Daniell's integration theory.

Math 212a

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Daniell's idea was to take the axiomatic properties of the integral as the starting point and develop integration for broader and broader classes of functions. Then derive measure theory as a consequence. Much of the presentation here is taken from the book *Abstract Harmonic Analysis* by Lynn Loomis. Some of the lemmas, propositions and theorems indicate the corresponding sections in Loomis's book.

Contents

1 The Daniell Integral

Let L be a vector space of *bounded* real valued functions on a set S closed under $\wedge$ and $\vee$. For example, S might be a complete metric space, and L might be the space of continuous functions of compact support on S.

A map

$$I : L \to \mathbf{R}$$

is called an **Integral** if

1. I is linear: $I(af + bg) = aI(f) + bI(g)$
2. I is non-negative: $f \geq 0 \Rightarrow I(f) \geq 0$ or equivalently $f \geq g \Rightarrow I(f) \geq I(g)$.
3. $f_n \searrow 0 \Rightarrow I(f_n) \searrow 0$.

For example, we might take $S = \mathbf{R}^n$, $L =$ the space of continuous functions of compact support on $\mathbf{R}^n$, and I to be the Riemann integral. The first two items on the above list are clearly satisfied. As to the third, we recall Dini's lemma from the notes on metric spaces, which says that a sequence of continuous functions of compact support $\{f_n\}$ on a metric space which satisfies $f_n \searrow 0$ actually converges uniformly to 0. Furthermore the supports of the f_n are all contained in a fixed compact set - for example the support of f_1. This establishes the third item.

The plan is now to successively increase the class of functions on which the integral is defined.

Define

$$U := \{\text{limits of monotone non-decreasing sequences of elements of } L\}.$$

We will use the word "increasing" as synonymous with "monotone non-decreasing" so as to simplify the language.

Lemma 1.1 *If f_n is an increasing sequence of elements of L and if $k \in L$ satisfies $k \leq \lim f_n$ then $\lim I(f_n) \geq I(k)$.*

Proof. If $k \in L$ and $\lim f_n \geq k$, then

$$f_n \wedge k \leq k \text{ and } f_n \geq f_n \wedge k$$

so $I(f_n) \geq I(f_n \wedge k)$ while

$$[k - (f_n \wedge k)] \searrow 0$$

so

$$I([k - f_n \wedge k]) \searrow 0$$

by 3) or

$$I(f_n \wedge k) \nearrow I(k).$$

Hence $\lim I(f_n) \geq \lim I(f_n \wedge k) = I(k)$. QED

Lemma 1.2 **[12C]** *If $\{f_n\}$ and $\{g_n\}$ are increasing sequences of elements of L and $\lim g_n \leq \lim f_n$ then $\lim I(g_n) \leq \lim I(f_n)$.*

Proof. Fix m and take $k = g_m$ in the previous lemma. Then $I(g_m) \leq \lim I(f_n)$. Now let $m \to \infty$. QED

Thus

$$f_n \nearrow f \text{ and } g_n \nearrow f \quad \Rightarrow \lim I(f_n) = \lim I(g_n)$$

so we may extend I to U by setting

$$I(f) := \lim I(f_n) \quad \text{for} \quad f_n \nearrow f.$$

If $f \in L$, this coincides with our original I, since we can take $g_n = f$ for all n in the preceding lemma.

We have now extended I from L to U. The next lemma shows that if we now start with I on U and apply the same procedure again, we do not get any further.

Lemma 1.3 [12D] *If $f_n \in U$ and $f_n \nearrow f$ then $f \in U$ and $I(f_n) \nearrow I(f)$.*

Proof. For each fixed n choose $g_n^m \nearrow_m f_n$. Set

$$h_n := g_1^n \vee \cdots \vee g_n^n$$

so

$$h_n \in L \quad \text{and } h_n \text{ is increasing}$$

with

$$g_i^n \leq h_n \leq f_n \quad \text{for } i \leq n.$$

Let $n \to \infty$. Then

$$f_i \leq \lim h_n \leq f.$$

Now let $i \to \infty$. We get

$$f \leq \lim h_n \leq f.$$

So we have written f as a limit of an increasing sequence of elements of L, So $f \in U$. Also

$$I(g_i^n) \leq I(h_n) \leq I(f)$$

so letting $n \to \infty$ we get

$$I(f_i) \leq I(f) \leq \lim I(f_n)$$

so passing to the limits gives $I(f) = \lim I(f_n)$. QED

We have

$$I(f+g) = I(f) + I(g) \quad \text{for} \quad f, g \in U.$$

Define

$$-U := \{-f |\ f \in U\}$$

and

$$I(f) := -I(-f) \qquad f \in -U.$$

If $f \in U$ and $-f \in U$ then $I(f)+I(-f) = I(f-f) = I(0) = 0$ so $I(-f) = -I(f)$ in this case. So the definition is consistent.

$-U$ is closed under monotone decreasing limits. etc.

If $g \in -U$ and $h \in U$ with $g \leq h$ then $-g \in U$ so $h - g \in U$ and $h - g \geq 0$ so $I(h) - I(g) = I(h + (-g)) = I(h - g) \geq 0$.

A function f is called **I-summable** if for every $\epsilon > 0$, $\exists\ g \in -U,\ h \in U$ with

$$g \leq f \leq h, \quad |I(g)| < \infty,\ |I(h)| < \infty \ \text{ and } I(h - g) \leq \epsilon.$$

For such f define

$$I(f) = \text{glb } I(h) = \text{lub } I(g).$$

If $f \in U$ take $h = f$ and $f_n \in L$ with $f_n \nearrow f$. Then $-f_n \in L \subset U$ so $f_n \in -U$. If $I(f) < \infty$ then we can choose n sufficiently large so that $I(f) - I(f_n) < \epsilon$. The space of summable functions is denoted by $\overline{L}_1$. It is clearly a vector space, and I satisfies conditions 1) and 2) above, i.e. is linear and non-negative.

