Haar measure.

Math 212a

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A **topological group** is a group G which is also a topological space such that the maps

$$G \times G \to G, \quad (x, y) \mapsto xy$$

and

$$G \to G, \quad x \mapsto x^{-1}$$

are continuous. If the topology on G is locally compact and Hausdorff, we say that G is a locally compact, Hausdorff,topological group. If $a \in G$ is fixed, then the map ℓ_a

$$\ell_a : G \to G, \quad \ell_a(x) = ax$$

is the composite of the multiplication map $G \times G \to G$ and the continuous map

$$G \to G \times G, \quad x \mapsto (a, x).$$

So ℓ_a is continuous, one to one, and with inverse $\ell_{a^{-1}}$. If μ is a measure G, then we can push it forward by ℓ_a, that is, consider the pushed forward measure $(\ell_a)_*\mu$. We say that the measure μ is **left invariant** if

$$(\ell_a)_*\mu = \mu \quad \forall\, a \in G.$$

The basic theorem on the subject, proved by Haar in 1933 is

Theorem 1 *If G is a locally compact Hausdorff topological group there exists a non-zero regular Borel measure μ which is left invariant. Any other such measure differs from μ by multiplication by a positive constant.*

This handout is devoted to the proof of this theorem and the derivation of some of its consequences.

Contents

1 Examples.

1.1 $\mathbf{R}^n$.

$\mathbf{R}^n$ is a group under addition, and Lebesgue measure is clearly left invariant. Similarly $\mathbf{T}^n$.

1.2 Discrete groups.

If G has the discrete topology then the counting measure which assigns the value one to every one element set $\{x\}$ is Haar measure.

1.3 Lie groups.

We can reformulate the condition of left invariance as follows: Let I denote the integral associated to the measure μ:

$$I(f) = \int f d\mu.$$

Then

$$\int f d(\ell_a)_* \mu = I(\ell_a^* f)$$

where

$$(\ell_a^* f)(x) = f(ax). \tag{1}$$

Indeed, this is most easily checked on indicator functions of sets, where

$$(\ell_a)^*\mathbf{1}_A = \mathbf{1}_{\ell_a^{-1}A}$$

and

$$\int \mathbf{1}_A d(\ell_a)_*\mu = ((\ell_a)_*\mu)(A) := \mu(\ell_a^{-1}A) = \int \mathbf{1}_{\ell_a^{-1}A} d\mu.$$

So the left invariance condition is

$$I(\ell_a^* f) = I(f) \quad \forall\ a \in G. \tag{2}$$

Suppose that G is a differentiable manifold and that the multiplication map $G \times G \to G$ and the inverse map $x \mapsto x^{-1}$ are differentiable.

Now if G is n-dimensional, and we could find an n-form Ω which does not vanish anywhere, and such that

$$\ell_a^*\Omega = \Omega$$

(in the sense of pull-back on forms) then we can choose an orientation relative to which Ω becomes identified with a density, and then

$$I(f) = \int_G f\Omega$$

is the desired integral. Indeed,

$$\begin{aligned} I((\ell_a)^*f) &= \int ((\ell_a)^*f)\Omega \\ &= \int ((\ell_a)^*f)((\ell_a)^*\Omega) \quad \text{since } ((\ell_a)^*\Omega) = \Omega \\ &= \int (\ell_a)^*(f\Omega) \\ &= \int f\Omega \\ &= I(f). \end{aligned}$$

We shall replace the problem of finding a left invariant n-form Ω by the apparently harder looking problem of finding n left invariant one-forms $\omega_1, \dots, \omega_n$ on G and then setting

$$\Omega := \omega_1 \wedge \cdots \wedge \omega_n.$$

The general theory of Lie groups says that such one forms (the Maurer-Cartan forms) always exist, but I want to sow how to compute them in important special cases.

Suppose that we can find a homomorphism

$$M : G \to Gl(d)$$

where $Gl(d)$ is the group of $d \times d$ invertible matrices (either real or complex). So M is a matrix valued function on G satisfying

$$M(e) = Id$$

where e is the identity element of G and Id is the identity matrix, and

$$M(xy) = M(x)M(y)$$

where multiplication on the left is group multiplication and multiplication on the right is matrix multiplication. We can think of M as a matrix valued function or as a matrix $M(x) = (M_{ij}(x))$ of real (or complex) valued functions. Suppose that all of these functions are differentiable. Then we can form

$$dM := (dM_{ij})$$

which is a matrix of linear differential forms on G, or, equivalently, a matrix valued linear differential form on G.

Finally, consider

$$M^{-1}dM.$$

Again, this is a matrix valued linear differential form on G (or what is the same thing a matrix of linear differential forms on G). Explicitly it is the matrix whose ik entry is

$$\sum_j (M(x))-1)_{ij} dM_{jk}.$$

I claim that every entry of this matrix is a left invariant linear differential form. Indeed,

$$(\ell_a^* M)(x) = M(ax) = M(a)M(x).$$

Let us write

$$A = M(a).$$

Since a is fixed, A is a constant matrix, and so

$$(\ell_a^* M)^{-1} = (AM)^{-1} = M^{-1}A^{-1}$$

while

$$\ell_a^* dM = d(AM) = AdM$$

since A is a constant. So

$$\ell_a^*(M^{-1}dM) = (M^{-1}A^{-1}AdM) = M^{-1}dM.$$

Of course, if the size of M is too small, there might not be enough linearly independent entries. (In the complex case we want to be able to choose the real and imaginary parts of these entries to be linearly independent.) But if, for example, the map $x \mapsto M(x)$ is an immersion, then there will be enough linearly independent entries to go around.

For example, consider the group of all two by two matrices of the form

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}.$$

This group is sometimes known as the "$ax + b$ group" since

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ 1 \end{pmatrix} = \begin{pmatrix} ax+b \\ 1 \end{pmatrix}.$$

In other words, G is the group of all translations and rescalings of the real line.

