Hilbert space and Compact Operators.

212

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Contents

1 Hilbert space.

1.1 Scalar products.

V is a complex vector space. A rule assigning to every pair of vectors $f, g \in V$ a complex number (f, g) is called a **semi-scalar product** if

1. (f, g) is linear in f when g is held fixed.
2. $(g, f) = \overline{(f, g)}$. This implies that (f, g) is anti-linear in g when f is held fixed. In other words. $(f, ag + bh) = \overline{a}(f, g) + \overline{b}(f, h)$. It also implies that (f, f) is real.
3. $(f, f) \geq 0$ for all $f \in V$.

If 3. is replaced by the stronger condition

4. $(f, f) > 0$ for all non-zero $f \in V$

then we say that $(\, , \,)$ is a **scalar product**.

Examples.

- $V = \mathbf{C}^n$, so an element $\mathbf{z}$ of V is a column vector of complex numbers:

$$\mathbf{z} = \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix}$$

  and $(\mathbf{z}, \mathbf{w})$ is given by

$$(\mathbf{z}, \mathbf{w}) := \sum_1^n z_i \overline{w_i}.$$

- V consists of all continuous (complex valued) functions on the real line which are periodic of period 2π and

$$(f, g) := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\overline{g(x)}dx.$$

  We will denote this space by $\mathcal{C}(\mathbf{T})$. Here the letter $\mathbf{T}$ stands for the one dimensional torus, i.e. the circle. We are identifying functions which are periodic with period 2π with functions which are defined on the circle $\mathbf{R}/2\pi\mathbf{Z}$.

- V consists of all doubly infinite sequences of complex numbers

$$\mathbf{a} = \ldots, a_{-2}, a_{-1}, a_0, a_1, a_2, \ldots$$

  which satisfy

$$\sum |a_i|^2 < \infty.$$

  Here

$$(\mathbf{a}, \mathbf{b}) := \sum a_i \overline{b}_i.$$

All three are examples examples of scalar products.

1.2 The Cauchy-Schwartz inequality.

This says that if (,) is a semi-scalar product then

$$|(f,g)| \leq (f,f)^{\frac{1}{2}}(g,g)^{\frac{1}{2}}. \tag{1}$$

Proof. For any real number t condition 3. above says that $(f - tg, f - tg) \geq 0$. Expanding out gives

$$0 \leq (f - tg, f - tg) = (f,f) - t[(f,g) + (g,f)] + t^2(g,g).$$

Since $(g,f) = \overline{(f,g)}$, the coefficient of t in the above expression is twice the real part of (f,g). So the real quadratic form

$$Q(t) := (f,f) - 2\mathrm{Re}(f,g)t + t^2(g.g)$$

is nowhere negative. So it can not have distinct real roots, and hence by the $b^2 - 4ac$ rule we get

$$4(\mathrm{Re}(f,g))^2 - 4(f,f)(g,g) \leq 0$$

or

$$(\mathrm{Re}(f,g))^2 \leq (f,f)(g,g). \tag{2}$$

This is useful and almost but not quite what we want. But we may apply this inequality to $h = e^{i\theta}g$ for any θ. Then $(h,h) = (g,g)$. Choose θ so that

$$(f,g) = re^{i\theta}$$

where $r = |(f,g)|$. Then

$$(f,h) = (f, e^{i\theta}g) = e^{-i\theta}(f,g) = |(f,g)|$$

and the preceding inequality with g replaced by h gives

$$|(f,g)|^2 \leq (f,f)(g,g)$$

and taking square roots gives (1).

1.3 The triangle inequality

For any semiscalar product define

$$\|f\| := (f, f)^{\frac{1}{2}}$$

so we can write the Cauchy-Schwartz inequality as

$$|(f, g)| \leq \|f\| \|g\|.$$

The **triangle inequality** says that

$$\|f + g\| \leq \|f\| + \|g\|. \tag{3}$$

Proof.

$$\begin{aligned} \|f + g\|^2 &= (f + g, f + g) \\ &= (f, f) + 2\mathrm{Re}(f, g) + (g, g) \\ &\leq (f, f) + 2\|f\|\|g\| + (g, g) \quad \text{by (2)} \\ &= \|f|^2 + 2\|f\|\|g\| + \|g\|^2 \\ &= (\|f\| + \|g\|)^2. \end{aligned}$$

Taking square roots gives the triangle inequality (3). Notice that

$$\|cf\| = |c|\|f\| \tag{4}$$

since $(cf, cf) = c\overline{c}(f, f) = |c|^2\|f\|^2$.

Suppose we try to define the distance between two elements of V by

$$d(f, g) := \|f - g\|.$$

Notice that then $d(f, f) = 0$, $d(f, g) = d(g, f)$ and for any three elements

$$d(f, h) \leq d(f, g) + d(g, h)$$

by virtue of the triangle inequality. The only trouble with this definition is that we might have two distinct elements at zero distance, i.e. $0 = d(f, g) = \|f - g\|$. But this can not happen if $(\ ,\ )$ is a scalar product, i.e. satisfies condition 4.

A complex vector space V endowed with a scalar product is called a **pre-Hilbert** space.

Let V be a complex vector space and let $\|\cdot\|$ be a map which assigns to any $f \in V$ a non-negative real $\|f\|$ number such that $\|f\| > 0$ for all non-zero f. If $\|\cdot\|$ satisfies the triangle inequality (3) and condition 4) it is called a **norm**. A vector space endowed with a norm is called a normed space. The pre-HIlbert spaces can be characterized among all normed spaces by the parallelogram law as we will discuss below.

Later on, we will have to weaken condition (4) in our general study. But it is too complicated to give the general definition right now.

1.4 Hilbert and pre-Hilbert spaces.

The reason for the prefix "pre" is the following: The distance d defined above has all the desired properties we might expect of a distance. In particular, we can define the notions of "limit" and of a "Cauchy sequence" as is done for the real numbers: If f_n is a sequence of elements of V, and $f \in V$ we say that f is the limit of the f_n and write

$$\lim_{n\to\infty} f_n = f, \quad \text{or} \quad f_n \to f$$

if, for any positive number ϵ there is an $N = N(\epsilon)$ such that

$$d(f_n, f) < \epsilon \quad \text{for all } n \geq N.$$

If a sequence converges to some limit f, then this limit is unique, since any limits must be at zero distance and hence equal.

We say that a sequence of elements is **Cauchy** if for any $\delta > 0$ there is an $K = K(\delta)$ such that

$$d(f_m, f_n) < \delta \quad \forall, m, n \geq K.$$

If the sequence f_n has a limit, then it is Cauchy - just choose $K(\delta) = N(\frac{1}{2}\delta)$ and use the triangle inequality.

But it is quite possible that a Cauchy sequence has no limit. As an example of this type of phenomenon, think of the rational numbers with $|r - s|$ as the distance. The whole point of introducing the real numbers is to guarantee that every Cauchy sequence has a limit.

So we say that a pre-Hilbert space is a **Hilbert space** if it is "complete" in the above sense - if every Cauchy sequence has a limit.

Since the complex numbers are complete (because the real numbers are), it follows that $\mathbf{C}^n$ is complete, i.e. is a Hilbert space. Indeed, we can say that any finite dimensional pre-Hilbert space is a Hilbert space because it is isomorphic (as a pre-Hilbert space) to $\mathbf{C}^n$ for some n. (See below when we discuss orthonormal bases.)

The trouble is in the infinite dimensional case, such as the space of continuous periodic functions. This space is not complete. For example, let f_n be the function which is equal to one on $(-\pi + \frac{1}{n}, -\frac{1}{n})$, equal to zero on $(\frac{1}{n}, \pi - \frac{1}{n})$ and extended linearly $-\frac{1}{n}$ to $\frac{1}{n}$ and from $\pi - \frac{1}{n}$ to $\pi + \frac{1}{n}$ so as to be continuous and then extended so as to be periodic.(Thus on the interval $(\pi - \frac{1}{n}, \pi + \frac{1}{n})$ the function is given by $f_n(x) = 2n(x - (\pi - \frac{1}{n}).)$ If $m \leq n$, the functions f_m and f_n agree outside two intervals of length $\frac{2}{m}$ and on these intervals $|f_m(x) - f_n(x)| \leq 1$. So $\|f_m - f_n\|^2 \leq \frac{1}{2\pi} \cdot 2/m$ showing that the sequence $\{f_n\}$ is Cauchy. But the limit would have to equal one on $(-\pi, 0)$ and equal zero on $(0, \pi)$ and so be discontinuous at the origin and at π. Thus the space of continuous periodic functions is not a Hilbert space, only a pre-Hilbert space.