Theorem 1.1 [12G] Monotone convergence theorem. *$f_n \in \overline{L}_1$, $f_n \nearrow f$ and $\lim I(f_n) < \infty \ \Rightarrow f \in \overline{L}_1$ and $I(f) = \lim I(f_n)$.*

Proof. Replacing f_n by $f_n - f_0$ we may assume that $f_0 = 0$. Choose

$$h_n \in U, \ \text{ such that } f_n - f_{n-1} \leq h_n \text{ and } I(h_n) \leq I(f_n - f_{n-1}) + \frac{\epsilon}{2^n}.$$

Then

$$f_n \leq \sum_1^n h_i \quad \text{and } \sum_{i=1}^n I(h_i) \leq I(f_n) + \epsilon.$$

Since U is closed under monotone increasing limits,

$$h := \sum_{i=1}^\infty h_i \in U, \quad f \leq h \quad \text{and } I(h) \leq \lim I(f_n) + \epsilon.$$

Since $f_m \in \overline{L}_1$ we can find a $g_m \in -U$ with $I(f_m) - I(g_m) < \epsilon$ and hence for m large enough $I(h) - I(g_m) < 2\epsilon$. So $f \in \overline{L}_1$ and $I(f) = \lim I(f_n)$. QED

2 Monotone class theorems.

A collection of functions which is closed under monotone increasing and monotone decreasing functions is called a **monotone class**. $\mathcal{B}$ is defined to be the smallest monotone class containing L.

Lemma 2.1 *Let $h \leq k$. If $\mathcal{M}$ is a monotone class which contains $(g \vee h) \wedge k$ for every $g \in L$, then $\mathcal{M}$ contains all $(f \vee h) \wedge k$ for all $f \in \mathcal{B}$.*

Proof. The set of f such that $(f \vee h) \wedge k \in \mathcal{M}$ is a monotone class containing L by the distributive laws.QED

Taking $h = k = 0$ this says that the smallest monotone class containing L^+, the set of non-negative functions in L, is the set $\mathcal{B}^+$, the set of non-negative functions in $\mathcal{B}$.

Here is a series of monotone class theorem style arguments:

Theorem 2.1 *$f, g \in \mathcal{B} \ \Rightarrow af + bg \in \mathcal{B}, f \vee g \in \mathcal{B}$ and $f \wedge g \in \mathcal{B}$.*

For $f \in \mathcal{B}$, let

$$\mathcal{M}(f) := \{g \in \mathcal{B} | f + g, f \vee g, f \wedge g \in \mathcal{B}\}.$$

$\mathcal{M}(f)$ is a monotone class. If $f \in L$ it includes all of L, hence all of $\mathcal{B}$. But

$$g \in \mathcal{M}(f) \ \Leftrightarrow \ f \in \mathcal{M}(g).$$

So $L \subset \mathcal{M}(g)$ for any $g \in \mathcal{B}$, and since it is a monotone class $\mathcal{B} \subset \mathcal{M}(g)$. This says that $f, g \in \mathcal{B} \Rightarrow f + g \in \mathcal{B}, \ f \wedge g \in \mathcal{B}$ and $f \vee g \in \mathcal{B}$. Similarly, let $\mathcal{M}$ be the class of functions for which $cf \in \mathcal{B}$ for all real c. This is a monotone class containing L hence contains $\mathcal{B}$. QED

Lemma 2.2 *If $f \in \mathcal{B}$ there exists a $g \in U$ such that $f \leq g$.*

Proof. The limit of a monotone increasing sequence of functions in U belongs to U. Hence the set of f for which the lemma is true is a monotone class which contains L. hence it contains $\mathcal{B}$. QED

A function f is L-**bounded** if there exists a $g \in L^+$ with $|f| \leq g$. A class $\mathcal{F}$ of functions is said to be L-monotone if $\mathcal{F}$ is closed under monotone limits of L-bounded functions.

Theorem 2.2 *The smallest L-monotone class including L^+ is $\mathcal{B}^+$.*

Proof. Call this smallest family $\mathcal{F}$. If $g \in L^+$, the set of all $f \in \mathcal{B}^+$ such that $f \wedge g \in \mathcal{F}$ form a monotone class containing L^+, hence containing $\mathcal{B}^+$ hence equal to $\mathcal{B}^+$. If $f \in \mathcal{B}^+$ and $f \leq g$ then $f \wedge g = f \in \mathcal{F}$. So $\mathcal{F}$ contains all L bounded functions belonging to $\mathcal{B}^+$. Let $f \in \mathcal{B}^+$. by the lemma, choose $g \in U$ such that $f \leq g$, and choose $g_n \in L^+$ with $g_n \nearrow g$. Then $f \wedge g_n \leq g_n$ and so is L bounded, so $f \wedge g_n \in \mathcal{F}$. Since $(f \wedge g_n) \to f$ we see that $f \in \mathcal{F}$. So

$$\mathcal{B}^+ \subset \mathcal{F}.$$

We know that $\mathcal{B}^+$ is a monotone class, in particular an L-monotone class. Hence $\mathcal{F} = \mathcal{B}^+$. QED

Define

$$L^1 := \overline{L}_1 \cap \mathcal{B}.$$

Since $\overline{L}_1$ and $\mathcal{B}$ are both closed under the lattice operations,

$$f \in L^1 \Rightarrow f^{\pm} \in L^1 \Rightarrow |f| \in L^1.$$

Theorem 2.3 *If $f \in \mathcal{B}$ then $f \in L^1 \Leftrightarrow \exists g \in L^1$ with $|f| \le g$.*

We have proved $\Rightarrow$: simply take $g = |f|$. For the converse we may assume that $f \geq 0$ by applying the result to f^+ and f^-. The family of all $h \in \mathcal{B}^+$ such that $h \wedge g \in L^1$ is monotone and includes L^+ so includes $\mathcal{B}^+$. So $f = f \wedge g \in L^1$. QED

Extend I to all of $\mathcal{B}^+$ be setting it $= \infty$ on functions which do not belong to L^1.