We have

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} a^{-1} & -a^{-1}b \\ 0 & 1 \end{pmatrix}$$

and

$$d\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} da & db \\ 0 & 0 \end{pmatrix}$$

so

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}^{-1} d\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a^{-1}da & a^{-1}db \\ 0 & 1 \end{pmatrix}$$

and the Haar measure is (proportional to)

$$\frac{dadb}{a^2} \tag{3}$$

As a second example, consider the group $SU(2)$ of all unitary two by two matrices with determinant one. Each column of a unitary matrix is a unit vector, and the columns are orthogonal. We can write the first column of the matrix as

$$\begin{pmatrix} \overline{\alpha} \\ \overline{\beta} \end{pmatrix}$$

where α and β are complex numbers with

$$|\alpha|^2 + |\beta|^2 = 1. \tag{4}$$

The second column must then be proportional to

$$\begin{pmatrix} -\beta \\ \alpha \end{pmatrix}$$

and the condition that the determinant be one fixes this constant of proportionality to be one. So we can write

$$M = \begin{pmatrix} \overline{\alpha} & -\beta \\ \overline{\beta} & \alpha \end{pmatrix}$$

where (4) is satisfied. So we can think of M as a complex matrix valued function on the group $SU(2)$. Since M is unitary, $M^{-1} = M^*$ so

$$M^{-1} = \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix}$$

and

$$M^{-1}dM = \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} \begin{pmatrix} d\overline{\alpha} & -d\beta \\ d\overline{\beta} & d\alpha \end{pmatrix} = \begin{pmatrix} \alpha d\overline{\alpha} + \beta d\overline{\beta} & -\alpha d\beta + \beta d\alpha \\ -\overline{\beta} d\overline{\alpha} + \overline{\alpha} d\overline{\beta} & \overline{\alpha} d\alpha + \overline{\beta} d\beta \end{pmatrix}.$$

Each of the real and imaginary parts of the entries is a left invariant one form. But let us multiply three of these entries directly:

$$\begin{aligned}
&(\overline{\alpha} d\alpha + \overline{\beta} d\beta) \wedge (-\alpha d\beta + \beta d\alpha) \wedge (-\overline{\beta} d\overline{\alpha} + \overline{\alpha} d\overline{\beta}) \\
&= -(|\alpha|^2 + |\beta|^2) d\alpha \wedge d\beta \wedge (-\overline{\beta} d\overline{\alpha} + \overline{\alpha} d\overline{\beta}) \\
&\quad -d\alpha \wedge d\beta \wedge (-\overline{\beta} d\overline{\alpha} + \overline{\alpha} d\overline{\beta}).
\end{aligned}$$

We can simplify this expression by differentiating the equation

$$\alpha\overline{\alpha} + \beta\overline{\beta} = 1$$

to get

$$\alpha d\overline{\alpha} + \overline{\alpha} d\alpha + \beta d\overline{\beta} + \overline{\beta} d\beta = 0.$$

So for $\beta \neq 0$ we can solve for $d\overline{\beta}$:

$$d\overline{\beta} = -\frac{1}{\beta}(\alpha d\overline{\alpha} + \overline{\alpha} d\alpha + \overline{\beta} d\beta).$$

When we multiply by $d\alpha \wedge d\beta$ the terms involving $d\alpha$ and $d\beta$ disappear. We thus get

$$-d\alpha \wedge d\beta \wedge (-\overline{\beta} d\overline{\alpha} + \overline{\alpha} d\overline{\beta}) = d\alpha \wedge d\beta \wedge (\overline{\beta} d\overline{\alpha} + \frac{\overline{\alpha}\alpha}{\beta} d\overline{\alpha}).$$

If we write

$$\overline{\beta} d\overline{\alpha} = \frac{\beta\overline{\beta}}{\beta} d\alpha$$

and use $|\alpha|^2 + |\beta|^2 = 1$ the above expression simplifies further to

$$\frac{1}{\beta} d\alpha \wedge d\beta \wedge d\overline{\alpha} \tag{5}$$

as a left invariant three form on $SU(2)$. You might think that this three form is complex valued, but we shall now give an alternative expression for it which will show that it is in fact real valued.

For this introduce polar coordinates in four dimensions as follows: Write

$$\begin{aligned}
\alpha &= w + iz \\
\beta &= x + iy \quad \text{so } x^2 + y^2 + z^2 + w^2 = 1, \\
w &= \cos\theta \\
z &= \sin\theta \ \cos\psi \\
x &= \sin\theta \sin\psi \cos\phi \\
y &= \sin\theta \sin\psi \sin\phi
\end{aligned}$$

$$0 \le \theta \le \pi, \ 0 \le \psi \le \pi, \ 0 \le \phi \le 2\pi.$$

Then

$$d\alpha \wedge d\overline{\alpha} = (dw + idz) \wedge (dw - idz) = -2idw \wedge dz$$

$$= -2id(\cos\theta) \wedge d(\sin\theta \cdot \cos\psi) = -2i \sin^2\theta d\theta \wedge d\psi.$$

Now

$$\beta = \sin\theta \sin\psi e^{i\phi}$$

so

$$d\beta = i\beta d\phi + \cdots$$

where the missing terms involve $d\theta$ and $d\psi$ and so will disappear when multiplied by $d\alpha \wedge d\overline{\alpha}$. Hence

$$d\alpha \wedge d\beta \wedge d\overline{\alpha} = -2\beta \sin^2\theta \sin\psi d\theta \wedge d\psi \wedge d\phi.$$

Finally, we see that the three form (5) when expressed in polar coordinates is

$$-2\sin^2\theta \sin\psi d\theta \wedge d\psi \wedge d\phi.$$

Of course we can multiply this by any constant. If we normalize so that $\mu(G) = 1$ the Haar measure is

$$\frac{1}{2\pi^2}\sin^2\theta \sin\psi d\theta d\psi d\phi.$$

2 Topological facts.

Since ℓ_a is a homeomorphism, if V is a neighborhood of the identity element e, then aV is a neighborhood of a, and if U is a neighborhood of U then $a^{-1}U$ is a neighborhood of e. Here we are using the obvious notation $aV = \ell_a(V)$ etc.

Suppose that U is a neighborhood of e. Then so is

$$U^{-1} := \{x^{-1} : \ x \in U\}$$

and hence so is

$$W = U \cap U^{-1}.$$

But

$$W^{-1} = W.$$

Proposition 1 *Every neighborhood of e contains a symmetric neighborhood, i.e. one that satisfies* $W^{-1} = W$.

Let U be a neighborhood of e. The inverse image of U under the multiplication map $G \times G \to G$ is a neighborhood of (e, e) in $G \times G$ and hence contains an open set of the form $V \times V$. Hence

Proposition 2 *Every neighborhood U of e contains a neighborhood V of e such that* $V^2 = V \cdot V$.

Here we are using the notation

$$A \cdot B = \{xy : \ x \in A, \ y \in B\}$$

where A and B are subsets of G.

If A and B are compact, so is $A \times B$ as a subset of $G \times G$, and since the image of a compact set under a continuous map is compact, we have

Proposition 3 *If A and B are compact, so is* $A \cdot B$.

Proposition 4 *If* $A \subset G$ *then* $\overline{A}$, *the closure of A, is given by*

$$\overline{A} = \bigcap_V AV$$

where V ranges over all neighborhoods of e.

