_n := - \sum_1^n x^i.$$

Then if  $m < n$

$$\|s_m - s_n\| \leq \sum_{m+1}^n \|x\|^i < \|x\|^m \frac{1}{1 - \|x\|} \rightarrow 0$$

as  $m \rightarrow \infty$ . Thus  $s_n$  is a Cauchy sequence and

$$x + s_n - xs_n = x^{n+1} \rightarrow 0.$$

Thus the series  $-\sum_1^\infty x^i$  as stated in the theorem converges and gives the adverse of  $x$  and is continuous function of  $x$ . The corresponding statements for  $(e - x)^{-1}$  now follow. QED

The proof shows that the adverse  $x'$  of  $x$  satisfies

$$\|x'\| \leq \frac{\|x\|}{1 - \|x\|}. \quad (6)$$

**Theorem 3.2** *Let  $y$  be an invertible element of  $A$  and set*

$$a := \frac{1}{\|y^{-1}\|}.$$

*Then  $y + x$  is invertible whenever*

$$\|x\| < a.$$

*Furthermore*

$$\|(x + y)^{-1} - y^{-1}\| \leq \frac{\|x\|}{(a - \|x\|)a}. \quad (7)$$

*Thus the set of elements having inverses is open and the map  $x \rightarrow x^{-1}$  is continuous on its domain of definition.*

**Proof.** If  $\|x\| < \|y^{-1}\|^{-1}$  then

$$\|y^{-1}x\| \leq \|y^{-1}\|\|x\| < 1.$$

Hence  $e + y^{-1}x$  has an inverse by the previous proposition. Hence  $y + x = y(e + y^{-1}x)$  has an inverse. Also

$$(y + x)^{-1} - y^{-1} = ((e + y^{-1}x)^{-1} - e) y^{-1} = -(-y^{-1}x)' y^{-1}$$

where  $(-y^{-1}x)'$  is the adverse of  $-y^{-1}x$ .

From (6) and the above expression for  $(x + y)^{-1} - y^{-1}$  we see that

$$\|(x + y)^{-1} - y^{-1}\| \leq \|(-y^{-1}x)'\| \|y^{-1}\| \leq \frac{\|x\| \|y^{-1}\|^2}{1 - \|x\| \|y^{-1}\|} = \frac{\|x\|}{a(a - \|x\|)}.$$

QED

**Proposition 3.3** *If  $I$  is a proper ideal then  $\|e - x\| \geq 1$  for all  $x \in I$ .*

**Proof.** Otherwise there would be some  $x \in I$  such that  $e - x$  has an adverse, i.e.  $x$  has an inverse which contradicts the hypothesis that  $I$  is proper.

**Proposition 3.4** *The closure of a proper ideal is proper. In particular, every maximal ideal is closed.*

**Proof.** The closure of an ideal  $I$  is clearly an ideal, and all elements in the closure still satisfy  $\|e - x\| \geq 1$  and so the closure is proper. QED

**Proposition 3.5** *If  $I$  is a closed ideal in  $A$  then  $A/I$  is again a Banach algebra.*

**Proof.** The quotient of a Banach space by a closed subspace is again a Banach space. The norm on  $A/I$  is given by

$$\|X\| = \min_{x \in X} \|x\|$$

where  $X$  is a coset of  $I$  in  $A$ . The product of two cosets  $X$  and  $Y$  is the coset containing  $xy$  for any  $x \in X$ ,  $y \in Y$ . Thus

$$\|XY\| = \min_{x \in X, y \in Y} \|xy\| \leq \min_{x \in X, y \in Y} \|x\| \|y\| = \|X\| \|Y\|.$$

Also, if  $E$  is the coset containing  $e$  then  $E$  is the identity element for  $A/I$  and so

$$\|E\| \leq 1.$$

But we know that this implies that  $\|E\| = 1$ . QED

Suppose that  $A$  is commutative and  $M$  is a maximal ideal of  $A$ . We know that  $A/M$  is a field, and the preceding proposition implies that  $A/M$  is a normed field containing the complex numbers. The following famous result implies that  $A/M$  is in fact norm isomorphic to  $\mathbf{C}$ . It deserves a subsection of its own:

### 3.1.1 The Gelfand-Mazur theorem.

A division algebra is a (possibly not commutative) algebra in which every non-zero element has an inverse.