3 Measure.

Loomis calls a set A **integrable** if $\mathbf{1}_A \in \mathcal{B}$. The monotone class properties of $\mathcal{B}$ imply that the integrable sets form a σ-field. Then define

$$\mu(A) := \int \mathbf{1}_A$$

and the monotone convergence theorem guarantees that μ is a measure.

Add **Stone's axiom**

$$f \in L \Rightarrow f \wedge \mathbf{1} \in L.$$

Then the monotone class property implies that this is true with L replaced by $\mathcal{B}$.

Theorem 3.1 *$f \in \mathcal{B}$ and $a > 0 \Rightarrow$ then*

$$A_a := \{p | f(p) > a\}$$

is an integrable set. If $f \in L^1$ then

$$\mu(A_a) < \infty.$$

Proof. Let

$$f_n := [n(f - f \wedge a)] \wedge 1 \in \mathcal{B}.$$

Then

$$f_n(x) = \begin{cases} 1 & \text{if} \quad f(x) \geq a + \frac{1}{n} \\ 0 & \text{if} \quad f(x) \leq a \\ n(f(x) - a) & \text{if} \quad a < f(x) < a + \frac{1}{n} \end{cases}.$$

We have

$$f_n \nearrow \mathbf{1}_{A_a}$$

so $\mathbf{1}_{A_a} \in \mathcal{B}$ and $0 \leq \mathbf{1}_{A_a} \leq \frac{1}{a} f^+$. QED

Theorem 3.2 *If $f \geq 0$ and A_a is integrable for all $a > 0$ then $f \in \mathcal{B}$.*

Proof. For $\delta > 1$ define

$$A_m^\delta := \{x | \delta^m < f(x) \leq \delta^{m+1}\}$$

and

$$f_\delta := \sum_m \delta^m \mathbf{1}_{A_m^\delta}.$$

Each $f_\delta \in \mathcal{B}$. Take

$$\delta_n = 2^{2^{-n}}.$$

Then each successive subdivision divides the previous one into "octaves" and $f_{\delta_m} \nearrow f$. QED

Also

$$f_\delta \leq f \leq \delta f_\delta$$

and

$$I(f_{\delta)} = \sum \delta^n \mu(A_m^\delta) = \int f_\delta d\mu.$$

So we have

$$I(f_\delta) \leq I(f) \leq \delta I(f_\delta)$$

and

$$\int f_\delta d\mu \leq \int f d\mu \leq \delta \int f_\delta d\mu.$$

So if either of $I(f)$ or $\int f d\mu$ is finite they both are and

$$\left| I(f) - \int f d\mu \right| \leq (\delta - 1) I(f_\delta) \leq (\delta - 1) I(f).$$

So

$$\int f d\mu = I(f).$$

If $f \in \mathcal{B}^+$ and $a > 0$ then

$$\{x | f(x)^a > b\} = \{x | f(x) > b^{\frac{1}{a}}\}.$$

So $f \in \mathcal{B}^+ \Rightarrow f^a \in \mathcal{B}^+$ and hence the product of two elements of $\mathcal{B}^+$ belongs to $\mathcal{B}^+$ because

$$fg = \frac{1}{4}\left[(f+g)^2 - (f-g)^2\right].$$

4 Hölder, Minkowski , L^p and L^q.

The numbers $p, q > 1$ are called **conjugate** if

$$\frac{1}{p} + \frac{1}{q} = 1.$$

This is the same as

$$pq = p + q$$

or

$$(p-1)(q-1)=1.$$

This last equation says that if

$$y = x^{p-1}$$

then

$$x = y^{q-1}.$$

The area under the curve $y = x^{p-1}$ from 0 to a is

$$A = \frac{a^p}{p}$$

while the area between the same curve and the y-axis up to $y = b$

$$B = \frac{b^q}{q}.$$

Suppose $b < a^{p-1}$ to fix the ideas. Then area ab of the rectangle is less than $A + B$ or

$$\frac{a^p}{p} + \frac{b^q}{q} \geq ab$$

with equality if and only if $b = a^{p-1}$. Replacing a by $a^{\frac{1}{p}}$ and b by $b^{\frac{1}{q}}$ gives

$$a^{\frac{1}{p}} b^{\frac{1}{q}} \leq \frac{a}{p} + \frac{b}{q}.$$

Let L^p denote the space of functions such that $|f|^p \in L^1$. For $f \in L^p$ define

$$\|f\|_p := \left(\int |f|^p d\mu \right)^{\frac{1}{p}}.$$

We will soon see that if $p \geq 1$ this is a (semi-)norm.

If $f \in L^p$ and $g \in L^q$ with $\|f\|_p \neq 0$ and $\|g\|_q \neq 0$ take

$$a = \frac{|f|^p}{\|f\|_p}, \quad b = \frac{|g|^q}{\|g\|_q}$$

as functions. Then

$$\int (|f||g|) d\mu \leq \|f\|_p \|g\|_q \left(\frac{1}{p} \frac{1}{\|f\|_p^p} \int |f|^p d\mu + \frac{1}{q} \frac{1}{\|g\|_q^q} \int |g|^q d\mu \right) = \|f\|_p \|g\|_q.$$

This shows that the left hand side is integrable and that

$$\left| \int fg d\mu \right| \leq \|f\|_p \|g\|_q \tag{1}$$

which is known as**Hölder's inequality**. (If either $\|f\|_p$ or $\|g\|_q = 0$ then $fg = 0$ a.e. and Hölder's inequality is trivial.)