Proof. If $a \in \overline{A}$ and V is a neighborhood of e, then aV^{-1} is an open set containing a, and hence containing a point of A. So $a \in AV$, and the left hand side of the equation in the proposition is contained in the right hand side. To show the reverse inclusion, suppose that x belongs to the right hand side. Then xV^{-1} intersects A for every V. But the sets xV^{-1} range over all neighborhoods of x. So $x \in \overline{A}$. QED

Recall that (following Loomis as we are) L denotes the space of continuous functions of compact support on G.

Proposition 5 *Suppose that G is locally compact. If* $f \in L$ *then f is uniformly left (and right) continuous. That is, given* $\epsilon > 0$ *there is a neighborhood V of e such that*

$$s \in V \ \Rightarrow \ |f(sx) - f(x)| < \epsilon.$$

Equivalently, this says that

$$xy^{-1} \in V \ \Rightarrow |f(x) - f(y)| < \epsilon.$$

Proof. Let

$$C := \mathrm{Supp}(f)$$

and let U be a symmetric compact neighborhood of e. Consider the set W of points s such that

$$|f(sx) - f(x)| < \epsilon \ \ \forall \, x \in UC.$$

I claime that this is an open neighborhood of e. Indeed, for each fixed $y \in UC$ the set of s satisfying this condition at y is an open neighborhood W_y of e, and this W_y works in some neighborhood O_y of y. Since UC is compact, finitely many of these O_y cover UC, and hence the intersection of the corresponding W_y form an open neighborhood W of e. Now take

$$V := U \cap W.$$

If $s \in V$ and $x \in UC$ then $|f(sx) - f(x)| < \epsilon$. If $x \notin UC$, then $sx \notin C$ (since we chose U to be symmetric) and $x \notin C$, so $f(sx) = 0$ and $f(x) = 0$, so $|f(sx) - f(x)| = 0 < \epsilon$. QED

In the construction of the Haar integral, we will need this proposition. So it is exactly at this point where the assumption that G is locally compact comes in.

3 Construction of the Haar integral.

Let f and g be non-zero elements of L^+ and let m_f and m_g be their respective maxima. At each

$$x \in \mathrm{Supp}(f),$$

we have

$$f(x) \leq \frac{m_f}{m_g} m_g$$

so if

$$c > \frac{m_f}{m_g}$$

and s is chosen so that g achieves its maximum at sx, then

$$f(y) \leq cg(sx)$$

in a neighborhood of x. Since $\mathrm{Supp}(f)$ is compact, we can cover it by finitely many such neighborhoods, so that there exist finitely many c_i and s_i such that

$$f(x) \leq \sum c_i g(s_i x) \quad \forall\, x. \tag{6}$$

If we choose x so that $f(x) = m_f$, then the right hand side is at most $\sum_i c_i m_g$ and thus we see that

$$\sum_i c_i \geq m_f/m_g > 0.$$

So let us define the "size of f relative to g" by

$$(f;g) := \mathrm{g.l.b.}\{\sum c_i : \exists s_i \text{ such that (6) holds}\}. \tag{7}$$

We have verified that

$$(f;g) \geq \frac{m_f}{m_g}. \tag{8}$$

It is clear that

$$\begin{aligned}
(\ell_a^* f; g) &= (f; g) \;\; \forall \, a \in G & (9)\\
(f_1 + f_2; g) &\leq (f_1, g) + (f_2; g) & (10)\\
(cf; g) &= c(f; g) \;\; \forall \, c > 0 & (11)\\
f_1 \leq f_2 &\Rightarrow (f_1; g) \leq (f_2; g). & (12)
\end{aligned}$$

If $f(x) \leq \sum c_i g(s_i x)$ for all x and $g(y) \leq \sum d_j g(t_j y)$ for all y then

$$f(x) \leq \sum_{ij} c_i d_j h(t_j s_i x) \;\forall\, x.$$

Taking greatest lower bounds gives

$$(f; h) \leq (f; g)(g; h). \tag{13}$$

To normalize our integral, fix some

$$f_0 \in L^+, \quad f_0 \neq 0.$$

Define

$$I_g(f) := \frac{(f; g)}{(f_0; g)}.$$

Since, according to (13) we have

$$(f_0; g) \leq (f_0; f)(f, g),$$

we see that

$$\frac{1}{(f_0; f)} \leq I_g(f).$$

Since (13) says that $(f; g) \leq (f; f_0)(f_0, g)$ we see that

$$I_g(f) \leq (f; f_0).$$

So for each non-zero $f \in L^+$ let S_f denote the closed interval

$$S_f := \left[\frac{1}{(f_0, f)}, (f, f_0)\right],$$

and let

$$S := \prod_{f \in L^+, f \neq 0} S_f.$$

This space is compact by Tychonoff. Each non-zero $g \in L^+$ determines a point $I_g \in S$ whose coordinate in S_f is $I_g(f)$.

For any neighborhood V of e, let C_V denote the closure in S of the set $I_g, g \in V$. We have

$$C_{V_1} \cap \cdots \cap C_{V_n} = C_{V_1 \cap \cdots \cap V_n} \neq \emptyset.$$

The C_V are all compact, and so there is a point I in the intersection of all the C_V:

$$I \in C := \bigcap_V C_V.$$

The idea is that I somehow is the limit of the I_g as we restrict the support of g to lie in smaller and smaller neighborhoods of the identity. We shall prove that as we make these neighborhoods smaller and small, the I_g are closer and closer to being additive, and so their limit I satisfies the conditions for being an invariant integral. Here are the details:

Lemma 1 *Given f_1 and $f_2$in L^+ and $\epsilon > 0$ there exists a neighborhood V of e such that*

$$I_g(f_1) + I_g(f_2) \leq I_g(f_1 + f_2) + \epsilon$$

for all g with $\mathrm{Supp}(g) \subset V$.

Proof. Choose $\phi \in L$ such that $\phi = 1$ on $\mathrm{Supp}(f_1 + f_2)$. For a given $\delta > 0$ to be chosen later, let

$$f := f_1 + f_2 + \delta\phi, \quad h_1 := \frac{f_1}{f}, \quad h_2 := \frac{f_2}{f}.$$

Here h_1 and h_2 were defined on $\mathrm{Supp}(f)$ and vanish outside $\mathrm{Supp}(f_1 + f_2)$, so extend them to be zero outside $\mathrm{Supp}(\phi)$. For an $\eta > 0$ and $\delta > 0$ to be chosen later, find a neighborhood $V = V_{\delta,\eta}$ so that

$$|h_1(x) - h_1(y)| < \eta \text{ and } |h_2(x) - h_2(y)| < \eta$$

when $x^{-1}y \in V$ which is possible by Prop. 5 if G is locally compact.