**Theorem 3.3** *Every normed division algebra over the complex numbers is isometrically isomorphic to the field of complex numbers.*

Let  $A$  be the normed division algebra and  $x \in A$ . We must show that  $x = \lambda e$  for some complex number  $\lambda$ . Suppose not. Then by the definition of a division algebra,  $(x - \lambda e)^{-1}$  exists for all  $\lambda \in \mathbf{C}$  and all these elements commute. Thus

$$(x - (\lambda + h)e)^{-1} - (x - \lambda e)^{-1} = h(x - (\lambda + h)e)^{-1}(x - \lambda e)^{-1}$$

as can be checked by multiplying both sides of this equation on the right by  $x - \lambda e$  and on the left by  $x - (\lambda + h)e$ . Thus the strong derivative of the function

$$\lambda \mapsto (x - \lambda e)^{-1}$$

exists and is given by the usual formula  $(x - \lambda e)^{-2}$ . In particular, for any  $\ell \in A^*$  the function

$$\lambda \mapsto \ell((x - \lambda e)^{-1})$$

is analytic on the entire complex plane. On the other hand for  $\lambda \neq 0$  we have

$$(x - \lambda e)^{-1} = \lambda^{-1} \left( \frac{1}{\lambda} x - e \right)^{-1}$$

and this approaches zero as  $\lambda \rightarrow \infty$ . Hence for any  $\ell \in A^*$  the function  $\lambda \mapsto \ell((x - \lambda e)^{-1})$  is an everywhere analytic function which vanishes at infinity, and hence is identically zero by Liouville's theorem. But this implies that  $(x - \lambda e)^{-1} \equiv 0$  by the Hahn Banach theorem, a contradiction. QED

### 3.2 The Gelfand representation for commutative Banach algebras.

Let  $A$  be a commutative Banach algebra. We know that every maximal ideal is the kernel of a homomorphism  $h$  of  $A$  onto the complex numbers. Conversely, suppose that  $h$  is such a homomorphism. We claim that

$$|h(x)| \leq \|x\|$$

for any  $x \in A$ . Indeed, suppose that  $|h(x)| > \|x\|$  for some  $x$ . Then

$$\|x/h(x)\| < 1$$

so  $e - x/h(x)$  is invertible; in particular  $h(e - x/h(x)) \neq 0$  which implies that  $1 = h(e) \neq h(x)/h(x)$ , a contradiction.

In short, we can identify  $\text{Mspec}(A)$  with  $\Delta$  and the map  $x \mapsto \hat{x}$  is a norm decreasing map of  $A$  onto a subalgebra  $\hat{A}$  of  $\mathcal{C}(\text{Mspec}(A))$  where we use the uniform norm  $\|\cdot\|_\infty$  on  $\mathcal{C}(\text{Mspec}(A))$ . A complex number is in the spectrum of an  $x \in A$  if and only if  $(x - \lambda e)$  belongs to some maximal ideal  $M$ , in which case  $\hat{x}(M) = \lambda$ . Thus

$$\|\hat{x}\|_\infty = \text{l.u.b. } \{|\lambda| : \lambda \in \text{Spec}(x)\}. \quad (8)$$

### 3.3 The spectral radius.

The right hand side of (8) makes sense in any algebra, and is called the **spectral radius** of  $x$  and is denoted by  $|x|_{sp}$ . We claim that

**Theorem 3.4** *In any Banach algebra we have*

$$|x|_{sp} = \lim_{n \rightarrow \infty} \|x^n\|^{\frac{1}{n}}. \quad (9)$$

**Proof.** If  $|\lambda| > \|x\|$  then  $e - x/\lambda$  is invertible, and therefore so is  $x - \lambda e$  so  $\lambda \notin \text{Spec}(x)$ . Thus

$$|x|_{sp} \leq \|x\|.$$

We know from (1) that  $\lambda \in \text{Spec}(x) \Rightarrow \lambda^n \in \text{Spec}(x^n)$ , so the previous inequality applied to  $x^n$  gives

$$|x|_{sp} \leq \|x^n\|^{\frac{1}{n}}$$

and so

$$|x|_{sp} \leq \liminf \|x^n\|^{\frac{1}{n}}.$$

We must prove the reverse inequality with  $\limsup$ . Suppose that  $|\mu| < 1/|x|_{sp}$  so that  $\mu := 1/\lambda$  satisfies  $|\lambda| > |x|_{sp}$  and hence  $e - \mu x$  is invertible. The formula for the adverse gives

$$(\mu x)' = - \sum_1^{\infty} (\mu x)^n$$

where we know that this converges in the open disk of radius  $1/\|\mu x\|$ . However, we know that  $(e - \mu x)^{-1}$  exists for  $|\mu| < 1/|x|_{sp}$ . In particular, for any  $\ell \in A^*$  the function  $\lambda \mapsto \ell((\mu x)')$  is analytic and hence its Taylor series

$$- \sum \ell(x^n) \mu^n$$

converges on this disk. Here we use the fact that the Taylor series of a function of a complex variable converges on any disk contained in the region where it is analytic. Thus

$$|\ell(\mu^n x^n)| \rightarrow 0$$

for each fixed  $\ell \in A^*$  if  $|\mu| < 1/|x|_{sp}$ . Considered as a family of linear functions of  $\ell$ , we see that

$$\ell \mapsto \ell(\mu^n x^n)$$

is bounded for each fixed  $\ell$ , and hence by the *uniform boundedness principle*, there exists a constant  $K$  such that

$$\|\mu^n x^n\| < K$$

for each  $\mu$  in this disk, in other words

$$\|x^n\|^{\frac{1}{n}} \leq K^{\frac{1}{n}} (1/|\mu|)$$

so

$$\limsup \|x^n\|^{\frac{1}{n}} \leq 1/|\mu| \text{ if } 1/|\mu| > |x|_{sp}.$$

QED

In a commutative Banach algebra we can combine (9) with (8) to conclude that

$$\|\hat{x}\|_{\infty} = \lim_{n \rightarrow \infty} \|x^n\|^{\frac{1}{n}}. \tag{10}$$