We write

$$(f,g) := \int fg d\mu.$$

Proposition 4.1 **[Minkowski's inequality]** *If* $f, g \in L^p,\ p \geq 1$ *then* $f + g \in L^p$ *and*

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p.$$

For $p = 1$ this is obvious. If $p > 1$

$$|f + g|^p \leq [2\max(|f|, |g|)]^p \leq 2^p\,[|f|^p + |g|^p]$$

implies that $f + g \in L^p$. Write

$$\|f + g\|_p^p \leq I(|f + g|^{p-1}|f|) + I(|f + g|^{p-1}|g|).$$

Now

$$q(p - 1) = qp - q = p$$

so

$$|f + g|^{p-1} \in L_q$$

and its $\|\cdot\|_q$ norm is

$$I(|f + g|^p)^{\frac{1}{q}} = I(|f + q|^p)^{1-\frac{1}{p}} = I(|f + g|^p)^{\frac{p-1}{p}} = \|f + g|_p^{p-1}.$$

So we can write the preceding inequality as

$$\|f + g\|_p^p \leq (|f|, |f + g|^{p-1}) + (|g|, |f + g|^{p-1}$$

and apply Hölder's inequality to conclude that

$$\|f + g\|^p \leq \|f + g\|^{p-1}(\|f\|_p + \|g\|_p).$$

We may divide by $\|f + g\|_p^{p-1}$ to get Minkowski's inequality unless $\|f + g\|_p = 0$ in which case it is obvious. QED

Theorem 4.1 L^p *is complete.*

Proof. Suppose $f_n \geq 0$, $f_n \in L^p$, and $\sum \|f_n\|_p < \infty$ Then

$$g_n := \sum_1^n f_j \in L^p$$

by Minkowski and since $g_n \nearrow f$ we have $|g_n|^p \nearrow f^p$ and hence by the monotone convergence theorem $f := \sum_{j=1}^\infty f_n \in L^p$ and $\|f\|_p = \lim \|g\|_p \leq \sum \|f_j\|_p$.

Now let $\{f_n\}$ be any Cauchy sequence in L^p. By passing to a subsequence we may assume that

$$\|f_{n+1} - f_n\|_p < \frac{1}{2^n}.$$

So $\sum_n^\infty |f_{i+1} - f_i| \in L^p$ and hence

$$g_n := f_n - \sum_n^\infty |f_{i+1} - f_i| \in L^p \ \text{ and } \ h_n := f_n + \sum_n^\infty |f_{i+1} - f_i| \in L^p.$$

We have

$$g_{n+1} - g_n = f_{n+1} - f_n + |f_{n+1} - f_n| \geq 0$$

so g_n is increasing and similarly h_n is decreasing. Hence $f := \lim g_n \in L^p$ and $\|f - f_n\|_p \leq \|h_n - g_n\|_p \leq 2^{-n+2} \to 0$. So the subsequence has a limit which then must be the limit of the original sequence. QED

Proposition 4.2 *L is dense in L^p for any $1 \leq p < \infty$.*

Proof. For $p = 1$ this was a defining property of L^1. More generally, suppose that $f \in L^p$ and that $f \geq 0$. Let

$$A_n := \{x : \frac{1}{n} < f(x) < n\},$$

and let

$$g_n := f \cdot \mathbf{1}_{A_n}.$$

Then $(f - g_n) \searrow 0$ as $n \to \infty$. Choose n sufficiently large so that $\|f - g_n\|_p < \epsilon/2$. Since

$$0 \leq g_n \leq n\mathbf{1}_{A_n} \quad \text{and} \quad \mu(A_n) < n^p I(|f|^p) < \infty$$

we conclude that

$$g_n \in L^1.$$

Now choose $h \in L^+$ so that

$$\|h - g_n\| < \left(\frac{\epsilon}{2n}\right)^p$$

and also so that $h \leq n$. Then

$$\begin{aligned}
\|h - g_n|_p &= (I(|h - g_n|^p))^{1/p} \\
&= \left(I(|h - g_n|^{p-1}|h - g_n|)\right)^{1/p} \\
&\leq \left(I(n^{p-1}|h - g_n|)\right)^{1/p} \\
&= \left(n^{p-1}\|h - g_n\|_1\right)^{1/p} \\
&< \epsilon/2.
\end{aligned}$$

So by the triangle inequality $\|f - h\| < \epsilon$. QED

In the above, we have not bothered to pass to the quotient by the elements of norm zero. In other words, we have not identified two functions which differ on a set of measure zero. We will continue with this ambiguity. But equally well, we could change our notation, and use L^p to denote the quotient space (as we did earlier in class) and denote the space before e pass to the quotient by $\mathcal{L}^p$ to conform with our earlier notation. I will continue to be sloppy on this point, in conformity to Loomis' notation.

5 $\|\cdot\|_\infty$ is the essential sup norm.

Suppose that $f \in \mathcal{B}$ has the property that it is equal almost everywhere to a function which is bounded above. We call such a function **essentially bounded** (from above). We can then define the **essential least upper bound** of f to be the smallest number which is an upper bound for a function which differs from f on a set of measure zero. If $|f|$ is essentially bounded, we denote its essential least upper bound by $\|f\|_\infty$. Otherwise we say that $\|f\|_\infty = \infty$. We let $\mathcal{L}^\infty$ denote the space of $f \in \mathcal{B}$ which have $\|f\|_\infty < \infty$. It is clear that $\|\cdot\|_\infty$ is a semi-norm on this space. The justification for this notation is

Theorem 5.1 **[14G]** *If $f \in L^p$ for some $p > 0$ then*

$$\|f\|_\infty = \lim_{q\to\infty} \|f\|_q. \tag{2}$$

Remark. In the statement of the theorem, both sides of (2) are allowed to be ∞.