Let g be a non-zero element of L^+ with $\mathrm{Supp}(g) \subset V$. If

$$f(x) \leq \sum c_j g(s_j x)$$

then $g(s_j x) \neq 0$ implies that

$$|h_i(x) - h_i(s_j^{-1})| < \eta, \quad i = 1, 2$$

so

$$f_i(x) = f(x)h_i(x) \leq \sum c_j g(s_j x)h_i(x) \leq \sum c_j g(s_j x)[h_i(s_j^{-1}) + \epsilon], \quad i = 1, 2.$$

This implies that

$$(f_i; g) \leq \sum_j [h_i(s_j^{-1}) + \eta]$$

and since $0 \leq h_i \leq 1$ by definition,

$$(f_1; g) + (f_2; g) \leq \sum c_j [1 + 2\eta].$$

We can choose the c_j and s_j so that $\sum c_j$ is as close as we like to $(f;g)$. Hence

$$(f_1,g)+(f_2;g)\leq(f;g)[1+2\eta].$$

Dividing by $(f_0;g)$ gives

$$I_g(f_1)+I_g(f_2)\leq I_g(f)[1+2\eta]$$

$$\leq[I_g(f_1+f_2)+\delta I_g(\phi)][1+2\eta],$$

where, in going from the second to the third inequality we have used the definition of f, (10) applied to (f_1+f_2) and $\delta\phi$ and (11). Now

$$I_g(f_1+f_2)\leq(f_1+f_2;f_0)$$

and $I_g(\phi)\leq(\phi,f_0)$. So choose δ and η so that

$$2\eta(f_1+f_2;f_0)+\delta(1+2\eta)(\phi;f_0)<\epsilon.$$

This completes the proof of the lemma.

For any finite number of $f_i\in L^+$ and any neighborhood V of the identity, there is a non-zero g with $\mathrm{Supp}(g)\in V$ and

$$|I(f_i)-I_g(f_i)|<\epsilon,\quad i=1,\ldots,n.$$

Applying this to f_1,f_2 and $f_3=f_1+f_2$ and the V supplied by the lemma, we get

$$I(f_1+f_2)-\epsilon\leq I_g(f_1+f_2)\leq I_g(f_1)+I_g(f_2)\leq I(f_1)+I(f_2)+2\epsilon$$

and

$$I(f_1)+I(f_2)\leq I_g(f_1)+I_g(f_2)+2\epsilon\leq I_g(f_1+f_2)+3\epsilon\leq I(f_1+f_2)+4\epsilon.$$

In short, I satisfies

$$I(f_1+f_2)=I(f_1)+I(f_2)$$

for all f_1,f_2 in L^+, is left invariant, and $I(cf)=cI(f)$ for $c\geq 0$. As usual, extend I to all of L by

$$I(f_1-f_2)=I(f_1)-I(f_2)$$

and this is well defined.

Since for $f\in L^+$ we have

$$I(f)\leq(f;f_0)\leq m_f/m_{f_0}=\|f\|_\infty/m_{f_0}$$

we see that I is bounded in the sup norm. So it is an integral (by Dini's lemma). Hence, by the Riesz representation theorem, if G is Hausdorff, we get a regular left invariant Borel measure. This completes the existence part of the main theorem.

From the fact that μ is regular, and not the zero measure, we conclude that there is some compact set K with $\mu(K) > 0$. Let U be any non-empty open set. The translates xU, $x \in K$ cover K, and since K is compact, a finite number, say n of them, cover K. But they all have the same measure, $\mu(U)$ since μ is left invariant. Thus

$$\mu(K) \leq n\mu(U)$$

implying

$$\mu(U) > 0 \text{ for any non-empty open set } U \tag{14}$$

if μ is a left invariant regular Borel measure.

If $f \in L^+$ and $f \neq 0$, then $f > \epsilon > 0$ on some non-empty open set U, and hence its integral is $> \epsilon\mu(U)$. So

$$f \in L^+,\ f \neq 0 \ \Rightarrow \int f d\mu > 0 \tag{15}$$

for any left invariant regular Borel measure μ.

4 Uniqueness.

Let μ and ν be two left invariant regular Borel measures on G. Pick some $g \in L^+, g \neq 0$ so that both $\int g d\mu$ and $\int g d\nu$ are positive. We are going to use Fubini to prove that for any $f \in L$ we have

$$\frac{\int f d\nu}{\int g d\nu} = \frac{\int f d\mu}{\int g d\mu}. \tag{16}$$

This clearly implies that $\nu = c\mu$ where

$$c = \frac{\int g d\nu}{\int g d\mu}.$$

To prove (16), it is enough to show that the left hand side can be expressed in terms of any left invariant regular Borel measure (say ν) because this implies that both sides do not depend on the choice of Haar measure. Define

$$h(x,y) := \frac{f(x)g(yx)}{\int g(tx)d\nu(t)}.$$

The integral in the denominator is positive for all x and by the left uniform continuity the integral is a continuous function of x. Thus h is continuous function of compact support in (x,y) so by Fubini,

$$\int\int h(x,y)d\nu(y)d\mu(x) = \int\int h(x,y)d\mu(x)d\nu(y).$$

In the inner integral on the right replace x by $y^{-1}x$ using the left invariance of μ. The right hand side becomes

$$\int h(y^{-1}x, y)d\mu(x)d\nu(y).$$

Use Fubini again so that this becomes

$$\int h(y^{-1}x, y)d\nu(y)d\mu(x).$$

Now use the left invariance of ν to replace y by xy. This last iterated integral becomes

$$\int h(y^{-1}, xy)d\nu(y)d\mu(x).$$

So we have

$$\int\int h(x,y)d\nu(y)d\mu(x) = \int h(y^{-1}, xy)d\nu(y)d\mu(x).$$

From the definition of h the left hand side is $\int f(x)d\mu(x)$. For the right hand side

$$h(y^{-1}, xy) = f(y^{-1})\frac{g(x)}{\int g(ty^{-1})d\nu(t)}.$$

Integrating this first with respect to $d\nu(y)$ gives

$$kg(x)$$

where k is the constant

$$k = \int \frac{f(y^{-1})}{\int g(ty^{-1})d\nu(t)}d\nu(y).$$

Now integrate with respect to μ. We get $\int fd\mu = k\int gd\mu$ so

$$\frac{\int fd\mu}{\int gd\mu},$$

the right hand side of (16), does not depend on μ, since it equals k which is expressed in terms of ν. QED

5 $\mu(G) < \infty$ if and only if G is compact.

Since μ is regular, the measure of any compact set is finite, so if G is compact then $\mu(G) < \infty$. We want to prove the converse. Let U be an open neighborhood of e with compact closure, K. So $\mu(K) > 0$. The fact that $\mu(G) < \infty$ implies that one can not have m disjoint sets of the form x_iK if

$$m > \frac{\mu(G)}{\mu(K)}.$$

Let n be such that we can find n disjoint sets of the form $x_i K$ but no $n+1$ disjoint sets of this form. This says that for any $x \in G$, xK can not be disjoint from all the $x_i K$. Thus

$$G = \left(\bigcup_i x_i K \right) \cdot K^{-1}$$

which is compact. QED

If G is compact, the Haar measure is usually normalized so that $\mu(G) = 1$.