We say that  $x$  is a **generalized nilpotent element** if  $\lim \|x^n\|^{\frac{1}{n}} = 0$ . From (9) we see that  $x$  is a generalized nilpotent element if and only if  $\hat{x} \equiv 0$ . This means that  $x$  belongs to all maximal ideals. The intersection of all maximal ideals is called the **radical** of the algebra. A Banach algebra is called **semi-simple** if its radical consists only of the 0 element.

### 3.4 The generalized Wiener theorem.

**Theorem 3.5** *Let  $A$  be a commutative Banach algebra. Then  $x \in A$  has an inverse if and only if  $\hat{x}$  never vanishes.*

**Proof.** If  $xy = e$  then  $\hat{x}\hat{y} \equiv 1$ . So if  $x$  has an inverse, then  $\hat{x}$  can not vanish anywhere. Conversely, suppose  $x$  does not have an inverse. Then  $Ax$  is a proper ideal. So  $x$  is contained in some maximal ideal  $M$  (by Zorn's lemma). So  $\hat{x}(M) = 0$ .

**Example.** Let  $G$  be a countable commutative group given the discrete topology. Then we may choose its Haar measure to be the counting measure. Thus  $L^1(G)$  consists of all complex valued functions on  $G$  which are absolutely summable, i.e. such that

$$\sum_{a \in G} |f(a)| < \infty.$$

Recall that  $L^1(G)$  is a Banach algebra under convolution:

$$f \star g(x) := \sum_{y \in G} f(y^{-1}x)g(y).$$

We repeat the proof: Since  $L^1(G) \subset L^2(G)$  this sum converges and

$$\begin{aligned} \sum_{x \in G} |(f \star g)(x)| &\leq \sum_{x, y \in G} |f(xy^{-1})| \cdot |g(y)| \\ &= \sum_{y \in G} |g(y)| \sum_{x \in G} |f(xy^{-1})| \\ &= \sum_{y \in G} |g(y)| \left( \sum_{w \in G} |f(w)| \right) \quad \text{i.e.} \\ \|f \star g\| &\leq \|f\| \|g\|. \end{aligned}$$

If  $\delta_x \in L^1(G)$  is defined by

$$\delta_x(t) = \begin{cases} 1 & \text{if } t = x \\ 0 & \text{otherwise} \end{cases}$$

then

$$\delta_x \star \delta_y = \delta_{xy}.$$

We know that the most general continuous linear function on  $L^1(G)$  is obtained from multiplication by an element of  $L^\infty(G)$  and then integrating = summing. That is it is given by

$$f \mapsto \sum_{x \in G} f(x)\rho(x)$$

where  $\rho$  is some bounded function. Under this linear function we have

$$\delta_x \mapsto \rho(x)$$

and so, if this linear function is to be multiplicative, we must have

$$\rho(xy) = \rho(x)\rho(y).$$

Since  $\rho(x^n) = \rho(x)^n$  and  $|\rho(x)|$  is to be bounded, we must have  $|\rho(x)| \equiv 1$ .

A function  $\rho$  satisfying these two conditions:

$$\rho(xy) = \rho(x)\rho(y) \quad \text{and} \quad |\rho(x)| \equiv 1$$

is called a **character** of the commutative group  $G$ . The space of characters is itself a group, denoted by  $\hat{G}$ .

We have shown that  $\text{Mspec}(L^1(G)) = \hat{G}$ . In particular we have a topology on  $\hat{G}$  and the Gelfand transform  $f \mapsto \hat{f}$  sends every element of  $L^1(G)$  to a continuous function on  $\hat{G}$ .

For example, if  $G = \mathbf{Z}$  under addition, the condition to be a character says that

$$\rho(m+n) = \rho(m)\rho(n), \quad |\rho| \equiv 1.$$

So

$$\rho(n) = \rho(1)^n$$

where

$$\rho(1) = e^{i\theta}$$

for some  $\theta \in \mathbf{R}/(2\pi\mathbf{Z})$ . Thus

$$\hat{f}(\theta) = \sum_{n \in \mathbf{Z}} f(n)e^{in\theta}$$

is just the Fourier series with coefficients  $f(n)$ . The image of the Gelfand transform is just the set of Fourier series which converge absolutely. We conclude from Theorem 3.5 that if  $F$  is an absolutely convergent Fourier series which vanishes nowhere, then  $1/F$  has an absolutely convergent Fourier series. Before Gelfand, this was a deep theorem of Wiener.