But just as we complete the rational numbers (by throwing in "ideal" elements) to get the real numbers, we may similarly complete any pre-Hilbert space to get a unique Hilbert space. See the section *Completion* in the chapter on metric spaces for a general discussion of how to complete any metric space. In particular, the completion of any normed vector space is a complete normed vector space. A complete normed space is called a **Banach** space. The general construction implies that any normed vector space can be completed to a Banach space. From the parallelogram law discussed below, it will follow that the completion of a pre-Hilbert space is a Hilbert space.

The completion of the space of continuous periodic functions will be denoted by $L^2(\mathbf{T})$.

1.5 The Pythagorean theorem.

Let V be a pre-Hilbert space. We have

$$\|f+g\|^2 = \|f\|^2 + 2\mathrm{Re}(f,g) + \|g\|^2.$$

So

$$\|f+g\|^2 = \|f\|^2 + \|g\|^2 \quad \Leftrightarrow \quad \mathrm{Re}\ (f,g) = 0. \tag{5}$$

We make the definition

$$f \perp g \ \Leftrightarrow \ (f,g) = 0$$

and say that f is perpendicular to g or that f is orthogonal to g. Notice that this is a stronger condition than the condition for the Pythagorean theorem, the right hand condition in (5). For example $\|f+if\|^2 = 2\|f\|^2$ but $(f,if) = -i\|f\|^2 \neq 0$ if $\|f\| \neq 0$.

If u_i is some finite collection of mutually orthogonal vectors, then so are $z_i u_i$ where the z_i are any complex numbers. So if

$$u = \sum_i z_i u_i$$

then by the Pythagorean theorem

$$\|u\|^2 = \sum_i |z_i|^2 \|u_i\|^2.$$

In particular, if the $u_i \neq 0$, then $u = 0 \Rightarrow z_i = 0$ for all i. This shows that any set of mutually orthogonal (non-zero) vectors is linearly independent.

Notice that the set of functions

$$e^{in\theta}$$

is an **orthonormal** set in the space of continuous periodic functions in that not only are they mutually orthogonal, but each has norm one.

1.6 The theorem of Apollonius.

Adding the equations

$$
\begin{aligned}
\|f+g\|^2 &= \|f\|^2 + 2\text{Re }(f,g) + \|g\|^2 &(6)\\
\|f-g\|^2 &= \|f\|^2 - 2\text{Re }(f,g) + \|g\|^2 &(7)
\end{aligned}
$$

gives

$$\|f+g\|^2 + \|f-g\|^2 = 2\left(\|f\|^2 + \|g\|^2\right). \tag{8}$$

This is known as the **parallelogram law**. It is the algebraic expression of the theorem of Apollonius which asserts that the sum of the areas of the squares on the sides of a parallelogram equals the sum of the areas of the squares on the diagonals.

If we subtract (7) from (6) we get

$$\text{Re}(f,g) = \frac{1}{4}\left(\|f+g\|^2 - \|f-g\|^2\right). \tag{9}$$

Now $(if,g) = i(f,g)$ and $\text{Re}\,\{i(f,g)\} = -\text{Im}(f,g)$ so

$$\text{Im}(f,g) = -\text{Re}(if,g) = \text{Re}(f,ig)$$

so

$$(f,g) = \frac{1}{4}\left(\|f+g\|^2 - \|f-g\|^2 + i\|f+ig\|^2 - i\|f-ig\|^2\right). \tag{10}$$

If we now complete a pre-HIlbert space, the right hand side of this equation is defined on the completion, and is a continuous function there. It therefore follows that the scalar product extends to the completion, and, by continuity, satisfies all the axioms for a scalar product, plus the completeness condition for the associated norm. In other words, the completion of a pre-Hilbert space is a Hilbert space.

1.7 The theorem of Jordan and von Neumann.

This is essentially a converse to the theorem of Apollonius. It says that if $\|\cdot\|$ is a norm on a (complex) vector space V which satisfies (8), then V is in fact a pre-Hilbert space with $\|f\|^2 = (f,f)$. If the theorem is true, then the scalar product must be given by (10). So we must prove that if we take (10) as the definition, then all the axioms on a scalar product hold. The easiest axiom to verify is

$$(g,f) = \overline{(f,g)}.$$

Indeed, the real part of the right hand side of (10) is unchanged under the interchange of f and g (since $g-f = -(f-g)$ and $\|-h\| = \|h\|$ for any h is one of the properties of a norm). Also $g+if = i(f-ig)$ and $\|ih\| = \|h\|$ so the last two terms on the right of (10) get interchanged, proving that $(g,f) = \overline{(f,g)}$.

It is just as easy to prove that

$$(if, g) = i(f, g).$$

Indeed replacing f by if sends $\|f + ig\|^2$ into $\|if + ig\|^2 = \|f + g\|^2$ and sends $\|f + g\|^2$ into $\|if + g\|^2 = \|i(f - ig)\|^2 = \|f - ig\|^2 = i(-i\|f - ig\|^2$ so has the effect of multiplying the sum of the first and fourth terms by i, and similarly for the sum of the second and third terms on the right hand side of (10).

Now (10) implies (9). Suppose we replace f, g in (8) by $f_1 + g, f_2$ and by $f_1 - g, f_2$ and subtract the second equation from the first. We get

$$\|f_1 + f_2 + g\|^2 - \|f_1 + f_2 - g\|^2 + \|f_1 - f_2 + g\|^2 - \|f_1 - f_2 - g\|^2$$
$$= 2\left(\|f_1 + g\|^2 - \|f_1 - g\|^2\right).$$

In view of (9) we can write this as

$$\mathrm{Re}\ (f_1 + f_2, g) + \mathrm{Re}\ (f_1 - f_2, g) = 2\mathrm{Re}\ (f_1, g). \tag{11}$$

Now the right hand side of (9) vanishes when $f = 0$ since $\|g\| = \|-g\|$. So if we take $f_1 = f_2 = f$ in (11) we get

$$\mathrm{Re}\ (2f, g) = 2\mathrm{Re}\ (f, g).$$

We can thus write (11) as

$$\mathrm{Re}\ (f_1 + f_2, g) + \mathrm{Re}\ (f_1 - f_2, g) = \mathrm{Re}\ (2f_1, g).$$

In this equation make the substitutions

$$f_1 \mapsto \frac{1}{2}(f_1 + f_2), \quad f_2 \mapsto \frac{1}{2}(f_1 - f_2).$$

This yields

$$\mathrm{Re}\ (f_1 + f_2, g) = \mathrm{Re}\ (f_1, g) + \mathrm{Re}\ (f_2, g).$$

Since it follows from (10) and (9) that

$$(f, g) = \mathrm{Re}\ (f, g) - i\mathrm{Re}\ (if, g)$$

we conclude that

$$(f_1 + f_2, g) = (f_1, g) + (f_2, g).$$

Taking $f_1 = -f_2$ shows that

$$(-f, g) = -(f, g).$$

Consider the collection $\mathcal{C}$ of complex numbers α which satisfy

$$(\alpha f, g) = \alpha(f, g)$$

(for all f, g). We know from $(f_1 + f_2, g) = (f_1, g) + (f_2, g)$ that

$$\alpha, \beta \in \mathcal{C} \Rightarrow \alpha + \beta \in \mathcal{C}.$$

So $\mathcal{C}$ contains all integers. If $0 \neq \beta \in \mathcal{C}$ then

$$(f, g) = (\beta \cdot (1/\beta) f, g) = \beta((1/\beta) f, g)$$

so $\beta^{-1} \in \mathcal{C}$. Thus $\mathcal{C}$ contains all (complex) rational numbers. The theorem will be proved if we can prove that $(\alpha f, g)$ is continuous in α. But the triangle inequality

$$\|f + g\| \leq \|f\| + \|g\|$$

applied to $f = f_2, g = f_1 - f_2$ implies that

$$\|f_1\| \leq \|f_1 - f_2\| + \|f_2\|$$

or

$$\|f_1\| - \|f_2\| \leq \|f_1\| - f_2\|.$$

Interchanging the role of f_1 and f_2 gives

$$|\ \|f_1\| - \|f_2\|\ | \leq \|f_1 - f_2\|.$$

Therefore

$$|\ \|\alpha f \pm g\| - \|\beta f \pm g\|\ | \leq \|(\alpha - \beta) f\|.$$

Since $\|(\alpha - \beta) f\| \to 0$ as $\alpha \to \beta$ this shows that the right hand side of (10) when applied to αf and g is a continuous function of α. Thus $\mathcal{C} = \mathbf{C}$. We have proved