6 The group algebra.

If $f, g \in L$ define their **convolution** by

$$(f \star g)(x) := \int f(xy) g(y^{-1}) d\mu(y), \tag{17}$$

where we have fixed, once and for all, a (left) Haar measure μ. The left invariance (under left multiplication by x^{-1}) implies that

$$f \star g = \int f(y) g(y^{-1}x) d\mu(y). \tag{18}$$

In what follows we will write dy instead of $d\mu(y)$ since we have chosen a fixed aar measure μ.

If $A := \mathrm{Supp}(f)$ and $B := \mathrm{Supp}(g)$ then $f(y)g(y^{-1}x)$ is continuous as a function of y for each fixed x and vanishes unless $y \in A$ and $y^{-1}x \in B$. Thus $f \star g$ vanishes unless $x \in AB$. Also

$$|f \star g(x_1) - f \star g(x_2) \leq \|\ell^*_{x_1} f - \ell^*_{x_2}\|_\infty \int |g(y^{-1})| dy.$$

Since $x \mapsto \ell^*_x f$ is continuous in the uniform norm, we conclude that

$$f, g \in L \Rightarrow f \star g \in L$$

and

$$\mathrm{Supp}(f \star g) \subset (\mathrm{Supp}(f)) \cdot (\mathrm{Supp}(g)). \tag{19}$$

I claim that we have the associative law: If, $f, g, h \in L$ then

$$((f \star g) \star h) = (f \star (g \star h) \tag{20}$$

Indeed, using the left invariance of the Haar measure and Fubini we have

$$\begin{aligned}((f\star g)\star h)(x) &:= \int (f\star g)(xy)h(y^{-1})dy\\ &= \int\int f(xyz)g(z^{-1})h(y^{-1})dzdy\\ &= \int\int f(xz)g(z^{-1}y)h(y^{-1})dzdy\\ &= \int\int f(xz)g(z^{-1}y)h(y^{-1})dydz\\ &= \int f(xz)(g\star h)(z^{-1})dz\\ &= (f\star(g\star h))(x).\end{aligned}$$

It is easy to check that $\star$ is commutative if and only if G is commutative.

I now want to extend the definition of $\star$ to all of L^1, and here I will follow Loomis and restrict our definition of L^1 so that our integrable functions belong to $\mathcal{B}$, the smallest monotone class containing L. When we were doing the Wiener integral, we needed all Borel sets. Here it is more convenient to operate with this smaller class, for technical reasons which will be practically invisible. For most groups one encounters in real life there is no difference between the Borel sets and the Baire sets. For example, if the Haar measure is σ-finite one can forget about these considerations.

If f and g are functions on G define the function $f\bullet g$ on $G\times G$ by

$$(f\bullet g)(x,y) := f(y)g(y^{-1}x).$$

Theorem 2 **[31A in Loomis]** *If $f,g\in\mathcal{B}^+$ then $f\bullet g\in\mathcal{B}^+(G\times G)$ and*

$$\|f\star g\|_p \le \|f\|_1\|g\|_p \tag{21}$$

for any p with $1\le p\le\infty$.

Proof. If $f\in L^+$ then the set of $g\in\mathcal{B}^+$ such that $f\bullet g\in\mathcal{B}^+(G\times G)$ is L-monotone and includes L^+, so includes $\mathcal{B}^+$. So if g is an L-bounded function in $\mathcal{B}^+$, the set of $f\in\mathcal{B}^+$ such that $f\bullet g\in\mathcal{B}^+(G\times G)$ includes L^+ and is L-monotone, and so includes $\mathcal{B}^+$. So $f\bullet g\in\mathcal{B}^+(G\times G)$ whenever f and g are L-bounded elements of $\mathcal{B}^+$. But the most general element of $\mathcal{B}^+$ can be written as the limit of an increasing sequence of L bounded elements of $\mathcal{B}^+$, and so the first assertion in the theorem follows.

As f and g are non-negative, Fubini asserts that the function $y\mapsto f(y)g(y^{-1}x)$ is integrable for each fixed x, that $f\star g$ is integrable as a function of x and that

$$\|f\star g\|_1 = \int\int f(y)g(y^{-1}x)dxdy = \|f\|_1\|g\|_1.$$

This proves (21) (with equality) for $p=1$. For $p=\infty$ (21) is obvious from the definitions.

For $1 < p < \infty$ we will use Hölder's inequality and the duality between L^p and L^2. We know that the product of two elements of $\mathcal{B}^+(G \times G)$ is an element of $\mathcal{B}^+(G \times G)$. So if $f, g, h \in \mathcal{B}^+(G)$ then the function $(x, y) \mapsto f(y)g(y^{-1}x)h(x)$ is an element of $\mathcal{B}^+(G \times G)$, and by Fubini

$$(f \star g, h) = \int f(y) \left[\int g(y^{-1}x)h(x)dx \right] dy.$$

We may apply Hölder's inequality to estimate the inner integral by $\|g\|_p \|h\|_q$. So for general $h \in L^q$ we have

$$|(f \star g, h)| \le \|f\|_1 \|g\|_p \|h\|_q.$$

If $\|f\|_1$ or $\|g\|_p$ are infinite, then (21) is trivial. If both are finite, then

$$h \mapsto (f \star g, h)$$

is a bounded linear functional on L^q and so by the isomorphism $L^p = (L^q)^*$ we conclude that $f \star g \in L^p$ and that (21) holds. QED

We define a **Banach algebra** to be an associative algebra (possibly without a unit element) which is also a Banach space, and such that

$$\|fg\| \le \|f\| \|g\|.$$

So the special case $p = 1$ of Theorem 21 asserts that $L^1(G)$ is a Banach algebra.

7 The involution.

7.1 The modular function.

Instead of considering the action of G on itself by left multiplication, $a \mapsto \ell_a$ we can consider right multiplication, r_b where

$$r_b(x) = xb^{-1}.$$

It is one of the basic principles of mathematics, that on account of the associative law, right and left multiplication commute:

$$\ell_a \circ r_b = r_b \circ \ell_a. \tag{22}$$

Indeed, both sides send $x \in G$ to

$$axb^{-1}.$$

If μ is a choice of left Haar measure, then it follows from (22) that $r_{b*}\mu$ is another choice of Haar measure, and so must be some positive multiple of μ. The function Δ on G defined by

$$r_{b*}\mu = \Delta(b)\mu$$

is called the **modular function** of G. It is immediate that this definition does not depend on the choice of μ.