Proof. If $\|f\|_\infty = 0$, then $\|f\|_q = 0$ for all $q > 0$ so the result is trivial in this case. So let us assume that $\|f\|_\infty > 0$ and let a be any positive number smaller that $\|f\|_\infty$. In other words,

$$0 < a < \|f\|_\infty.$$

Let

$$A_a := \|x : |f(x)| > a\}.$$

This set has positive measure by the choice of a, and its measure is finite since $f \in L^p$. Also

$$\|f\|_q \geq \left(\int_{A_a} |f|^q\right)^{1/q} \geq a\mu(A_a)^{1/q}.$$

Letting $q \to \infty$ gives

$$\lim \inf_{q\to\infty} \geq a$$

and since a can be any number $< \|f\|_\infty$ we conclude that

$$\lim \inf_{q\to\infty} \geq \|f\|_\infty.$$

So we need to prove that

$$\lim_{q\to\infty} \sum \leq \|f\|_\infty.$$

This is obvious if $\|f\|_\infty = \infty$. So suppose that $\|f\|_\infty$ is finite. Then for $q > p$ we have

$$|f|^q \leq |f|^p(\|f\|_\infty)^{q-p}$$

almost everywhere. Integrating and taking the q-th root gives

$$\|f\|_q \leq (\|f\|_p)^{\frac{p}{q}}(\|f\|_\infty)^{1-\frac{p}{q}}.$$

Letting $q \to \infty$ gives the desired result. QED

6 The Radon-Nikodym Theorem.

Suppose we are given two integrals, I and J on the same space L. That is, both I and J satisfy the three conditions of linearity, positivity, and the monotone limit property that went into our definition of the term "integral". We say that J is **absolutely continuous** with respect to I if every set which is I null (i.e. has measure zero with respect to the measure associated to I) is J null.

The integral I is said to be **bounded** if

$$I(\mathbf{1}) < \infty,$$

or, what amounts to the same thing, that

$$\mu_I(S) < \infty$$

where μ_I is the measure associated to I.

We will first formulate the Radon-Nikodym theorem for the case of bounded integrals, where there is a very clever proof due to von-Neumman which reduces it to the Riesz representation theorem in Hilbert space theory.

Theorem 6.1 [**Radon-Nikodym**] *Let I and J be bounded integrals, and suppose that J is absolutely continuous with respect to I. Then there exists an element $f_0 \in \mathcal{L}^1(I)$ such that*

$$J(f) = I(ff_0) \quad \forall\ f \in \mathcal{L}^1(J). \tag{3}$$

The element f_0 is unique up to equality almost everywhere (with respect to μ_I).

Proof.(After von-Neumann.) Consider the linear function

$$K := I + J$$

on L. Then K satisfies all three conditions in our definition of an integral, and in addition is bounded. We know from the case $p = 2$ of Theorem 4.1 that $L^2(K)$ is a (real) Hilbert space. (Assume for this argument that we have passed to the quotient space so an element of $L^2(K)$ is an equivalence class of of functions.) The fact that K is bounded, says that $\mathbf{1} := \mathbf{1}_S$. If $f \in L^2(K)$ then the Cauchy-Schwartz inequality says that

$$K(|f|) = K(|f| \cdot \mathbf{1}) = (|f|, \mathbf{1})_{2,K} \leq \|f\|_{2,K} \|\mathbf{1}\|_{2,K} < \infty$$

so $|f|$ and hence f are elements of $L^2(K)$.

Furthermore,

$$|J(f)| \leq J(|f|) \leq K(|f|) \leq \|f\|_{2,K} \|\mathbf{1}\|_{2,K}$$

for all $f \in L$. Since we know that L is dense in $L^2(K)$ by Proposition 4.2, J extends to a unique continuous linear functional on $L^2(K)$. We conclude from the real version of the Riesz representation theorem, that there exists a unique $g \in L^2(K)$ such that

$$J(f) = (f, g)_{2,K} = K(fg).$$

If A is any subset of S of positive measure, then $J(\mathbf{1}_A) = K(\mathbf{1}_A g)$ so g is non-negative. (More precisely, g is equivalent almost everywhere to a function which is non-negative.) We obtain inductively

$$\begin{aligned} J(f) &= K(fg) = \\ I(fg) + J(fg) &= I(fg) + I(fg^2 + J(fg^2 = \\ &\vdots \\ &= I\left(f \cdot \sum_{i=1}^{n} g^i\right) + J(fg^n). \end{aligned}$$

Let N be the set of all x where $g(x) \geq 1$. Taking $f = \mathbf{1}_N$ in the preceding string of equalities shows that

$$J(\mathbf{1}_N) \geq nI(\mathbf{1}_N).$$

Since n is arbitrary, we have proved

Lemma 6.1 *The set where $g \geq 1$ has I measure zero.*

We have not yet used the assumption that J is absolutely continuous with respect to I. Let us now use this assumption to conclude that N is also J-null. This means that if $f \geq 0$ and $f \in L^1(J)$ then $fg^n \searrow 0$ almost everywhere (J), and hence by the dominated convergence theorem

$$J(fg^n) \searrow 0.$$

Plugging this back into the above string of equalities shows (by the monotone convergence theorem for I) that

$$f \sum_{i=1}^{\infty} g^n$$

converges in the $L^1(I)$ norm to $J(f)$. In particular, since $J(\mathbf{1}) < \infty$, we may take $f = \mathbf{1}$ and conclude that $\sum_{i=1}^{\infty}$ converges in $L^1(I)$. So set

$$f_0 := \sum_{i=1}^{\infty} g^i \in L^1(I).$$

We have

$$f_0 = \frac{1}{1-g} \quad \text{almost everywhere}$$

so

$$g = \frac{f_0 - 1}{f_0} \quad \text{almost everywhere}$$

and

$$J(f) = I(ff_0)$$

for $f \geq 0,\ f \in L^1(J)$. By breaking any $f \in L^1(J)$ into the difference of its positive and negative parts, we conclude that (3) holds for all $f \in L^1(J)$. The uniqueness of f_0 (almost everywhere (I)) follows from the uniqueness of g in $L^2(K)$. QED

The Radon Nikodym theorem can be extended in two directions. First of all, let us continue with our assumption that I and J are bounded, but drop the absolute continuity requirement. Let us say that an integral H is **absolutely singular** with respect to I if there is a set N of I-measure zero such that $J(h) = 0$ for any h vanishing on N.