To deal with the version of this theorem which treats the Fourier transform rather than Fourier series, we would have to consider algebras which do not have an identity element. Most of what we did goes through with only mild modifications, but I do not go into this, as my goals are elsewhere.

## 4 Self-adjoint algebras.

Let  $A$  be a semi-simple commutative Banach algebra. Since “semi-simple” means that the radical is  $\{0\}$ , we know that the Gelfand transform is injective.  $A$  is called **self adjoint** if for every  $x \in A$  there exists an  $x^* \in A$  such that

$$(x^*)^\wedge = \overline{\hat{x}}.$$

By the injectivity of the Gelfand transform, the element  $x^*$  is uniquely specified by this equation.

In general, for any Banach algebra, a map  $f \mapsto f^\dagger$  is called an **involution anti-automorphism** if

- $(fg)^\dagger = g^\dagger f^\dagger$
- $(f + g)^\dagger = f^\dagger + g^\dagger$
- $(\lambda f)^\dagger = \overline{\lambda} f^\dagger$  and
- $(f^\dagger)^\dagger = f$ .

For example, if  $A$  is the algebra of bounded operators on a Hilbert space, then the map  $T \mapsto T^*$  sending every operator to its adjoint is an example of an involutory anti-automorphism. Another example is  $L^1(G)$  under convolution, for a locally compact Hausdorff group  $G$  where the involution was the map  $f \mapsto \tilde{f}$ .

If  $A$  is a semi-simple self-adjoint commutative Banach algebra, the map  $x \mapsto x^*$  is an involutory anti-automorphism. It has this further property:

$$f = f^* \Rightarrow 1 + f^2 \text{ is invertible.}$$

indeed, if  $f = f^*$  then  $\hat{f}$  is real valued, so  $1 + \hat{f}^2$  vanishes nowhere, and so  $1 + f^2$  is invertible by Theorem 3.5. Conversely

**Theorem 4.1** *Let  $A$  be a commutative semi-simple Banach algebra with an involutory anti-automorphism  $f \mapsto f^\dagger$  such that  $1 + f^2$  is invertible whenever  $f = f^\dagger$ . Then  $A$  is self-adjoint and  $\dagger = *$ .*

**Proof.** We must show that  $(f^\dagger)^\dagger = \overline{f}$ . We first prove that if we set

$$g := f + f^\dagger$$

then  $\hat{g}$  is real valued. Suppose the contrary, that

$$\hat{g}(M) = a + ib, \quad b \neq 0 \quad \text{for some } M \in \text{Mspec}(A).$$

Now  $g^\dagger = f^\dagger + (f^\dagger)^\dagger = g$  and hence  $(g^2)^\dagger = g^2$  so

$$h := \frac{ag^2 - (a^2 - b^2)g}{b(a^2 + b^2)}$$

satisfies

$$h^\dagger = h.$$

We have

$$h(M) = \frac{a(a + ib)^2 - (a^2 - b^2)(a + ib)}{b(a^2 + b^2)} = i.$$

So

$$1 + h(M)^2 = 0$$

contradicting the hypothesis that  $1 + h^2$  is invertible. Now let us apply this result to  $\frac{1}{2}f$  and to  $\frac{1}{2i}f$ . We have

$$f = g + ih \quad \text{where } g = \frac{1}{2}(f + f^\dagger), \quad h = \frac{1}{2i}(f - f^\dagger)$$

and we know that  $\hat{g}$  and  $\hat{h}$  are real and satisfy  $g^\dagger = g$  and  $h^\dagger = h$ . So

$$\overline{\hat{f}} = \overline{\hat{g} + i\hat{h}} = \hat{g} - i\hat{h} = (f^\dagger)^\wedge.$$

QED

**Theorem 4.2** *Let  $A$  be a commutative Banach algebra with an involutory anti-automorphism  $\dagger$  which satisfies the condition*

$$\|ff^\dagger\| = \|f\|^2 \quad \forall f \in A. \quad (11)$$

*Then the Gelfand transform  $f \mapsto \hat{f}$  is a norm preserving surjective isomorphism which satisfies*

$$(f^\dagger)^\wedge = \overline{\hat{f}}.$$

*In particular,  $A$  is semi-simple and self-adjoint.*

**Proof.**  $\|f\|^2 = \|ff^\dagger\| \leq \|f\|\|f^\dagger\|$  so  $\|f\| \leq \|f^\dagger\|$ . Replacing  $f$  by  $f^\dagger$  gives  $\|f^\dagger\| \leq \|f\|$ . so

$$\|f\| = \|f^\dagger\|$$

and

$$\|ff^\dagger\| = \|f\|^2 = \|f\|\|f^\dagger\|. \quad (12)$$

Now since  $A$  is commutative,

$$ff^\dagger = f^\dagger f$$

and

$$f^2(f^2)^\dagger = f^2(f^\dagger)^2 = (ff^\dagger)(ff^\dagger)^\dagger \quad (13)$$

and so applying (12) to  $f^2$  and then applying it once again to  $f$  we get

$$\|f^2\|\|(f^\dagger)^2\| = \|ff^\dagger(ff^\dagger)^\dagger\| = \|ff^\dagger\|\|ff^\dagger\| = \|f\|^2\|f^\dagger\|^2$$

or

$$\|f^2\|^2 = \|f\|^4.$$

Thus

$$\|f^2\| = \|f\|^2,$$

and therefore

$$\|f^4\| = \|f^2\|^2 = \|f\|^4$$

and by induction

$$\|f^{2^k}\| = \|f\|^{2^k}$$

for all non-negative integers  $k$ .