Theorem 1 **[P. Jordan and J. von Neumann]** *If V is a normed space whose norm satisfies (8) then V is a pre-Hilbert space.*

Notice that the condition (8) involves only two vectors at a time. So we conclude as an immediate consequence of this theorem that

Corollary 1 *A normed vector space is pre-Hilbert space if and only if every two dimensional subspace is a Hilbert space in the induced norm.*

Actually, a weaker version of this corollary, with two replaced by three had been proved by Fréchet, *Annals of Mathematics*, July 1935, who raised the problem of giving an abstract characterization of those norms on vector spaces which come from scalar products. In the immediately following paper Jordan and von Neumann proved the theorem above leading to the stronger corollary that two dimensions suffice.

1.8 Orthogonal projection.

We continue with the assumption that V is pre-Hilbert space. If A and B are two subsets of V, we write $A \perp B$ if $u \in A$ and $v \in B \Rightarrow u \perp v$, in other words if every element of A is perpendicular to every element of B. Similarly, we will write $v \perp A$ if the element v is perpendicular to all elements of B. Finally, we will write $A^\perp$ for the set of all v which satisfy $v \perp A$. Notice that $A^\perp$ is always a linear subspace of V, for any A.

Now let M be a (linear) subspace of V. Let v be some element of V, not necessarily belonging to M. We want to investigate the problem of finding a $w \in M$ such that $(v - w) \perp M$. Of course, if $v \in M$ then the only choice is to take $w = v$. So the interesting problem is when $v \notin M$. Suppose that such a w exists, and let x be any (other) point of M. Then by the Pythagorean theorem,

$$\|v - x\|^2 = \|(v - w) + (w - x)\|^2 = \|v - w\|^2 + \|w - x\|^2$$

since $(v - w) \perp M$ and $(w - x) \in M$. So

$$\|v - w\| \leq \|v - x\|$$

and this inequality is strict if $x \neq w$. In words: if we can find a $w \in M$ such that $(v - w) \perp M$ then w is the unique solution of the problem of finding the point in M which is closest to v. Conversely, suppose we found a $w \in M$ which has this minimization property, and let x be any element of M. Then for any real number t we have

$$\|v - w\|^2 \leq \|(v - w) + tx\|^2 = \|v - w\|^2 + 2t\text{Re }(v - w, x) + t^2\|x\|^2.$$

Since the minimum of this quadratic polynomial in t occurring on the right is achieved at $t = 0$, we conclude (by differentiating with respect to t and setting $t = 0$, for example) that

$$\text{Re }(v - w, x) = 0.$$

By our usual trick of replacing x by $e^{i\theta}x$ we conclude that

$$(v - w, x) = 0.$$

Since this holds for all $x \in N$, we conclude that $(v - w) \perp M$. So to find w we search for the minimum of $\|v - x\|,\ x \in M$.

Now $\|v - x\| \geq 0$ and is some finite number for any $x \in M$. So there will be some real number m such that $m \leq \|v - x\|$ for $x \in M$, and such that no strictly larger real number will have this property. (m is known as the "greatest lower bound" of the values $\|v - x\|,\ x \in M$.) So we can find a sequence of vectors $x_n \in M$ such that

$$\|v - x_n\| \to m.$$

We claim that the x_n form a Cauchy sequence. Indeed,

$$\|x_i - x_j\|^2 = \|(v - x_j) - (v - x_i)\|^2$$

and by the parallelogram law this equals

$$2\left(\|v - x_i\|^2 + \|v - x_j\|^2\right) - \|2v - (x_i + x_j)\|^2.$$

Now the expression in parenthesis converges to $2m^2$. The last term on the right is

$$-\|2(v - \frac{1}{2}(x_i + x_j)\|^2.$$

Since $\frac{1}{2}(x_i + x_j) \in M$, we conclude that

$$\|2v - (x_i + x_j)\|^2 \geq 4m^2$$

so

$$\|x_i - x_j\|^2 \leq 4(m + \epsilon)^2 - 4m^2$$

for i and j large enough that $\|v - x_i\| \leq m + \epsilon$ and $\|v - x_j\| \leq m + \epsilon$. This proves that the sequence x_n is Cauchy.

Here is the crux of the matter: If M is complete, then we can conclude that the x_n converge to a limit w which is then the unique element in M such that $(v - w) \perp M$. It is at this point that completeness plays such an important role.

Put another way, we can say that if M is a subspace of V which is complete (under the scalar product $(\ ,\ )$ restricted to M) then we have the orthogonal direct sum decomposition

$$V = M \oplus M^{\perp},$$

which says that every element of V can be uniquely decomposed into the sum of an element of M and a vector perpendicular to M.

For example, if M is the one dimensional subspace consisting of all (complex) multiples of a non-zero vector y, then M is complete, since $\mathbf{C}$ is complete. So w exists. Since all elements of M are of the form ay, we can write $w = ay$ for some complex number a. Then $(v - ay, y) = 0$ or

$$(v, y) = a\|y\|^2$$

so

$$a = \frac{(v, y)}{\|y\|^2}.$$

We call a the **Fourier coefficient** of v with respect to y. Particularly useful is the case where $\|y\| = 1$ and we can write

$$a = (v, y). \tag{12}$$

Getting back to the general case, if $V = M \oplus M^{\perp}$ holds, so that every v corresponds to a unique $w \in M$ satisfying $(v - w) \in M^{\perp}$ the map $v \mapsto w$ is called orthogonal projection of V onto M and will be denoted by π_M.

1.9 The Riesz representation theorem.

Let V and W be two complex vector spaces. A map

$$T : V \to W$$

is called **linear** if

$$T(\lambda x + \mu y) = \lambda T(x) + \mu T(Y) \quad \forall\ x, y \in V, \quad \lambda, \mu \in \mathbf{C}$$

and is called **anti-linear** if

$$T(\lambda x + \mu y) = \overline{\lambda} T(x) + \overline{\mu} T(Y) \quad \forall\ x, y \in V \quad \lambda, \mu \in \mathbf{C}.$$

If $\ell : V \to \mathbf{C}$ is a linear map, (also known as a linear function) then

$$\ker\ \ell := \{x \in V |\ \ell(x) = 0\}$$

has codimension one (unless $\ell \equiv 0$). Indeed, if

$$\ell(y) \neq 0$$

then

$$\ell(x) = 1 \ \text{ where } x = \frac{1}{\ell(y)} y$$

and for any $z \in V$,

$$z - \ell(z)x \in \ker\ \ell.$$

If V is a normed space and ℓ is continuous, then ker (ℓ) is a closed subspace. The space of continuous linear functions is denoted by V^*. It has its own norm defined by

$$\|\ell\| := \sup_{x \in V, \|x\| \neq 0} |\ell(x)|/\|x\|.$$

Suppose that H is a pre-hilbert space. Then we have an antilinear map

$$\phi : H \to H^*, \quad (\phi(g))(f) := (f, g).$$

The Cauchy-Schwartz inequality implies that

$$\|\phi(g)\| \leq \|g\|$$

and in fact

$$(g, g) = \|g\|^2$$

shows that

$$\|\phi(g)\| = \|g\|.$$

In particular the map ϕ is injective.