Let us now go back to Lemma 6.1. Define J_{sing} by

$$J_{sing}(f) = J(\mathbf{1}_N f).$$

Then J_{sing} is singular with respect to I, and we can write

$$J = J_{cont} + J_{sing}$$

where

$$J_{cont} = J - J_{sing} = J(\mathbf{1}_{N^c}\cdot).$$

Then we can apply the rest of the proof of the Radon Nikodym theorem to J_{cont} to conclude that

$$J_{cont}(f) = I(ff_0)$$

where $f_0 = \sum_{i=1}^{\infty}(\mathbf{1}_{N^c}g)^i$ is an element of $L^1(I)$ as before. In particular, J_{cont} is absolutely continuous with respect to I.

A second extension is to certain situations where S is not of finite measure. We say that a function f is **locally** L^1 if $f\mathbf{1}_A \in L^1$ for every set A with $\mu(A) < \infty$. We say that S is σ**-finite** with respect to μ if S is a countable union of sets of finite μ measure. This is the same as saying that $\mathbf{1} = \mathbf{1}_S \in \mathcal{B}$. If S is σ-finite then it can be written as a disjoint union of sets of finite measure. If S is σ-finite with respect to both I and J it can be written as the disjoint union of countably many sets which are both I and J finite. So if J is absolutely continuous with respect I, we can apply the Radon-Nikodym theorem to each of these sets of finite measure, and conclude that there is an f_0 which is locally L^1 with respect to I, such that $J(f) = I(ff_0)$ for all $f \in L^1(J)$, and f_0 is unique up to almost everywhere equality.

7 The dual space of L^p.

Recall that Hölder's inequality (1) says that

$$\left|\int fg d\mu\right| \leq \|f\|_p \|g\|_q$$

if $f \in L^p$ and $g \in L^q$ where

$$\frac{1}{p} + \frac{1}{q} = 1.$$

For the rest of this section we will assume without further mention that this relation between p and q holds. Hölder's inequality implies that we have a map from

$$L^q \to (L^p)^*$$

sending $g \in L^p$ to the continuous linear function on L^p which sends

$$f \mapsto I(fg) = \int fg d\mu.$$

Furthermore, Hölder's inequality says that the norm of this map from $L^q \to (L^p)^*$ is ≤ 1. In particular, this map is injective.

The theorem we want to prove is that under suitable conditions on S and I (which are more general even that σ-finiteness) this map is surjective for $1 \leq p < \infty$.

We will first prove the theorem in the case where $\mu(S) < \infty$, that is when I is a bounded integral. For this we will will need a lemma:

7.1 The variations of a bounded functional.

Suppose we start with an arbitrary L and I. For each $1 \leq p \leq \infty$ we have the norm $\|\cdot\|_p$ on L which makes L into a real normed linear space. Let F be a linear function on L which is bounded with respect to this norm, so that

$$|F(f)| \leq C\|f\|_p$$

for all $f \in L$ where C is some non-negative constant. The least upper bound of the set of C which work is called $\|F\|_p$ as usual. If $f \geq 0 \in L$, define

$$F^+(f) := \text{lub}\{F(g) : 0 \leq g \leq f, \quad g \in L\}.$$

Then

$$F^+(f) \geq 0$$

and

$$F^+(f) \leq \|F\|_p \|f\|_p$$

since $F(g) \leq |F(g)| \leq \|F\|_p\|g\|_p \leq \|F\|_p\|f\|_p$ for all $0 \leq g \leq f$, $g \in L$, since $0 \leq g \leq f$ implies $|g|^p \leq |f|^p$ for $1 \leq p < \infty$ and also implies $\|g\|_\infty \leq \|f\|_\infty$. Also

$$F^+(cf) = cF^+(f) \quad \forall\ c \geq 0$$

as follows directly from the definition. Suppose that f_1 and f_2 are both non-negative elements of L. If $g_1,\ g_2 \in L$ with

$$0 \leq g_1 \leq f_1 \text{ and } 0 \leq g_2 \leq f_2$$

then

$$F^+(f_1+f_2) \geq \text{lub}\, F(g_1+g_1) = \text{lub}\, F(g_1) + \text{lub}\, F(g_2) = F^+(f_1) + F^+(f_2).$$

On the other hand, if $g \in L$ satisfies $0 \leq g \leq (f_1+f_2)$ then $0 \leq g \wedge f_1 \leq f_1$, and $g \wedge f_1 \in L$. Also $g - g \wedge f_1 \in L$ and vanishes at points x where $g(x) \leq f_1(x)$ while at points where $g(x) > f_1(x)$ we have $g(x) - g \wedge f_1(x) = g(x) - f_{1(x)} \leq f_2(x)$. So

$$g - g \wedge f_1 leq f_2$$

and so

$$F^+(f_1+f_2) = \text{lub}\, F(g) \leq lubF(g \wedge f_1) + \text{lub}\, F(g - g \wedge f_1) \leq F^+(f_1) + F^+(f_2).$$

So

$$F^+(f_1+f_2) = F^+(f_1) + F^+(f_2)$$

if both f_1 and f_2 are non-negative elements of L. Now write any $f \in L$ as $f = f_1 - g_1$ where f_1 and g_1 are non-negative. (For example we could take $f_1 = f^+$ and $g_1 = f^-$.) Define

$$F^+(f) = F^+(f_1) - F^+(g_1).$$

This is well defined, for if we also had $f = f_2 - g_2$ then $f_1 + g_2 = f_2 + g_1$ so

$$F^+(f_1) + F^+(g_2) = F^+(f_1+g_2) = F^+(f_2+g_1) = F^+(f_2) + F^+(g_1)$$

so

$$F^+(f_1) - F^+(g_1) = F^+(f_2) - F^+(g_2).$$

From this it follows that F^+ so extended is linear, and

$$|F^+(f)| \leq F^+(|f|) \leq \|F\|_p Q \|f_p$$

so F^+ is bounded.