Example. In the case that we are dealing with manifolds and integration of n-forms,

$$I(f) = \int f\Omega,$$

then the push-forward of the measure associated to I under a diffeomorphism ϕ assigns to any function f the integral

$$I(\phi^* f) = \int (\phi^* f)\Omega = \int \phi^*(f(\phi^{-1})^*\Omega) = \int f(\phi^{-1})^*\Omega.$$

So the push forward measure corresponds to the form

$$(\phi^{-1})^*\Omega.$$

Thus in computing $\Delta(z)$ using differential forms, we have to compute the pull-back under right multiplication by z, not z^{-1}. For example, in the $ax+b$ group, we have

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} ax & ay+b \\ 0 & 1 \end{pmatrix}$$

so

$$\frac{da \wedge db}{a^2} \mapsto \frac{x da \wedge db}{x^2 a^2}$$

and hence the modular function is given by

$$\Delta\left(\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}\right) = \frac{1}{x}.$$

In all cases the modular function is continuous (as follows from the uniform right continuity, Proposition, 5), and from its definition, it follows that

$$\Delta(st) = \Delta(s)\Delta(t).$$

In other words, Δ is a continuous homomorphism from G to the multiplicative group of positive real numbers.

The group G is called **unimodular** if $\Delta \equiv 1$. For example, a commutative group is obviously unimodular. Also, a compact group is unimodular, because G has finite measure, and is carried into itself by right multiplication so

$$\Delta(s)\mu(G) = (r_{s*}(\mu)(G) = \mu(r_s^{-1}(G)) = \mu(G).$$

7.2 Definition of the involution.

For any complex valued continuous function f of compact support define $\tilde{f}$ by

$$\tilde{f}(x) := \overline{f(x^{-1})}\Delta(x^{-1}). \tag{23}$$

It follows immediately from the definition that

$$(\tilde{f})\,\tilde{} = f.$$

That is, applying $\tilde{}$ twice is the identity transformation. Also,

$$(r_s(\tilde{f}))(x) = \overline{f(sx^{-1})}\Delta(x^{-1})\Delta(s)$$

so

$$r_s\tilde{f} = \Delta(s)(\ell_s f)\tilde{}. \tag{24}$$

Similarly,

$$(r_s f)\tilde{}(x) = \overline{f(x^{-1}s^{-1})}\Delta(x^{-1}) = \Delta(s)\ell_s(\tilde{f})$$

or

$$(r_s f)\tilde{} = \Delta(s)\ell_s\tilde{f}. \tag{25}$$

Suppose that f is real valued, and consider the functional

$$J(f) := I(\tilde{f}).$$

Then from (24) and the definition of Δ we have

$$J(\ell_s f) = \Delta(s^{-1})I(r_s\tilde{f}) = \Delta(s^{-1})\Delta(s)I(\tilde{f}) = I(\tilde{f}) = J(f).$$

In other words, J is a left invariant integral on real valued functions, and hence must be some constant multiple of I,

$$J = cI.$$

Let V be a symmetric neighborhood of e chosen so small that $|1 - \Delta(s)| < \epsilon$ which is possible for any given $\epsilon > 0$, since $\Delta(e) = 1$ and Δ is continuous. If we take $f = \mathbf{1}_V$ then $f(x) = f(x^{-1})$ and

$$|J(f) - I(f)| \leq \epsilon I(f).$$

Dividing by $I(f)$ shows that $|c - 1| < \epsilon$, and since ϵ is arbitrary, we have proved that

$$I(\tilde{f}) = \overline{I(f)}. \tag{26}$$

We can derive two immediate consequences:

Proposition 6 *Haar measure is inverse invariant if and only if G is unimodular,*

and

Proposition 7 *The involution $f \mapsto \tilde{f}$ extends to an anti-linear isometry of $L^1_{\mathbf{C}}$.*

7.3 Relation to convolution.

We claim that

$$(f \star g)\tilde{} = \tilde{g} \star \tilde{f} \tag{27}$$

Proof.

$$\begin{aligned}
(f \star g)\tilde{}(x) &= \int \overline{f(x^{-1}y)g(y^{-1})} dy \Delta(x^{-1}) \\
&= \int \overline{g(y^{-1})} \Delta(y^{-1}) \overline{f((y^{-1}x)^{-1}) \Delta((y^{-1}x)^{-1})} dy \\
&= (\tilde{g} \star \tilde{f})(x). \text{ QED}
\end{aligned}$$

7.4 Banach algebras with involutions.

For a general Banach algebra B (over the complex numbers) a map

$$x \mapsto x^\dagger$$

is called an involution if it is antilinear and anti-multiplicative, i.e. satisfies

$$(xy)^\dagger = y^\dagger x^\dagger.$$

Thus the map $f \mapsto \tilde{f}$ is an involution on $L^1(G)$.

8 The algebra of finite measures.

In general, the algebra $L^1(G)$ will not have an identity element, since the only candidate for the identity element would be the δ-"function"

$$\langle \delta, f \rangle = f(e),$$

and this will not be an honest function unless the topology of G is discrete.

So we need to introduce a different algebra if we want to have an algebra with identity. If G were a Lie group we could consider the algebra of all distributions. For a general locally compact Hausdorff group we can proceed as follows: Let $\mathcal{M}(G)$ denote the space of all finite complex measures on G: A non-negative measure ν is called **finite** if $\mu(G) < \infty$. A real valued measure is called finite if its positive and negative parts are finite, and a complex valued measure is called finite if its real and imaginary parts are finite.

Given two finite measures μ and ν on G we can form the product measure $\mu \otimes \nu$ on $G \times G$ and then push this measure forward under the multiplication map

$$m : G \times G \to G,$$

and so define their convolution by

$$\mu \star \nu := m_*(\mu \otimes \nu).$$

One checks that the convolution of two regular Borel measures is again a regular Borel measure, and that on measures which are absolutely continuous with respect to Haar measure, this coincides with the convolution as previously defined. One can also make the algebra of regular finite Borel measures under convolution into a Banach algebra (under the "total variation norm"). This algebra does include the δ-function (which is a measure!) and so has an identity. I will not go into this matter here except to make a number of vague but important points.

8.1 Algebras and coalgebras.

An algebra A is a vector space (over the complex numbers) together with a map

$$m : A \otimes A \to A$$

which is subject to various conditions (perhaps the associative law, perhaps the commutative law, perhaps the existence of the identity, etc.). The "dual" object would be a co-algebra, consisting of a vector space C and a map

$$c : C \to C \otimes C$$

subjects to a series of conditions dual to those listed above. If A is finite dimensional, then we have an identification of $(A \otimes A)^*$ with $A^\star \otimes A^\star$, and so the dual space of a finite dimensional algebra is a coalgebra and vice versa. For infinite dimensional algebras or coalgebras we have to pass to certain topological completions.