The Riesz representation theorem says that if H is a Hilbert space, then this map is surjective:

Theorem 2 *Every continuous linear function on H is given by scalar product by some element of H.*

The proof is a consequence of the theorem about projections applied to

$$N := \ker \ell :$$

If $\ell = 0$ there is nothing to prove. If $\ell \neq 0$ then N is a closed subspace of codimension one. Choose $v \notin N$. Then there is an $x \in N$ with $(v - x) \perp N$. Let

$$y := \frac{1}{\|v - x\|}(v - x).$$

Then

$$y \perp N$$

and

$$\|y\| = 1.$$

For any $f \in H$,

$$[f - (f, y)y] \perp y$$

so

$$f - (f, y)y \in N$$

or

$$\ell(f) = (f, y)\ell(y),$$

so if we set

$$g := \overline{\ell(y)}y$$

then

$$(f, g) = \ell(f)$$

for all $f \in H$. QED

1.10 What is $L^2(\mathbf{T})$?

We have defined the space $L^2(\mathbf{T})$ to be the completion of the space $\mathcal{C}(\mathbf{T})$ under the L_2 norm $\|f\|_2 = (f, f)^{\frac{1}{2}}$. In particular, every linear function on $\mathcal{C}(\mathbf{T})$ which is continuous with respect to the this L_2 norm extends to a unique continuous linear function on $L_2(\mathbf{T}$. By the Riesz representation theorem we know that every such continuous linear function is give by scalar product by an element of $L_2(T)$. Thus we may think of the elements of $L_2(\mathbf{T})$ as being the linear functions on $\mathcal{C}(\mathbf{T})$ which are continuous with respect to the L_2. An element of $L_2(\mathbf{T})$ should not be thought of as a function, but rather as a linear function on the space of continuous functions.

1.11 Projection onto a direct sum.

Suppose that the closed subspace M of a pre-Hilbert space is the orthogonal direct sum of a finite number of subspaces

$$M = \bigoplus_i M_i$$

meaning that the M_i are mutually perpendicular and every element x of M can be written as

$$x = \sum x_i, \quad x_i \in M_i.$$

(The orthogonality guarantees that such a decomposition is unique.) Suppose further that each M_i is such that the projection π_{M_i} exists. Then π_M exists and

$$\pi_M(v) = \sum \pi_{M_i}(v). \tag{13}$$

Proof. Clearly the right hand side belongs to M. We must show $v - \sum_i \pi_{M_i}(v)$ is orthogonal to every element of M. For this it is enough to show that it is orthogonal to each M_j since every element of M is a sum of elements of the M_j. So suppose $x_j \in M_j$. But $(\pi_{M_i} v, x_j) = 0$ if $i \neq j$. So

$$(v - \sum \pi_{M_i}(v), x_j) = (v - \pi_{M_j}(v), x_j) = 0$$

by the defining property of π_{M_j}.

1.12 Projection onto a finite dimensional subspace.

We now will put the equations (12) and (13) together: Suppose that M is a finite dimensional subspace with an orthonormal basis ϕ_i. This implies that M is an orthogonal direct sum of the one dimensional spaces spanned by the ϕ_i and hence π_M exists and is given by

$$\pi_M(v) = \sum a_i \phi_i \quad \text{where} \quad a_i = (v, \phi_i). \tag{14}$$

1.13 Bessel's inequality.

We now look at the infinite dimensional situation and suppose that we are given an orthonormal sequence $\{\phi_i\}_1^\infty$. Any $v \in V$ has its Fourier coefficients

$$a_i = (v, \phi_i)$$

relative to the members of this sequence. Bessel's inequality asserts that

$$\sum_1^\infty |a_i|^2 \leq \|v\|^2, \tag{15}$$

in particular the sum on the left converges.

Proof. Let

$$v_n := \sum_{i=1}^{n} a_i \phi_i,$$

so that v_n is the projection of v onto the subspace spanned by the first n of the ϕ_i. In any event, $(v - v_n) \perp v_n$ so by the Pythagorean Theorem

$$\|v\|^2 = \|v - v_n\|^2 + \|v_n\|^2 = \|v - v_n\|^2 + \sum_{i=1}^{n} |a_i|^2.$$

This implies that

$$\sum_{i=1}^{n} |a_i|^2 \leq \|v\|^2$$

and letting $n \to \infty$ shows that the series on the left of Bessel's inequality converges and that Bessel's inequality holds.

1.14 Parseval's equation.

Continuing the above argument, observe that

$$\|v - v_n\|^2 \to 0 \;\Leftrightarrow\; \sum |a_i|^2 = \|v\|^2.$$

But $\|v - v_n\|^2 \to 0$ if and only if $\|v - v_n\| \to 0$ which is the same as saying that $v_n \to v$. But v_n is the n-th partial sum of the series $\sum a_i \phi_i$, and in the language of series, we say that a series converges to a limit v and write $\sum a_i \phi_i = v$ if and only if the partial sums approach v. So

$$\sum a_i \phi_i = v \quad \Leftrightarrow \quad \sum_i |a_i|^2 = \|v\|^2. \tag{16}$$

In general, we will call the series $\sum_i a_i \phi_i$ the Fourier series of v (relative to the given orthonormal sequence) whether or not it converges to v. Thus Parseval's equality says that the Fourier series of v converges to v if and only if $\sum |a_i|^2 = \|v\|^2$.

1.15 Orthonormal bases.

We still suppose that V is merely a pre-Hilbert space. We say that an orthonormal sequence $\{\phi_i\}$ is a **basis** of V if every element of V is the sum of its Fourier series. For example, one of our tasks will be to show that the exponentials $\{e^{inx}\}_{n=-\infty}^{\infty}$ form a basis of $\mathcal{C}(\mathbf{T})$.

If the orthonormal sequence ϕ_i is a basis, then any v can be approximated as closely as we like by finite linear combinations of the ϕ_i, in fact by the partial sums of its Fourier series. We say that the finite linear combinations of the ϕ_i are *dense* in V. Conversely, suppose that the finite linear combinations of the

ϕ_i are dense in V. This means that for any v and any $\epsilon > 0$ we can find an n and a set of n complex numbers b_i such that

$$\|v - \sum b_i \phi_i\| \leq \epsilon.$$

But we know that v_n is the closest vector to v among all the linear combinations of the first n of the ϕ_i. so we must have

$$\|v - v_n\| \leq \epsilon.$$

But this says that the Fourier series of v converges to v, i.e. that the ϕ_i form a basis. For example, we know from Fejer's theorem that the exponentials e^{ikx} are dense in $\mathcal{C}(\mathbf{T})$. Hence we know that they form a basis of the pre-Hilbert space $\mathcal{C}(\mathbf{T})$. We will give some alternative proofs of this fact below.

In the case that V is actually a Hilbert space, and not merely a pre-Hilbert space, there is an alternative and very useful criterion for an orthonormal sequence to be a basis: Let M be the set of all limits of finite linear combinations of the ϕ_i. Any Cauchy sequence in M converges (in V) since V is a Hilbert space, and this limit belongs to M since it is itself a limit of finite linear combinations of the ϕ_i (by the diagonal argument for example). Thus $V = M \oplus M^{\perp}$, and the ϕ_i form a basis of M. So the ϕ_i form a basis of V if and only if $M^{\perp} = \{0\}$. But this is the same as saying that no non-zero vector is orthogonal to all the ϕ_i. So we have proved

Proposition 1 *In a Hilbert space, the orthonormal set $\{\phi_i\}$ is a basis if and only if no non-zero vector is orthogonal to all the ϕ_i.*

2 Self-adjoint transformations.

We continue to let V denote a pre-Hilbert space. Let T be a linear transformation of V into itself. This means that for every $v \in V$ the vector $Tv \in V$ is defined and that Tv depends linearly on v : $T(av + bw) = aTv + bTw$ for any two vectors v and w and any two complex numbers a and b. We recall from linear algebra that a non-zero vector v is called an eigenvector of T if Tv is a scalar times v, in other words if $Tv = \lambda v$ where the number λ is called the corresponding eigenvalue.

A linear transformation T on V is called **symmetric** if for any pair of elements v and w of V we have

$$(Tv, w) = (v, Tw).$$

Notice that if v is an eigenvector of a symmetric transformation T with eigenvalue λ, then

$$\lambda(v, v) = (\lambda v, v) = (Tv, v) = (v, Tv) = (v, \lambda v) = \overline{\lambda}(v, v),$$

so $\lambda = \overline{\lambda}$. In other words, all eigenvalues of a symmetric transformation are real.

We will let $\mathbf{S} = \mathbf{S}(V)$ denote the "unit sphere" of V, i.e. $\mathbf{S}$ denotes the set of all $\phi \in V$ such that $\|\phi\| = 1$. A linear transformation T is called **bounded** if $\|T\phi\|$ is bounded as ϕ ranges over all of $\mathbf{S}$. If T is bounded, we let

$$\|T\| := \max_{\phi \in \mathbf{S}} \|T\phi\|.$$

Then

$$\|Tv\| \leq \|T\|\|v\|$$

for all $v \in V$. A linear transformation on a finite dimensional space is automatically bounded, but not so for an infinite dimensional space.