Define F^- by

$$F^-(f) := F^+(f) - F(f).$$

As F^- is the difference of two linear functions it is linear. Since by its definition, $F^+(f) \geq F(f)$ if $f \geq 0$, we see that $F^-(f) \geq 0$ if $f \geq 0$. Clearly $\|F^-\| \leq \|F^+\|_p + \|F\| \leq 2\|F\|_p$. We have proved:

Proposition 7.1 *Every linear function on L which is bounded with respect to the $\|\cdot\|_p$ norm can be written as the difference $F = F^+ - F^-$ of two linear functions which are bounded and take non-negative values on non-negative functions.*

In fact, we could formulate this proposition more abstractly as dealing with a normed vector space which has an order relation consistent with its metric but we shall refrain from this more abstract formulation.

7.2 Duality of L^p and L^q when $\mu(S) < \infty$.

Theorem 7.1 *Suppose that $\mu(S) < \infty$ and that F is a bounded linear function on L^p with $1 \le p < \infty$. Then there exists a unique $g \in L^q$ such that*

$$F(f) = (f, g) = I(fg).$$

Here $q = p/(p-1)$ if $p > 1$ and $q = \infty$ if $p = 1$.

Proof. Consider the restriction of F to L. We know that $F = F^+ - F^-$ where both F^+ and F^- are linear and non-negative and are bounded with respect to the $\|\cdot\|_p$ norm on L. The monotone convergence theorem implies that if $f_n \searrow 0$ then $\|f_n\|_p \to 0$ and the boundedness of F^+ with respect to the $\|\cdot\|_p$ says that

$$\|f_n\|_p \to 0 \quad \Rightarrow \quad F^+(f_n) \to 0.$$

So F^+ satisfies all the axioms for an integral, and so does F^-. If f vanishes outside a set of I measure zero, then $\|f\|_p = 0$. Applied to a function of the form $f = \mathbf{1}_A$ we conclude that if A has $\mu = \mu_I$ measure zero, then A has measure zero with respect to the measures determined by F^+ or F^-. We can apply the Radon-Nikodym theorem to conclude that there are functions g^+ and g^- which belong to $L^1(I)$ and such that

$$F^{\pm}(f) = I(fg^{\pm})$$

for every f which belongs to $L^1(F^{\pm})$. In particular, if we set $g := g^+ - g^-$ then

$$F(f) = I(fg)$$

for every function f which is integrable with respect to both F^+ and F^-, in particular for any $f \in L^p(I)$. We must show that $g \in L^q$.

We first treat the case where $p > 1$. Suppose that $0 \le f \le |g|$ and that f is bounded. Then

$$I(f^q) \le I(f^{q-1} \cdot \mathrm{sgn}(g)g) = F(f^{q-1} \cdot \mathrm{sgn}(g)) \le \|F\|_p \|f^{q-1}\|_p.$$

So

$$I(f^q) \le \|F\|_p (I(f^{(q-1)p}))^{\frac{1}{p}}.$$

Now $(q-1)p = q$ so we have

$$I(f^q) \le \|F\|_p I(f^q)^{\frac{1}{p}} = \|F\|_q I(f^q)^{1-\frac{1}{q}}.$$

This gives

$$\|f\|_q \le \|F\|_p$$

for all $0 \le f \le |g|$ with f bounded. We can choose such functions f_n with $f_n \nearrow |g|$. It follows from the monotone convergence theorem that $|g|$ and hence $g \in L^q(I)$. This proves the theorem for $p > 1$.

Let us now give the argument for $p = 1$. We want to show that $\|g\|_\infty \leq \|F\|_1$. Suppose that $\|g\|_\infty \geq \|F\|_1 + \epsilon$ where $\epsilon > 0$. Consider the function $\mathbf{1}_A$ where

$$A := \{x : |g(x) \geq \|F\|_1 + \frac{\epsilon}{2}\}.$$

Then

$$(\|F\|_1 + \frac{\epsilon}{2})\mu(A) \leq I(\mathbf{1}_A|g|) = I(\mathbf{1}_A \mathrm{sgn}(g)g) = F(\mathbf{1}_A \mathrm{sgn}(g))$$

$$\leq \|F\|_1 \|\mathbf{1}_A \mathrm{sgn}(g)|_1 = \|F\|_1 \mu(A)$$

which is impossible unless $\mu(A) = 0$, contrary to our assumption. QED

7.3 The case where $\mu(S) = \infty$.

Here the cases $p > 1$ and $p = 1$ may be different, depending on "how infinite S is".

Let us first consider the case where $p > 1$. If we restrict the functional F to any subspace of L^p its norm can only decrease. Consider a subspace consisting of all functions which vanish outside a subset S_1 where $\mu(S_1) < \infty$. We get a corresponding function g_1 defined on S_1 (and set equal to zero off S_1 with $\|g_1\|_q \leq \|F\|_p$ and $F(f) = I(fg_1)$ for all f belonging to this subspace. If (S_2, g_2) is a second such pair, then the uniqueness part of the theorem shows that $g_1 = g_2$ almost everywhere on $S_1 \cap S_2$. Thus we can consistently define g_{12} on $S_1 \cup S_2$. Let

$$b := \mathrm{lub}\{\|g_\alpha\|_q\}$$

taken over all such g_α. Since this set of numbers is bounded by $\|F\|_p$ this least upper bound is finite. We can therefore find a nested sequence of sets S_n and corresponding functions g_n such that

$$\|g_n\|_q \nearrow b.$$

By the triangle inequality, if $n > m$ then

$$\|g_n - g_m\|_q \leq \|g_n\| - \|g_m\|_q$$

and so, as in your proof of the L^2 Martingale convergence theorem, this sequence is Cauchy in the $\|\cdot\|_q$ norm. Hence there is a limit $g \in L^q$ and g is supported on

$$S_0 := \bigcup S_n.$$

There can be no pair (S', g') with S disjoint from S_0 and $g' \neq 0$ on a subset of positive measure of S'. Indeed, if this were the case, then we could consider $g + g'$ on $S \cup S'$ and this would have a strictly larger $\|\cdot\|_q$ norm than $\|g\|_q = b$, contradicting the definition of b. (It is at this point in the argument that we use $q < \infty$ which is the same as $p > 1$.) Thus F vanishes on any function which is supported outside S_0. We have thus reduced the theorem to the case where S is σ-finite.