For example, consider the space $C_b(G)$ denote the space of continuous bounded functions on G endowed with the uniform norm

$$\|f\|_\infty = \text{l.u.b.}_{x \in G}\{|f(x)|\}.$$

We have a bounded linear map

$$c : C_b(G) \to C_b(G \times G)$$

given by

$$c(f)(x, y) := f(xy).$$

In the case that G is finite, and endowed with the discrete topology, the space $C_b(G)$ is just the space of all functions on G, and $C_b(G \times G) = C_b(G) \otimes C_b(G)$ where $C_b(G) \otimes C_b(G)$ can be identified with the space of all functions on $G \times G$ of the form

$$(x, y) \mapsto \sum_i f_i(x) g_i(y)$$

where the sum is finite. In the general case, not every bounded continuous function on $G \times G$ can be written in the above form, but, by Stone-Weierstrass, the space of such functions is dense in $C_b(G \times G)$. So we can say that $C_b(G)$ is "almost" a co-algebra, or a "co-algebra in the topological sense", in that the map c does not carry C into $C \otimes C$ but rather into the completion of $C \otimes C$. If A denotes the dual space of C, then A becomes an (honest) algebra. To make all this work in the case at hand, we need yet another version of the Riesz representation theorem. I will state and prove the appropriate theorem, but not go into the further details:

Let X be a topological space, let $C_b := C_b(X, \mathbf{R})$ be the space of bounded continuous real valued functions on X. For any $f \in C_b$ and any subset $A \subset X$ let

$$\|f\|_{\infty,A} := \text{l.u.b.}_{x \in A}\{|f(x)|\}.$$

So

$$\|f\|_\infty = \|f\|_{\infty,X}.$$

A continuous linear function ℓ is called **tight** if for every $\delta > 0$ there is a compact set K_δ and a positive number A_δ such that

$$|\ell(f)| \le A_\delta \|f\|_{\infty,K_\delta} + \delta \|f\|_\infty.$$

Theorem 3 **[Yet another Riesz representation theorem.]** *If $\ell \in C_b^*$ is a tight non-negative linear functional, then there is a finite non-negative measure μ on $(X, \mathcal{B}(\mathcal{X})$ such that*

$$\langle \ell, f \rangle = \int_X f d\mu$$

for all $f \in C_b$.

Proof. We need to show that $f_n \searrow 0 \Rightarrow \langle \ell, f_n \rangle \searrow 0$. Given $\epsilon > 0$, choose

$$\delta := \frac{\epsilon}{1 + 2\|f_1\|_\infty}.$$

This same inequality then holds with f_1 replaced by f_n since the f_n are monotone decreasing. We have the K_δ as in the definition of tightness, and by Dini's lemma, we can choose N so that

$$\|f_n|_{K_\delta} \le \frac{\epsilon}{2A_\delta} \quad \forall\, n > N.$$

Then $|\langle \ell, f_n \rangle| \le \epsilon$ for all $n > N$. QED

9 Invariant and relatively invariant measures on homogeneous spaces.

Let G be a locally compact Hausdorff topological group, and let H be a closed subgroup. Then G acts on the quotient space G/H by left multiplication, the element a sending the coset xH into axH. By abuse of language, we will continue to denote this action by ℓ_a. So

$$\ell_a(xH) := (ax)H.$$

We can consider the corresponding action on measures

$$\kappa \mapsto \ell_{a*}\kappa.$$

The measure κ is said to be invariant if

$$\ell_{a*}\kappa = \kappa \quad \forall\, a \in G.$$

The measure κ on G/H is said to be **relatively invariant** with **modulus** D if D is a function on G such that

$$\ell_{a*}\kappa = D(a)\kappa \quad \forall\, a \in G.$$

From its definition it follows that

$$D(ab) = D(a)D(b),$$

and it is not hard to see from the ensuing discussion that D is continuous. We will only deal with positive measures here, so D is continuous homomorphism of G into the multiplicative group of real numbers. We call such an object a positive character. The questions we want to address in this section are what are the possible invariant measures or relatively invariant measures on G/H, and what are their modular functions.

For example, consider the $ax + b$ group acting on the real line. So G is the $ax+b$ group, and H is the subgroup consisting of those elements with $b = 0$, the "pure rescalings". So H is the subgroup fixing the origin in the real line, and we can identify G/H with the real line. Let $N \subset G$ be the subgroup consisting of pure translations, so N consists of those elements of G with $a = 1$. The group N acts as translations of the line, and (up to scalar multiple) the only measure on the real line invariant under all translations is Lebesgue measure, dx. But

$$h = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$$

acts on the real line by sending $x \mapsto ax$ and hence

$$\ell_{h*}(dx) = a^{-1}dx.$$

(The push forward of the measure μ under the map ϕ assigns the measure $\mu(\phi^{-1}(A))$ to the set A.) So there is no measure on the real line invariant under G. On the other hand, the above formula shows that dx is relatively invariant with modular function

$$D\left(\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}\right) = a^{-1}.$$

Notice that this is the same as the modular function Δ, of the group G, and that the modular function δ for the subgroup H is the trivial function $\delta \equiv 1$ since H is commutative.

We now turn to the general study, and will follow Loomis in dealing with the integrals rather than the measures, and so denote the Haar integral of G

by I, with Δ its modular function, denote the Haar integral of H by J and its modular function by χ. We will let K denote an integral on G/H, and D a positive character on G.

If $f \in C_0(G)$, then we will let $J_t f(xt)$ denote that function of x obtained by integrating the function $t \mapsto f(xt)$ on H with respect to J. By the left invariance of J, we see that

$$J_t(xst) = J_t(xt).$$

In other words, this function is constant on cosets and hence defines a function on G/H (which is easily seen to be continuous and of compact support). Thus J defines a map

$$\mathbf{J}: \; C_0(G) \to C_0(G/H).$$

We will prove below that this map is surjective.

The main result we are aiming for in this section (due to A. Weil) is

Theorem 4 *In order that D be the modular function of a relatively invariant integral K on G/H it is necessary and sufficient that*

$$D(s) = \frac{\Delta(s)}{\chi(s)} \quad \forall \; s \in H. \tag{28}$$

If this happens, then K is uniquely determined up to scalar multiple, in fact,

$$K(\mathbf{J}(fD)) = cI(f) \quad \forall \; f \in C_0(G) \tag{29}$$

where c is some positive constant.

We begin with some preliminaries. Let $\pi : G \to G/H$ denote the projection map which sends each element $x \in G$ into its right coset

$$\pi(x) = xH.$$

The topology on G/H is defined by declaring a set $U \subset (G/H)$ to be open if and only if $\pi^{-1}(U)$ is open. The map π is then not only continuous (by definition) but also open, i.e. sends open sets into open sets. Indeed, if $O \subset G$ is an open subset, then

$$\pi^{-1}(\pi(O)) = \bigcup_{h \in H} Oh$$

which is a union of open sets, hence open, hence $\pi(O)$ is open.