Also, for any linear transformation T, we will let $N(T)$ denote the kernel of T, so

$$N(T) = \{v \in V | Tv = 0\}$$

and $R(T)$ denote the range of T, so

$$R(T) := \{v | v = Tw \text{ for some } w \in V\}.$$

Both $N(T)$ and $R(T)$ are linear subspaces of V.

For bounded transformations, the phrase "self-adjoint" is synonymous with "symmetric". Later on we will need to study non-bounded (not everywhere defined) symmetric transformations, and then a rather subtle and important distinction will be made between self-adjoint transformations and those which are merely symmetric. But for the rest of this section we will only be considering bounded linear transformations, and so we will freely use the phrase "self-adjoint", and (usually) drop the adjective "bounded" since all our transformations will be assumed to be bounded.

We denote the set of all (bounded) self-adjoint transformations by $\mathcal{A}$, or by $\mathcal{A}(V)$ if we need to make V explicit.

2.1 Non-negative self-adjoint transformations.

If T is a self-adjoint transformation, then

$$(Tv, v) = (v, Tv) = \overline{(Tv, v)}$$

so (Tv, v) is always a real number. More generally, for any pair of elements v and w,

$$(Tv, w) = \overline{(Tw, v)}.$$

Since (Tv, w) depends linearly on v for fixed w, we see that the rule which assigns to every pair of elements v and w the number (Tv, w) satisfies the first two conditions in our definition of a semi-scalar product. Since (Tv, v) might be negative, condition 3. of the definition need not be satisfied. This leads to the following definition:

A self-adjoint transformation T is called **non-negative** if

$$(Tv, v) \geq 0 \quad \forall\, v \in V.$$

So if T is a non-negative self-adjoint transformation, then the rule which assigns to every pair of elements v and w the number (Tv, w) *is* a semi-scalar product to which we may apply the Cauchy-Schwartz inequality and conclude that

$$|(Tv, w)| \leq (Tv, v)^{\frac{1}{2}}(Tw, w)^{\frac{1}{2}}.$$

Now let us assume in addition that T is bounded with norm $\|T\|$. Let us take $w = Tv$ in the preceding inequality. We get

$$\|Tv\|^2 = |(Tv, Tv)| \leq (Tv, v)^{\frac{1}{2}}(TTv, Tv)^{\frac{1}{2}}.$$

Now apply the Cauchy-Schwartz inequality for the original scalar product to the last factor on the right:

$$(TTv, Tv)^{\frac{1}{2}} \leq \|TTv\|^{\frac{1}{2}}\|Tv\|^{\frac{1}{2}} \leq \|T\|^{\frac{1}{2}}\|Tv\|^{\frac{1}{2}}\|Tv\|^{\frac{1}{2}} = \|T\|^{\frac{1}{2}}\|Tv\|,$$

where we have used the defining property of $\|T\|$ in the form $\|TTv\| \leq \|T\|\|Tv\|$. Substituting this into the previous inequality we get

$$\|Tv\|^2 \leq (Tv, v)^{\frac{1}{2}}\|T\|\|Tv\|.$$

If $\|Tv\| \neq 0$ we may divide this inequality by $\|Tv\|$ to obtain

$$\|Tv\| \leq \|T\|^{\frac{1}{2}}(Tv, v)^{\frac{1}{2}}. \tag{17}$$

This inequality is clearly true if $\|Tv\| = 0$ and so holds in all cases.

We will make much use of this inequality. For example, it follows from (17) that

$$(Tv, v) = 0 \quad \Rightarrow \quad Tv = 0. \tag{18}$$

It also follows from (17) that if we have a sequence $\{v_n\}$ of vectors with $(Tv_n, v_n) \to 0$ then $\|Tv_v\| \to 0$ and so

$$(Tv_n, v_n) \to 0 \quad \Rightarrow \quad Tv_n \to 0. \tag{19}$$

Notice that if T is a bounded self adjoint transformation, not necessarily non-negative, then $rI - T$ is a non-negative self-adjoint transformation if $r \geq \|T\|$: Indeed,

$$((rI - T)v, v) = r(v, v) - (Tv, v) \geq (r - \|T\|)(v, v) \geq 0$$

since, by Cauchy-Schwartz,

$$(Tv, v) \leq |(Tv, v)| \leq \|Tv\|\|v\| \leq \|T\|\|v\|^2 = \|T\|(v, v).$$

So we may apply the preceding results to $rI - T$.

3 Compact self-adjoint transformations.

We say that the self-adjoint transformation T is **compact** if it has the following property: Given any sequence of elements $u_n \in \mathbf{S}$, we can choose a subsequence u_{n_i} such that the sequence Tu_{n_i} converges to a limit in V.

Some remarks about this complicated looking definition: In case V is finite dimensional, every linear transformation is bounded, hence the sequence Tu_n lies in a bounded region of our finite dimensional space, and hence by the completeness property of the real (and hence complex) numbers, we can always find such a convergent subsequence. So in finite dimensions every T is compact. More generally, the same argument shows that if $R(T)$ is finite dimensional and T is bounded then T is compact. So the definition is of interest essentially in the case when $R(T)$ is infinite dimensional.

Also notice that if T is compact, then T is bounded. Otherwise we could find a sequence u_n of elements of $\mathbf{S}$ such that $\|Tu_n\| \geq n$ and so no subsequence Tu_{n_i} can converge.

We now come to the key result which we will use over and over again:

Theorem 3 *Let T be a compact self-adjoint operator. Then $R(T)$ has an orthonormal basis $\{\phi_i\}$ consisting of eigenvectors of T and if $R(T)$ is infinite dimensional then the corresponding sequence $\{r_n\}$ of eigenvalues converges to 0.*

Proof. We know that T is bounded. If $T = 0$ there is nothing to prove. So assume that $T \neq 0$ and let

$$m_1 := \|T\| > 0.$$

By the definition of $\|T\|$ we can find a sequence of vectors $u_n \in \mathbf{S}$ such that $\|Tu_n\| \to \|T\|$. By the definition of compactness we can find a subsequence of this sequence so that $Tu_{n_i} \to w$ for some $w \in V$. On the other hand, the transformation T^2 is self-adjoint and bounded by $\|T\|^2$. Hence $\|T\|^2 I - T^2$ is non-negative, and

$$((\|T\|^2 I - T^2)u_n, u_n) = \|T\|^2 - \|Tu_n\|^2 \to 0.$$

So we know from (19) that

$$\|T\|^2 u_n - T^2 u_n \to 0.$$

Passing to the subsequence we have $T^2 u_{n_i} = T(Tu_{n_i}) \to Tw$ and so

$$\|T\|^2 u_{n_i} \to Tw$$

or

$$u_{n_i} \to \frac{1}{m_1^2} Tw.$$

Applying T to this we get

$$Tu_{n_i} \to \frac{1}{m_1^2} T^2 w$$

or

$$T^2 w = m_1^2 w.$$

Also $\|w\| = \|T\| = m_1 \neq 0$. So $w \neq 0$. So w is an eigenvector of T^2 with eigenvalue m_1^2. We have

$$0 = (T^2 - m_1^2)w = (T + m_1)(T - m_1)w.$$

If $(T - m_1)w = 0$, then w is an eigenvector of T with eigenvalue m_1 and we normalize by setting

$$\phi_1 := \frac{1}{\|w\|} w.$$

Then $\|\phi_1\| = 1$ and

$$T\phi_1 = m_1 \phi_1.$$

If $(T - m_1)w \neq 0$ then $y := (T - m_1)w$ is an eigenvector of T with eigenvalue $-m_1$ and again we normalize by setting

$$\phi_1 := \frac{1}{\|y\|} y.$$

So we have found a unit vector $\phi_1 \in R(T)$ which is an eigenvector of T with eigenvalue $r_1 = \pm m_1$.