Hence letting  $n = 2^k$  in the right hand side of (10) we see that  $\|\hat{f}\|_\infty = \|f\|$  so the Gelfand transform is norm preserving, and hence injective. To show that  $\dagger = *$  it is enough to show that if  $f = f^\dagger$  then  $\hat{f}$  is real valued, as in the proof

of the preceding theorem. Suppose not, so  $\hat{f}(M) = a + ib$ ,  $b \neq 0$ . For any real number  $c$  we have

$$(f + ice)^\wedge(M) = a + i(b + c)$$

so

$$\begin{aligned} |(f + ice)^\wedge(M)|^2 &= a^2 + (b + c)^2 \leq \|f + ice\|^2 = \|(f + ice)(f - ice)\| \\ &= \|f^2 + c^2e\| \leq \|f\|^2 + c^2. \end{aligned}$$

This says that

$$a^2 + b^2 + 2bc + c^2 \leq \|f\|^2 + c^2$$

which is impossible if we choose  $c$  so that  $2bc > \|f\|^2$ .

So we have proved that  $\dagger = *$ . Now by definition, if  $f(M) = f(N)$  for all  $f \in A$ , the maximal ideals  $M$  and  $N$  coincide. So the image of elements of  $A$  under the Gelfand transform separate points of  $\text{Mspec}(A)$ . But every  $f \in A$  can be written as

$$f = \frac{1}{2}(f + f^*) + i\frac{1}{2i}(f - f^*)$$

i.e. as a sum  $g + ih$  where  $\hat{g}$  and  $\hat{h}$  are real valued. Hence the real valued functions of the form  $\hat{g}$  separate points of  $\text{Mspec}(A)$ . Hence by the *Stone Weierstrass theorem* we know that the image of the Gelfand transform is dense in  $\mathcal{C}(\text{Mspec}(A))$ . Since  $A$  is complete and the Gelfand transform is norm preserving, we conclude that the Gelfand transform is surjective. QED

#### 4.1 An important generalization.

A Banach algebra with an involution  $\dagger$  such that (11) holds is called a  *$C^*$ -algebra*. Notice that we are *not* assuming that this Banach algebra is commutative. But an element  $x$  of such an algebra is called **normal** if

$$xx^\dagger = x^\dagger x,$$

in other words if  $x$  *does* commute with  $x^\dagger$ . Then we can repeat the argument at the beginning of the proof of Theorem 4.2 to conclude that if  $x$  is a normal element of a  $C^*$  algebra, then

$$\|x^{2^k}\| = \|x\|^{2^k}$$

and hence by (9)

$$|x|_{sp} = \|x\|. \tag{14}$$

An element  $x$  of an algebra with involution is called **self-adjoint** if  $x^\dagger = x$ . In particular, every self-adjoint element is normal.

Again, a rerun of a previous argument shows that if  $x$  is self-adjoint, meaning that  $x^* = x$  then

$$\text{Spec}(x) \subset \mathbf{R} \tag{15}$$

Indeed, suppose that  $a + ib \in \text{Spec}(x)$  with  $b \neq 0$ , and let

$$y := \frac{1}{b}(x - ae)$$

so that  $y = y^*$  and  $i \in \text{Spec}(y)$ . So  $e + iy$  is not invertible. So for any real number  $r$ ,

$$(r + 1)e - (re - iy) = e + iy$$

is not invertible. This implies that

$$|r + 1| \leq \|re - iy\|$$

and so

$$(r + 1)^2 \leq \|re - iy\|^2 = \|(re - iy)(re + iy)\|^2.$$

by (11). Thus

$$(r + 1)^2 \leq \|r^2e + y^2\|^2 \leq r^2 + \|y\|^2$$

which is not possible if  $2r - 1 > \|y\|^2$ .