Lemma 2 *If B is a compact closed subset of G/H then there exists a compact set $A \subset B$ such that*

$$\pi(A) = B,$$

Proof. Since we are assuming that G is locally compact, we can find an open neighborhood O of e in G whose closure C is compact. The sets $\pi(xO), \; x \in G$

are all open subsets of G/H since π is open, and there images cover all of G/H. In particular, since B is compact, finitely many of them cover B, so

$$B \subset \bigcup_i \pi(x_i U) \subset \bigcup_i \pi(x_i C) = \pi\left(\bigcup_i x_i C\right)$$

the unions being finite. The set

$$K = \bigcup_i x_i C$$

is compact, being the finite union of compact sets. The set $\pi^{-1}(B)$ is closed (since its complement is the inverse image of an open set, hence open). So

$$A := K \cap \pi^{-1}(B)$$

is compact, and its image is B. QED

Proposition 8 $\mathbf{J}$ *is surjective.*

Let $F \in C_0(G/H)$ and let $B = \mathrm{Supp}(F)$. Choose a compact set $A \subset G$ with $\pi(A) = B$ as in the lemma. Choose $\phi \in C_0(G)$ with $\phi \geq 0$ and $\phi > 0$ on A. If

$$x \in AH = \pi^{-1}(B)$$

then $\phi(xh) > 0$ for some $h \in H$, and so $\mathbf{J}(\phi) > 0$ on B. So we may extend the function

$$z \mapsto \frac{F(z)}{\mathbf{J}(h)(z)}$$

to a continuous function, call it γ, by defining it to be zero outside $B = \mathrm{Supp}(F)$. The function $g = \pi^*\gamma$, i.e.

$$g(x) = \gamma(\pi(x))$$

is hence a continuous function on G, and hence

$$f := g\phi$$

is a continuous function of compact support on G. Since g is constant on H cosets,

$$\mathbf{J}(f)(z) = \gamma(z)J(h)(z) = F(z). \text{ QED}$$

Now to the proof of the theorem. Suppose that K is an integral on $C_0(G/H)$ with modular function D. Define

$$M(f) = K(\mathbf{J}(fD))$$

for $f \in C_0(G)$. By applying the monotone convergence theorem for J and K we see that M is an integral. We must check that it is left invariant, and hence determines a Haar measure which is a multiple of the given Haar measure.

$$\begin{aligned} M(\ell_a^*(f) &= K(\mathbf{J}((\ell_a^* f)D)) \\ &= D(a)^{-1}K((\mathbf{J}(\ell_a^*(fD)))) \\ &= D(a)^{-1}K(\ell_a^*(\mathbf{J}(fD)))) \\ &= D(a)^{-1}D(a)K(\mathbf{J}(fD))) \\ &= M(f). \end{aligned}$$

This shows that if K is relatively invariant with modular function D then it is unique up to scalar factor. Let us multiply K be a scalar if necessary (which does not change D) so that

$$I(f) = K(\mathbf{J}(fD)).$$

We now argue more or less as before: Let $h \in H$. Then

$$\begin{aligned} \Delta(h)I(f) &= I(r_h^* f) \\ &= K(\mathbf{J}((r_h^* f)D)) \\ &= D(h)K(\mathbf{J}(r_h^*(fD))) \\ &= D(h)\chi(h)K(\mathbf{J}(fd)) \\ &= D(h)\chi(h)I(f), \end{aligned}$$

proving that (28) holds.

Conversely, suppose that (28) holds, and try to define K by

$$K(\mathbf{J}(f)) = I(fD^{-1}).$$

Since $\mathbf{J}$ is surjective, this will define an integral on $C_0(G/H)$ once we show that it is well defined, i.e. once we show that

$$\mathbf{J}(f) = 0 \Rightarrow I(fD^{-1}) = 0.$$

Suppose that $\mathbf{J}(f) = 0$, and let $\phi \in C_0(G)$. Then

$$\phi(x)D(x)^{-1}\pi^*(\mathbf{J}(f))(x) = 0$$

for all $x \in G$, and so taking I of the above expression will also vanish. We will now use Fubini: We have

$$0 = I_x\left(\phi(x)D^{-1}(x)J_h(f(xh))\right) = I_x J_h\left(\phi(x)D^{-1}(x)(f(xh))\right) =$$

$$J_h I_x\left(\phi(x)D^{-1}(x)(f(xh))\right)$$

We can write the expression that is inside the last I_x integral as

$$r_{h^{-1}}^*(\phi(xh^{-1})D^{-1}(xh^{-1})(f(x)))$$

and hence

$$J_h I_x \left(\phi(x) D^{-1}(x)(f(xh))\right) = J_h I_x \left(\phi(xh^{-1}) D^{-1}(xh^{-1})(f(x)\Delta(h^{-1}))\right)$$

by the defining properties of Δ. Now use the hypothesis that $\Delta = D\chi$ to get

$$J_h I_x \left(\chi(h^{-1})\phi(xh^{-1}) D^{-1}(x)(f(x))\right)$$

and apply Fubini again to write this as

$$I_x \left(D^{-1}(x) f(x) J_h(\chi(h^{-1}\phi(xh^{-1})))\right).$$

By equation (26) applied to the group H, we can replace the J integral above by $J_h(\phi(xh))$ so finally we conclude that

$$I_x \left(D^{-1}(x) f(x) J_h(\phi(xh))\right) = 0$$

for any $\phi \in C_0(G)$. Now choose $\psi \in C_0(G/H)$ which is non-negative and identically one on $\pi(\mathrm{Supp}(f))$, and choose $\phi \in C_0(G)$ with $\mathbf{J}(\phi) = \psi$. Then the above expression is $I(D^{-1}f)$. So we have proved that

$$\mathbf{J}(f) = 0 \Rightarrow I(fD^{-1}) = 0,$$

and hence that $K : C_0(G/H) \to \mathbf{C}$ defined by

$$K(F) = I(D^{-1}f) \quad \text{if} \quad \mathbf{J}(f) = F$$

is well defined. We still must show that K defined this way is relatively invariant with modular function K. We compute

$$\begin{aligned} K(\ell_a^* F) &= I(D^{-1}\ell_a^*(f)) \\ &= D(a) I(\ell_a^*(D^{-1}f)) \\ &= D(a) I(D^{-1}f) \\ &= D(a) K(F). \quad \text{QED} \end{aligned}$$

Of particular importance is the case where G and H are unimodular, for example compact. Then, up to scalar factor, there is a unique measure on G/H invariant under the action of G.