Now let

$$V_2 := \phi_1^{\perp}.$$

If $(w, \phi_1) = 0$, then

$$(Tw, \phi_1) = (w, T\phi_1) = r_1(w, \phi_1) = 0.$$

In other words,

$$T(V_2) \subset V_2$$

and we can consider the linear transformation T restricted to V_2 which is again compact. If we let m_2 denote the norm of the linear transformation T when restricted to V_2 then $m_2 \leq m_1$ and we can apply the preceding procedure to find a unit eigenvector ϕ_2 with eigenvalue $\pm m_2$.

We proceed inductively, letting

$$V_n := \{\phi_1, \ldots, \phi_{n-1}\}^{\perp}$$

and find an eigenvector ϕ_n of T restricted to V_n with eigenvalue $\pm m_n \neq 0$ if the restriction of T to V_n is not zero. So there are two alternatives:

- after some finite stage the restriction of T to V_n is zero. In this case $R(T)$ is finite dimensional with orthonormal basis $\phi_1, \ldots, \phi_{n-1}$. Or
- The process continues indefinitely so that at each stage the restriction of T to V_n is not zero and we get an infinite sequence of eigenvectors and eigenvalues r_i with $|r_i| \geq |r_{i+1}|$.

The first case is one of the alternatives in the theorem, so we need to look at the second alternative.

We first prove that $|r_n| \to 0$. If not, there is some $c > 0$ such that $|r_n| \geq c$ for all n (since the $|r_n|$ are decreasing). If $i \neq j$,then by the Pythagorean theorem we have

$$\|T\phi_i - T\phi_j\|^2 = \|r_i\phi_i - r_j\phi_j\|^2 = r_i^2\|\phi_i\|^2 + r_j^2\|\phi_j\|^2.$$

Since $\|\phi_i\| = \|\phi_j| = 1$ this gives

$$\|T\phi_i - T\phi_j\|^2 = r_i^2 + r_j^2 \geq 2c^2.$$

Hence no subsequence of the $T\phi_i$ can converge, since all these vectors are at least a distance $c\sqrt{2}$ apart. This contradicts the compactness of T.

To complete the proof of the theorem we must show that the ϕ_i form a basis of $R(T)$. So if $w = Tv$ we must show that the Fourier series of w with respect to the ϕ_i converges to w. We begin with the Fourier coefficients of v relative to the ϕ_i which are given by

$$a_n = (v, \phi_n).$$

Then the Fourier coefficients of w are given by

$$b_i = (w, \phi_i) = (Tv, \phi_i) = (v, T\phi_i) = (v, r_i\phi_i) = r_i a_i.$$

So

$$w - \sum_{i=1}^{n} b_i\phi_i = Tv - \sum_{i=1}^{n} a_i r_i \phi_i = T(v - \sum_{i=1}^{n} a_i\phi_i).$$

Now $v - \sum_{i=1}^{n} a_i\phi_i$ is orthogonal to $\phi_1, \ldots, \phi_n$ and hence belongs to V_{n+1}. So

$$\|T(v - \sum_{i=1}^{n} a_i\phi_i)\| \leq |r_{n+1}|\|(v - \sum_{i=1}^{n} a_i\phi_i)\|.$$

By the Pythagorean theorem,

$$\|(v - \sum_{i=1}^{n} a_i\phi_i)\| \leq \|v\|.$$

Putting the two previous inequalities together we get

$$\|w - \sum_{i=1}^{n} b_i\phi_i\| = \|T(v - \sum_{i=1}^{n} a_i\phi_i)\| \leq |r_{n+1}|\|v\| \to 0.$$

This proves that the Fourier series of w converges to w concluding the proof of the theorem.

The "converse" of the above result is easy. Here is a version: Suppose that $\mathbf{H}$ is a Hilbert space with an orthonormal basis $\{\phi_i\}$ consisting of eigenvectors of an operator T, so $T\phi_i = \lambda_i \phi_i$, and suppose that $\lambda_i \to 0$ as $i \to \infty$. Then T is compact. Indeed, for each j we can find an $N = N(j)$ such that

$$|\lambda_r| < \frac{1}{j} \quad \forall\, r > N(j).$$

We can then let $\mathbf{H}_j$ denote the closed subspace spanned by all the eigenvectors $\phi_r, r > N(j)$, so that

$$\mathbf{H} = \mathbf{H}_j^\perp \oplus \mathbf{H}_j$$

is an orthogonal decomposition and $\mathbf{H}_j^\perp$ is finite dimensional, in fact is spanned the first $N(j)$ eigenvectors of T.

Now let $\{u_i\}$ be a sequence of vectors with $\|u_i\| \le 1$ say. We decompose each element as

$$u_i = u_i' \oplus u_i'', \qquad u_i' \in \mathbf{H}_1^\perp,\ u_i'' \in \mathbf{H}_j.$$

We can choose a subsequence so that u'_{i_k} converges, because they all belong to a finite dimensional space, and hence so does Tu_{i_k} since T is bounded. We can decompose every element of this subsequence into its $\mathbf{H}_2^\perp$ and $\mathbf{H}_2$ components, and choose a subsequence so that the first component converges. Proceeding in this way, and then using the Cantor diagonal trick of choosing the k-th term of the k-th selected subsequence, we have found a subsequence such that for any fixed j, the (now relabeled) subsequence, the $\mathbf{H}_j)^\perp$ component of Tu_j converges. But the $\mathbf{H}_j$ component of Tu_j has norm less than $1/j$, and so the sequence converges by the triangle inequality.

4 Fourier's Fourier series.

We want to apply the theorem about compact self-adjoint operators that we proved in the preceding section to conclude that the functions e^{inx} form an orthonormal basis of the space $\mathcal{C}(\mathbf{T})$. In fact, a direct proof of this fact is elementary, using integration by parts. So we will pause to given this direct proof. Then we will go back and give a (more complicated) proof of the same fact using our theorem on compact operators. The reason for giving the more complicated proof is that it extends to far more general situations.

4.1 Proof by integration by parts.

We have let $\mathcal{C}(\mathbf{T})$ denote the space of continuous functions on the real line which are periodic with period 2π. We will let $\mathcal{C}^1(\mathbf{T})$ denote the space of periodic functions which have a continuous first derivative (necessarily periodic) and by $\mathcal{C}^2(\mathbf{T})$ the space of periodic functions with two continuous derivatives. If f and g both belong to $\mathcal{C}^1(\mathbf{T})$ then integration by parts gives

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f'\overline{g}dx = -\frac{1}{2\pi}\int_{-\pi}^{\pi} f\overline{g}'dx$$

since the boundary terms, which normally arise in the integration by parts formula, cancel, due to the periodicity of f and g. If we take $g = e^{inx}/(in), n \neq 0$ the integral on the right hand side of this equation is the Fourier coefficient:

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx.$$

We thus obtain

$$c_n = \frac{1}{in}\frac{1}{2\pi}\int_{-\pi}^{\pi} f'(x)e^{-inx}dx$$

so, for $n \neq 0$,

$$|c_n| \leq \frac{A}{n} \quad \text{where } A := \frac{1}{2\pi}\int_{-\pi}^{\pi} |f'(x)|dx$$

is a constant independent of n (but depending on f).

If $f \in \mathcal{C}^2(\mathbf{T})$ we can take $g(x) = -e^{inx}/n^2$ and integrate by parts twice. We conclude that (for $n \neq 0$)

$$|c_n| \leq \frac{B}{n^2} \quad \text{where } \; B := \frac{1}{2\pi}\int_{-\pi}^{\pi} |f''(x)|^2$$

is again independent of n. But this proves that the Fourier series of f,

$$\sum c_n e^{inx}$$

converges uniformly and absolutely for and $f \in \mathcal{C}^2(\mathbf{T})$. The limit of this series is therefore some continuous periodic function. We must prove that this limit equals f. So we must prove that at each point f

$$\sum c_n e^{iny} \to f(y).$$

Replacing $f(x)$ by $f(x-y)$ it is enough to prove this formula for the case $y = 0$. So we must prove that for any $f \in \mathcal{C}^2(\mathbf{T})$ we have

$$\lim_{N,M\to\infty} \sum_{-N}^{M} c_n \to f(0).$$

Write $f(x) = (f(x) - f(0)) + f(0)$. The Fourier coefficients of any constant function c all vanish except for the c_0 term which equals c. So the above limit is trivially true when f is a constant. Hence, in proving the above formula, it is enough to prove it under the additional assumption that $f(0) = 0$, and we need to prove that in this case

$$\lim_{N,M\to\infty} (c_{-N} + c_{-N+1} + \cdots + c_M) \to 0.$$

The expression in parenthesis is

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\overline{g_{N,M}(x)}dx$$

where

$$g_{N,M}(x) = e^{-iNx} + e^{-i(N-1)x} + \cdots + e^{iMx} = e^{-iNx}\left(1 + e^{ix} + \cdots + e^{i(M+N)x}\right) =$$

$$e^{-iNx}\frac{1 - e^{i(M+N+1)x}}{1 - e^{ix}} = \frac{e^{-iNx} - e^{i(M+1)x}}{1 - e^{ix}}, \quad x \neq 0$$

where we have used the formula for a geometric sum. By l'Hôpital's rule, this extends continuously to the value $M + N + 1$ for $x = 0$. Now $f(0) = 0$, and since f has two continuous derivatives, the function

$$h(x) := \frac{f(x)}{1 - e^{-ix}}$$

defined for $x \neq 0$ (or any multiple of 2π) extends, by l'Hôpital's rule, to a function defined at all values, and which is continuously differentiable and periodic. Hence the limit we are computing is