So we have proved:

**Theorem 4.3** *Let  $A$  be a  $C^*$  algebra. If  $x \in A$  is normal, then*

$$|x|_{sp} = \|x\|.$$

*If  $x \in A$  is self-adjoint, then*

$$\text{Spec}(x) \subset \mathbf{R}.$$

## 4.2 An important application.

**Proposition 4.1** *If  $T$  is a bounded linear operator on a Hilbert space, then*

$$\|TT^*\| = \|T\|^2. \tag{16}$$

*In other words, the algebra of all bounded operators on a Hilbert space is a  $C^*$ -algebra under the involution  $T \mapsto T^*$ .*

**Proof.**

$$\begin{aligned} \|TT^*\| &= \sup_{\|\phi\|=1} \|TT^*\phi\| \\ &= \sup_{\|\phi\|=1, \|\psi\|=1} |(TT^*\phi, \psi)| \\ &= \sup_{\|\phi\|=1, \|\psi\|=1} |(T^*\phi, T^*\psi)| \\ &\geq \sup_{\|\phi\|=1} (T^*\phi, T^*\phi) \\ &= \|T^*\|^2 \end{aligned}$$

so

$$\|T^*\|^2 \leq \|TT^*\| \leq \|T\|\|T^*\|$$

so

$$\|T^*\| \leq \|T\|.$$

Reversing the role of  $T$  and  $T^*$  gives the reverse inequality so  $\|T\| = \|T^*\|$ . Inserting into the preceding inequality gives

$$\|T^2\| \leq \|TT^*\| \leq \|T\|^2$$

so we have the equality (16). QED

Thus the map  $T \mapsto T^*$  sending every bounded operator on a Hilbert space into its adjoint is an anti-involution on the Banach algebra of all bounded operators, and it satisfies (11). We can thus apply Theorem 4.2 to conclude:

**Theorem 4.4** *Let  $B$  be any commutative subalgebra of the algebra of bounded operators on a Hilbert space which is closed in the strong topology and with the property that  $T \in B \Rightarrow T^* \in B$ . Then the Gelfand transform  $T \mapsto \hat{T}$  gives a norm preserving isomorphism of  $B$  with  $\mathcal{C}(\mathcal{M})$  where  $\mathcal{M} = \text{Mspec}(B)$ . Furthermore,  $(T^*)^\wedge = \overline{\hat{T}}$  for all  $T \in B$ . In particular, if  $T$  is self-adjoint, then  $\hat{T}$  is real valued.*

## 5 The Spectral Theorem for Bounded Normal Operators, Functional Calculus Form.

As a special case of the definition we gave earlier, a (bounded) operator  $T$  on a Hilbert space  $\mathbf{H}$  is called **normal** if

$$TT^* = T^*T.$$

We can then consider the subalgebra of the algebra of all bounded operators which is generated by  $e$ , the identity operator,  $T$  and  $T^*$ . Take the closure,  $B$ , of this algebra in the strong topology. We can apply the preceding theorem to  $B$  to conclude that  $B$  is isometrically isomorphic to the algebra of all continuous functions on the compact Hausdorff space

$$\mathcal{M} = \text{Mspec}(B).$$

Remember that a point of  $\mathcal{M}$  is a homomorphism  $h : B \rightarrow \mathbf{C}$  and that

$$h(T^*) = (T^*)^\wedge(h) = \overline{\hat{T}(h)} = \overline{h(T)}.$$

Since  $h$  is a homomorphism, we see that  $h$  is determined on the algebra generated by  $e, T$  and  $T^*$  by the value  $h(T)$ , and since it is continuous, it is determined on all of  $B$  by the knowledge of  $h(T)$ . We thus get a map

$$\mathcal{M} \rightarrow \mathbf{C}, \quad \mathcal{M} \ni h \mapsto h(T), \tag{17}$$

and we know that this map is injective. Now

$$h(T - h(T)e) = h(T) - h(T) = 0$$

so  $T - h(T)e$  belongs to the maximal ideal  $h$ , and hence is not invertible in the algebra  $B$ . Thus the image of our map lies in  $\text{Spec}_B(T)$ . Here I have added the subscript  $B$ , because our general definition of the spectrum of an element  $T$  in an algebra  $B$  consists of those complex numbers  $z$  such that  $T - ze$  does not have an inverse in  $B$ . If we let  $A$  denote the algebra of *all* bounded operators on  $\mathbf{H}$ , it is logically conceivable that  $T - ze$  has an inverse in  $A$  which does not lie in  $B$ . In fact, this can not happen. But this requires a proof, which I will give later. So for the time being I will stick with the temporary notation  $\text{Spec}_B$ .

So the map  $h \mapsto h(T)$  actually maps  $\mathcal{M}$  to the subset  $\text{Spec}_B(T)$  of the complex plane. If  $\lambda \in \text{Spec}_B(T)$ , then by definition,  $T - \lambda e$  is not invertible in  $B$ , so lies in some maximal ideal  $h$ , so  $\lambda \in \text{Spec}_B(T)$ . So the map (17) maps  $\mathcal{M}$  onto  $\text{Spec}_B(T)$ . Since the topology on  $\mathcal{M}$  is inherited from the weak topology, the map  $h \mapsto h(T)$  is continuous. Since  $\mathcal{M}$  is compact and  $\mathbf{C}$  is Hausdorff, the fact that  $h$  is bijective and continuous implies that  $h^{-1}$  is also continuous. Indeed we must show that if  $U$  is an open subset of  $\mathcal{M}$ , then  $h(U)$  is open in  $\text{Spec}_B(T)$ . But  $\mathcal{M} \setminus U$  is compact, hence its image under the continuous map  $h$  is compact, hence closed, and so the complement of this image which is  $h(U)$  is open.