If S is σ-finite, decompose S into a disjoint union of sets A_i of finite measure. Let f_m denote the restriction of $f \in L^p$ to A_m and let h_m denote the restriction of g to A_m. Then

$$\sum_{m=1}^{\infty} f_m = f$$

as a convergent series in L^p and so

$$F(f) = \sum_m F(f_m) = \sum_m \int_{A_m} f_m h_m$$

and this last series converges to $I(fg)$ in L^1.

So we have proved that $(L^p)^* = L^q$ in complete generality when $p > 1$, and for σ-finite S when $p = 1$.

It may happen (and will happen when we consider the Haar integral on the most general locally compact group) that we don't even have σ-finiteness. But we will have the following more complicated condition: Recall that a set A is called **integrable** (by Loomis) if $\mathbf{1}_A \in \mathcal{B}$. Now suppose that

$$S = \bigcup_\alpha S_\alpha$$

where this union is disjoint, but possibly uncountable, of integrable sets, and with the property that every integrable set is contained in at most a countable union of the S_α. A set A is called **measurable** if the intersections $A \cap S_\alpha$ are all integrable, and a function is called **measurable** if its restriction to each S_α has the property that the restriction of f to each S_α belongs to $\mathcal{B}$, and further, that either the restriction of f^+ to every S_α or the restriction of f^- to every S_α belongs to L^1.

If we find ourselves in this situation, then we can find a g_α on each S_α since S_α is σ-finite, and piece these all together to get a g defined on all of S. If $f \in L^1$ then the set where $f \neq 0$ can have intersections with positive measure with only countably many of the S_α and so we can apply the result for the σ-finite case for $p = 1$ to this more general case as well.

8 Integration on locally compact Hausdorff spaces.

Suppose that S is a locally compact Hausdorff space. As in the case of $\mathbf{R}^n$, we can (and will) take L to be the space of continuous functions of compact support. Dini's lemma then says that if $f_n \in L \searrow 0$ then $f_n \to 0$ in the uniform topology.

If A is any subset of S we will denote the set of $f \in L$ whose support is contained in A by L_A.

Lemma 8.1 *A non-negative linear function I is bounded in the uniform norm on L_C whenever C is compact.*

Proof. Choose $g \geq 0 \in L$ so that $g(x) \geq 1$ for $x \in C$. If $f \in L_C$ then

$$|f| \leq \|f\|_\infty g$$

so

$$|I(f)| \leq I(|f|) \leq I(g) \cdot \|f\|_\infty. \text{ QED.}$$

8.1 Riesz representation theorems.

This is the same Riesz, but two more theorems.

Theorem 8.1 *Every non-negative linear functional I on L is an integral.*

Proof. This is Dini's lemma together with the preceding lemma. Indeed, by Dini we know that $f_n \in L \searrow 0$ implies that $\|f_n\|_\infty \searrow 0$. Since f_1 has compact support, let C be its support, a compact set. All the succeeding f_n are then also supported in C and so by the preceding lemma $I(f_n) \searrow 0$. QED

Theorem 8.2 *Let F be a bounded linear function on L (with respect to the uniform norm). Then there are two integrals I^+ and I^- such that*

$$F(f) = I^+(f) - I^-(f).$$

Proof. We apply Proposition 7.1 to the case of our L and with the uniform norm, $\|\cdot\|_\infty$. We get

$$F = F^+ - F^-$$

and an examination of he proof will show that in fact

$$\|F^\pm\|_\infty \leq \|F\|_\infty.$$

By the preceding theorem, $F^\pm$ are both integrals. QED

8.2 Fubini's theorem.

Theorem 8.3 *Let S_1 and S_2 be locally compact Hausdorff spaces and let I and J be non-negative linear functionals on $L(S_1)$ and $L(S_2)$ respectively. Then*

$$I_x(J_y h(x,y)) = J_y(I_x(h(x,y))$$

for every $h \in L(S_1 \times S_2)$ in the obvious notation, and this common value is an integral on $L(S_1 \times S_2)$.

Proof via Stone-Weierstrass. The equation in the theorem is clearly true if $h(x,y) = f(x)g(y)$ where $f \in L(S_1)$ and $g \in L(S_2)$ and so it is true for any h which can be written as a finite sum of such functions. Let h be a general element of $L(S_1 \times S_2)$. then we can find compact subsets $C_1 \subset S_1$ and $C_2 \subset S_2$ such that h is supported in the compact set $C_1 \times C_2$. The functions of the form

$$\sum f_i(x)g_i(y)$$

where the f_i are all supported in C_1 and the g_i in C_2, and the sum is finite, form an algebra which separates points. So for any $\epsilon > 0$ we can find a k of the above form with

$$\|h - k\|_\infty < \epsilon.$$

Let B_1 and B_2 be bounds for I on $L(C_1$ and J on $L(C_2$ as provided by Lemma 8.1. Then

$$|J_y h(x,y) - \sum J(g_i) f_i(x)| = |J_y(f - k)| < \epsilon B_2.$$

This shows that $J_y h(x,y)$ is the uniform limit of continuous functions supported in C_1 and so $J_y h(x,y)$ is itself continuous and supported in C_1. It then follows that $I_x(J_y(h)$ is defined, and that

$$|I_x(J_y h(x,y) - \sum I(f)_i J(g_i)| \leq \epsilon B_1 B_2.$$

Doing things in the reverse order shows that

$$I_x(J_y h(x,y)) - J_y(I_x(h(x,y))| \leq 2\epsilon B_1 B_2.$$

Since ϵ is arbitrary, this gives the equality in the theorem. Since this (same) functional is non-negative, it is an integral by the first of the Riesz representation theorems above. QED