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} h(x)e^{iNx}dx - \frac{1}{2\pi}\int_{-\pi}^{\pi} h(x)e^{-i(M+1)x}dx$$

and we know that each of these terms tends to zero.

We have thus proved that the Fourier series of any twice differentiable periodic function converges uniformly and absolutely to that function. If we consider the space $\mathcal{C}^2(\mathbf{T})$ with our usual scalar product

$$(f, g) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f\overline{g}dx$$

then the functions e^{inx} are dense in this space, since uniform convergence implies convergence in the $\| \ \|$ norm associated to $(\ , \ )$. So, on general principles, Bessel's inequality and Parseval's equation hold.

It is not true in general that the Fourier series of a continuous function converges uniformly to that function (or converges at all in the sense of uniform convergence). However it is true that we *do* have convergence in the L_2 norm, i.e. the Hilbert space $\| \ \|$ norm on $\mathcal{C}(\mathbf{T})$. To prove this, we need only prove that the exponential functions e^{inx} are dense, and since they are dense in $\mathcal{C}^2(\mathbf{T})$, it is enough to prove that $\mathcal{C}^2(\mathbf{T})$ is dense in $\mathcal{C}(\mathbf{T})$. For this, let ϕ be a function defined on the line with at least two continuous bounded derivatives with $\phi(0) = 1$ and of total integral equal to one and which vanishes rapidly at infinity. A favorite is the Gauss normal function

$$\phi(x) := \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$$

Equally well, we could take ϕ to be a function which actually vanishes outside of some neighborhood of the origin. Let

$$\phi_t(x) := \frac{1}{t}\phi\left(\frac{x}{t}\right).$$

As $t \to 0$ the function ϕ_t becomes more and more concentrated about the origin, but still has total integral one. Hence, for any bounded continuous function f, the function $\phi_t \star f$ defined by

$$(\phi_t \star f)(x) := \int_{-\infty}^{\infty} f(x-y)\phi_t(y)dy = \int_{-\infty}^{\infty} f(u)\phi_t(x-u)du.$$

satisfies $\phi_t \star f \to f$ uniformly on any finite interval. From the rightmost expression for $\phi_t \star f$ above we see that $\phi_t \star f$ has two continuous derivatives. From the first expression we see that $\phi_t \star f$ is periodic if f is. This proves that $\mathcal{C}^2(\mathbf{T})$ is dense in $\mathcal{C}(\mathbf{T})$. We have thus proved convergence in the L_2 norm.

4.2 Relation to the operator $\frac{d}{dx}$.

Each of the functions $e^{inx}$is an eigenvector of the operator

$$D = \frac{d}{dx}$$

in that

$$D\left(e^{inx}\right) = ine^{inx}.$$

So they are also eigenvalues of the operator D^2 with eigenvalues $-n^2$. Also, on the space of twice differentiable periodic functions the operator D^2 satisfies

$$(D^2 f, g) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f''(x)\overline{g(x)}dx = f'(x)\overline{g(x)}\Big|_{-\pi}^{\pi} - \frac{1}{2\pi}\int_{-\pi}^{\pi} f'(x)\overline{g'(x)}dx$$

by integration by parts. Since f' and g are assumed to be periodic, the end point terms cancel, and integration by parts once more shows that

$$(D^2 f, g) = (f, D^2 g) = -(f', g').$$

But of course D and certainly D^2 is not defined on $\mathcal{C}(\mathbf{T})$ since some of the functions belonging to this space are not differentiable. Furthermore, the eigenvalues of D^2 are tending to infinity rather than to zero. So somehow the operator D^2 must be replaced with something like its inverse. In fact, we will work with the inverse of $D^2 - 1$, but first some preliminaries.

We will let $\mathcal{C}^2([-\pi,\pi])$ denote the functions defined on $[-\pi.\pi]$ and twice differentiable there, with continuous second derivatives up to the boundary. We denote by $\mathcal{C}([-\pi,\pi])$ the space of functions defined on $[-\pi,\pi]$ which are continuous up to the boundary. We can regard $\mathcal{C}(\mathbf{T})$ as the subspace of $\mathcal{C}([-\pi,\pi])$ consisting of those functions which satisfy the boundary conditions $f(\pi) = f(-\pi)$ (and then extended to the whole line by periodicity).

We regard $\mathcal{C}([-\pi,\pi])$ as a pre-Hilbert space with the same scalar product that we have been using:

$$(f, g) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\overline{g(x)}dx.$$

If we can show that every element of $\mathcal{C}([-\pi,\pi])$ is a sum of its Fourier series (in the pre-Hilbert space sense) then the same will be true for $\mathcal{C}(\mathbf{T})$. So we will work with $\mathcal{C}([-\pi,\pi])$.

We can consider the operator $D^2 - 1$ as a linear map

$$D^2 - 1 : \mathcal{C}^2([-\pi,\pi]) \to \mathcal{C}([-\pi,\pi]).$$

This map is surjective, meaning that given any continuous function g we can find a twice differentiable function f satisfying the differential equation

$$f'' - f = g.$$

In fact we can find a whole two dimensional family of solutions because we can add any solution of the homogeneous equation

$$h'' - h = 0$$

to f and still obtain a solution. We could write down an explicit solution for the equation $f'' - f = g$, but we will not need to. It is enough for us to know that the solution exists, which follows from the general theory of ordinary differential equations.

The general solution of the homogeneous equation is given by

$$h(x) = ae^x + be^{-x}.$$

Let

$$M \subset \mathcal{C}^2([-\pi,\pi])$$

be the subspace consisting of those functions which satisfy the "periodic boundary conditions"

$$f(\pi) = f(-\pi), \quad f'(\pi) = f'(-\pi).$$

Given any f we can always find a solution of the homogeneous equation such that $f - h \in M$. Indeed, we need to choose the complex numbers a and b such that if h is as given above, then

$$h(\pi) - h(-\pi) = f(\pi) - f(-\pi), \text{ and } h'(\pi) - h'(-\pi) = f'(\pi) - f'(-\pi).$$

Collecting coefficients and denoting the right hand side of these equations by c and d we get the linear equations

$$(e^\pi - e^{-\pi})(a - b) = c, (e^\pi - e^{-\pi})(a + b) = d$$

which has a unique solution.

So there exists a unique operator

$$T : \mathcal{C}([-\pi,\pi]) \to M$$

with the property that

$$(D^2 - I) \circ T = I.$$

We will prove that

$$T \text{ is self adjoint and compact.} \tag{20}$$

Once we will have proved this fact, then we know every element of M can be expanded in terms of a series consisting of eigenvectors of T with non-zero eigenvalues. But if

$$Tw = \lambda w$$

then

$$D^2 w = (D^2 - I)w + w = \frac{1}{\lambda}[(D^2 - I) \circ T]w + w = \left(\frac{1}{\lambda} + 1\right) w.$$

So w must be an eigenvector of D^2; it must satisfy

$$w'' = \mu w.$$

So if $\mu = 0$ then $w =$ a constant is a solution. If $\mu = r^2 > 0$ then w is a linear combination of e^{rx} and e^{-rx} and as we showed above, no non-zero such combination can belong to M. If $\mu = -r^2$ then the solution is a linear combination of e^{irx} and e^{-irx} and the above argument shows that r must be such that $e^{ir\pi} = e^{-ir\pi}$ so $r = n$ is an integer.