Thus  $h$  is a homeomorphism, and hence we have a norm-preserving  $*$ -isomorphism  $T \mapsto \hat{T}$  of  $B$  with the algebra of continuous functions on  $\text{Spec}_B(T)$ . Furthermore the element  $T$  corresponds the function  $z \mapsto z$  (restricted to  $\text{Spec}_B(T)$ ).

Since  $B$  is determined by  $T$ , let me use the notation  $\sigma(T)$  for  $\text{Spec}_B(T)$ , postponing until later the proof that  $\sigma(T) = \text{Spec}_A(T)$ . Let me set

$$\phi = h^{-1}$$

and now follow the customary notation in Hilbert space theory and use  $I$  to denote the identity operator.

### 5.1 Statement of the theorem.

**Theorem 5.1** *Let  $T$  be a bounded normal operator on a Hilbert space  $\mathbf{H}$ . Let  $B(T)$  denote the closure in the strong topology of the algebra generated by  $I, T$  and  $T^*$ . Then there is a unique continuous  $*$  isomorphism*

$$\phi : \sigma(T) \rightarrow B(T)$$

such that

$$\phi(\mathbf{1}) = I$$

and

$$\phi(z \mapsto z) = T.$$

Furthermore,

$$T\psi = \lambda\psi, \quad \psi \in \mathbf{H} \Rightarrow \phi(f)\psi = f(\lambda)\psi. \quad (18)$$

If  $f \in C(\sigma(T))$  is real valued then  $\phi(f)$  is self-adjoint, and if  $f \geq 0$  then  $\phi(f) \geq 0$  as an operator.

The only facts that we have not yet proved (aside from the big issue of proving that  $\sigma(T) = \text{Spec}_A(T)$ ) are (18) and the assertions which follow it. Now (18) is clearly true if we take  $f$  to be a polynomial, in which case  $\phi(f) = f(T)$ . Then just apply the Stone-Weierstrass theorem to conclude (18) for all  $f$ . If  $f$  is real then  $f = \bar{f}$  and therefore  $\phi(f) = \phi(f)^*$ . If  $f \geq 0$  then we can find a real valued  $g \in C(\sigma(T))$  such that  $f = g^2$  and the square of a self-adjoint operator is non-negative. QED

In view of this theorem, there is a more suggestive notation for the map  $\phi$ . Since the image of the monomial  $z$  is  $T$ , and since the image of any polynomial  $P$  (thought of as a function on  $\sigma(T)$ ) is  $P(T)$ , we are safe in using the notation

$$f(T) := \phi(f)$$

for any  $f \in C(\sigma(T))$ .

## 5.2 $\text{Spec}_B(T) = \text{Spec}_A(T)$ .

Here is the main result of this section:

**Theorem 5.2** *Let  $A$  be a  $C^*$  algebra and let  $B$  be a subalgebra of  $A$  which is closed under the involution. Then for any  $x \in B$  we have*

$$\text{Spec}_B(x) = \text{Spec}_A(x). \quad (19)$$

**Remarks:**

1. Applied to the case where  $A$  is the algebra of all bounded operators on a Hilbert space, and where  $B$  is the closed subalgebra by  $I, T$  and  $T^*$  we get the spectral theorem for normal operators as promised.
2. If  $T - ze$  has no inverse in  $A$  it has no inverse in  $B$ . So

$$\text{Spec}_A(x) \subset \text{Spec}_B(x).$$

We must show the reverse inclusion. We begin by formulating some general results and introducing some notation.

For any associative algebra  $A$  we let  $G(A)$  denote the set of elements of  $A$  which are invertible (the “group-like” elements).

**Proposition 5.1** *Let  $B$  be a Banach algebra, and let  $x_n \in G(B)$  be such that  $x_n \rightarrow x$  and  $x \notin G(B)$ . Then*

$$\|x_n^{-1}\| \rightarrow \infty.$$

**Proof.** Suppose not. Then there is some  $C > 0$  and a subsequence of elements (which we will relabel as  $x_n$ ) such that

$$\|x_n^{-1}\| < C.$$

Then

$$x = x_n(e + x_n^{-1}(x - x_n))$$

with  $x - x_n \rightarrow 0$ . In particular, for  $n$  large enough

$$\|x_n^{-1}\| \cdot \|x - x_n\| < 1,$$

so  $(e + x_n^{-1}(x - x_n))$  is invertible as is  $x_n$  and so  $x$  is invertible contrary to hypothesis. QED

**Proposition 5.2** *Let  $B$  be a closed subalgebra of a Banach algebra  $A$  containing the unit  $e$ . Then*

- $G(B)$  is the union of some of the components of  $B \cap G(A)$ .
- If  $x \in B$  then  $\text{Spec}_B(x)$  is the union of  $\text{Spec}_A(x)$  and a (possibly empty) collection of bounded components of  $\mathbf{C} \setminus \text{Spec}_A(x)$ .
- If  $\text{Spec}_A(x)$  does not separate the complex plane then

$$\text{Spec}_B(x) = \text{Spec}_A(x).$$

**Proof.** We know that  $G(B) \subset G(A) \cap B$  and both are open subsets of  $B$ . We claim that  $G(A) \cap B$  contains no boundary points of  $G(B)$ . If  $x$  were such a boundary point, then  $x = \lim x_n$  for  $x_n \in G(B)$ , and by the continuity of the map  $y \mapsto y^{-1}$  in  $A$  we conclude that  $x_n^{-1} \rightarrow x^{-1}$  in  $A$ , so in particular the  $\|x_n\|^{-1}$  are bounded which is impossible by Proposition 5.1. Let  $O$  be a component of  $B \cap G(A)$  which intersects  $G(B)$ , and let  $U$  be the complement of  $\overline{G(B)}$ . Then  $O \cap G(B)$  and  $O \cap U$  are open subsets of  $B$  and since  $O$  does not intersect  $\partial G(B)$ , the union of these two disjoint open sets is  $O$ . So  $O \cap U$  is empty since we assumed that  $O$  is connected. Hence  $O \subset G(B)$ . This proves the first assertion.

For the second assertion, let us fix the element  $x$ , and let

$$G_A(x) = \mathbf{C} \setminus \text{Spec}_A(x)$$

so that  $G_A(x)$  consists of those complex numbers  $z$  for which  $x - ze$  is invertible in  $A$ , with a similar notation for  $G_B(x)$ . Both of these are open subsets of  $\mathbf{C}$  and  $G_B(x) \subset G_A(x)$ . Furthermore, as before,  $G_B(x) \subset G_A(x)$  and  $G_A(x)$  can not contain any boundary points of  $G_B(x)$ . So again,  $G_B(x)$  is a union of some of the connected components of  $G_A(x)$ . Therefore  $\text{Spec}_B(x)$  is the union of  $\text{Spec}_A(x)$  and the remaining components. Since  $\text{Spec}_B(x)$  is bounded, it will not contain any unbounded components.

The third assertion follows immediately from the second. QED

But now we can prove Theorem 5.2. We need to show that if  $x \in B$  is invertible in  $A$  then it is invertible in  $B$ . If  $x$  is invertible in  $A$  then so are  $x^*$  and  $xx^*$ . But  $xx^*$  is self-adjoint, hence its spectrum is a bounded subset of  $\mathbf{R}$ , so does not separate  $\mathbf{C}$ . Since  $0 \notin \text{Spec}_A(xx^*)$  we conclude from the last assertion of the proposition that  $0 \notin \text{Spec}_B(xx^*)$  so  $xx^*$  has an inverse in  $B$ . But then

$$x^*(xx^*)^{-1} \in B$$

and

$$x(x^*(xx^*)^{-1}) = e.$$

QED.

### 5.3 A direct proof of the spectral theorem.

I started out this handout with the general theory of Banach algebras, went to the Gelfand representation theorem, the special properties of  $C^*$  algebras and then some general facts about how the spectrum of an element can vary with the algebra containing it. I took this route because of the impact the Gelfand representation theorem had on the course of mathematics, especially in algebraic geometry. But the key ideas are

- (9), which, for a bounded operator  $T$  on a Banach space says that

$$\max_{\lambda \in \text{Spec}(T)} |\lambda| = \lim_{n \rightarrow \infty} \|T^n\|^{\frac{1}{n}}$$

,

- (16) which says that if  $T$  is a bounded operator on a Hilbert space then  $\|TT^*\| = \|T\|^2$ , and
- If  $T$  is a bounded operator on a Hilbert space and  $TT^* = T^*T$  then it follows from (16) that

$$\|T^{2^k}\| = \|T\|^{2^k}.$$

We could prove these facts by the arguments given above and conclude that if  $T$  is a normal bounded operator on a Hilbert space then

$$\max_{\lambda \in \text{Spec}(T)} |\lambda| = \|T\|. \tag{20}$$

Suppose (for simplicity) that  $T$  is self-adjoint:  $T = T^*$ . Then the argument given several times above shows that  $\text{Spec}(T) \subset \mathbf{R}$ . Let  $P$  be a polynomial. Then (1) combined with the preceding equation says that

$$\|P(T)\| = \max_{\lambda \in \text{Spec}(T)} |P(\lambda)|. \tag{21}$$

The norm on the right is the restriction to polynomials of the uniform norm  $\|\cdot\|_\infty$  on the space  $C(\text{Spec}(T))$ .

Now the map

$$P \mapsto P(T)$$

is a homomorphism of the ring of polynomials into bounded normal operators on our Hilbert space satisfying

$$\overline{P} \mapsto P(T)^*$$

and

$$\|P(T)\| = \|P\|_{\infty, \text{Spec}(T)}.$$

The Weierstrass approximation theorem then allows us to conclude that this homomorphism extends to the ring of continuous functions on  $\text{Spec}(T)$  with all the properties stated in Theorem 5.1 .