Thus (20) will show that the e^{inx} are a basis of M, and a little more work that we will do at the end will show that they are in fact also a basis of $\mathcal{C}([-\pi, \pi])$. But first let us work on (20).

It is easy to see that T is self adjoint. Indeed, let $f = Tu$ and $g = Tv$ so that f and g are in M and

$$(u, Tv) = ([D^2 - 1]f, g) = -(f', g') - (f, g) = (f, [D^2 - 1]g) = (Tu, v)$$

where we have used integration by parts and the boundary conditions defining M for the two middle equalities.

4.3 Gårding's inequality, special case.

We now turn to the compactness. We have already verified that for any $f \in M$ we have

$$([D^2 - 1]f, f) = -(f', f') - (f, f).$$

Taking absolute values we get

$$\|f'\|^2 + \|f\|^2 \leq |([D^2 - 1]f, f)|. \tag{21}$$

(We actually get equality here, the more general version of this that we will develop later will be an inequality.)

Let $u = [D^2 - 1]f$ and use the Cauchy-Schwartz inequality

$$|([D^2 - 1]f, f)| = |(u, f)| \leq \|u\|\|f\|$$

on the right hand side of (21) to conclude that

$$\|f\|^2 \leq \|u\|\|f\|$$

or

$$\|f\| \leq \|u\|.$$

Use (21) again to conclude that

$$\|f'\|^2 \leq \|u\|\|f\| \leq \|u\|^2$$

by the preceding inequality. We have $f = Tu$, and let us now suppose that u lies on the unit sphere i.e. that $\|u\| = 1$. Then we have proved that

$$\|f\| \leq 1, \quad \text{and} \quad \|f'\| \leq 1. \tag{22}$$

We wish to show that from any sequence of functions satisfying these two conditions we can extract a subsequence which converges. Here convergence means, of course, with respect to the norm given by

$$\|f\|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2 dx.$$

In fact, we will prove something stronger: that given any sequence of functions satisfying (22) we can find a subsequence which converges in the uniform norm

$$\|f\|_\infty := \max_{x\in[-\pi,\pi]} |f(x)|.$$

Notice that

$$\|f\| = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2 dx\right)^{\frac{1}{2}} \leq \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |f|_\infty^2 dx\right)^{\frac{1}{2}} = \|f\|_\infty$$

so convergence in the uniform norm implies convergence in the norm we have been using.

To prove our result, notice that for any $\pi \leq a < b \leq \pi$ we have

$$|f(b) - f(a)| = \left|\int_a^b f'(x)dx\right| \leq \int_a^b |f'(x)|dx = 2\pi(|f'|, \mathbf{1}_{[a,b]})$$

where $\mathbf{1}_{[a,b]}$ is the function which is one on $[a, b]$ and zero elsewhere. Apply Cauchy-Schwartz to conclude that

$$|(|f'|, \mathbf{1}_{[a,b]})| \leq \|\ |f'|\ \| \cdot \|\mathbf{1}_{[a,b]}\|.$$

But

$$\|\mathbf{1}_{[a,b]}\|^2 = \frac{1}{2\pi}|b - a|$$

and

$$\|\ |f'|\| = \|f'\| \leq 1.$$

We conclude that

$$|f(b) - f(a)| \le (2\pi)^{\frac{1}{2}} |b-a|^{\frac{1}{2}}. \qquad (23)$$

In this inequality, let us take b to be a point where $|f|$ takes on its maximum value, so that $|f(b)| = \|f\|_\infty$. Let a be a point where $|f|$ takes on its minimum value. (If necessary interchange the role of a and b to arrange that $a < b$ or observe that the condition $a < b$ was not needed in the above proof.) Then (23) implies that

$$\|f\|_\infty - \min |f| \le (2\pi)^{\frac{1}{2}} |b-a|^{\frac{1}{2}}.$$

But

$$1 \ge \|f\| = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |f|^2(x)dx\right)^{\frac{1}{2}} \ge \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} (\min|f|)^2 dx\right)^{\frac{1}{2}} = \min|f|$$

and $|b-a| \le 2\pi$ so

$$\|f\|_\infty \le 1 + 2\pi.$$

Thus the values of all the $f \in T[S]$ are all uniformly bounded - (they take values in a circle of radius $1+2\pi$) and they are equicontinuous in that (23) holds. This is enough to guarantee that out of every sequence of such f we can choose a uniformly convergent subsequence.

(We recall how the proof of this goes: Since all the values of all the f are bounded, at any point we can choose a subsequence so that the values of the f at that point converge, and, by passing to a succession of subsequences (and passing to a diagonal), we can arrange that this holds at any countable set of points. In particular, we may choose say the rational points in $[-\pi, \pi]$. Suppose that f_n is this subsequence. We claim that (23) then implies that the f_n form a Cauchy sequence in the uniform norm and hence converge in the uniform norm to some continuous function. Indeed, for any ϵ choose δ such that

$$(2\pi)^{\frac{1}{2}}\delta^{\frac{1}{2}} < \frac{1}{3}\epsilon,$$

choose a finite number of rational points which are within δ distance of any point of $[-\pi, \pi]$ and choose N sufficiently large that $|f_i - f_j| < \frac{1}{3}\epsilon$ at each of these points, r. when i and j are $\ge N$. Then at any $x \in [-\pi, \pi]$

$$|f_i(x) - f_j(x)| \le |f_i(x) - f_i(r)| + |f_j(x) - f_j(x)| + |f_i(r) - f_j(r)| \le \epsilon$$

since each of the three terms is $\le \frac{1}{3}\epsilon$.)

5 The Heisenberg uncertainty principle.

In this section we show how the arguments leading to the Cauchy-Schwartz inequality give one of the most important discoveries of twentieth century physics, the Heisenberg uncertainty principle.

Let V be a pre-Hilbert space, and S denote the unit sphere in V. If ϕ and ψ are two unit vectors (i.e. elements of S) their scalar product (ϕ, ψ) is a complex number with $0 \leq |(\phi, \psi)|^2 \leq 1$. In quantum mechanics, this number is taken as a probability. Although in the "real world" V is usually infinite dimensional, we will warm up by considering the case where V is finite dimensional.

Given a $\phi \in S$ and an orthonormal basis $\phi_1, \ldots, \phi_n$ of V, we have

$$1 = \|\phi\|^2 = |(\phi, \phi_1)|^2 + \cdots + |(\phi, \phi_n|^2.$$

The says that the various probabilities $|(\phi, \phi_i)|^2$ add up to one. We recall some language from elementary probability theory: Suppose we have an experiment resulting in a finite number of measured numerical outcomes λ_i, each with probability p_i of occurring. Then the mean $\langle \lambda \rangle$ is defined by

$$\langle \lambda \rangle := \lambda_1 p_1 + \cdots + \lambda_n p_n$$

and its variance $(\Delta\lambda)^2$

$$(\Delta\lambda)^2 := (\lambda_1 - \langle \lambda \rangle)^2 p_1 + \cdots + (\lambda_n - \langle \lambda \rangle)^2 p_n$$

and an immediate computation shows that

$$(\Delta\lambda)^2 = \langle \lambda^2 \rangle - \langle \lambda \rangle^2.$$

The square root $\Delta\lambda$ of the variance is called the standard deviation. The variance (or the standard deviation) measures the "spread" of the possible values of λ. To understand its meaning we have Chebychev's inequality which estimates the probability that λ_k can deviate from $\langle \lambda \rangle$ by as much as $r\Delta\lambda$ for any positive number r. Chebychev's inequality says that this probability is $\leq 1/r^2$. In symbols

$$\text{Prob } (|\lambda_k - \langle \lambda \rangle| \geq r\Delta\lambda) \leq \frac{1}{r^2}.$$

Indeed, the probability on the left is the sum of all the p_k such that $|\lambda_k - \langle \lambda \rangle| \geq r\Delta$. Denoting this sum by $\sum_r$ we have

$$\sum\nolimits_r p_k \leq \sum\nolimits_r p_k \frac{(\lambda - \langle \lambda \rangle)^2}{r^2(\Delta\lambda)^2} \leq$$

$$\leq \sum_{\text{all } k} p_k \frac{(\lambda - \langle \lambda \rangle)^2}{r^2(\Delta\lambda)^2} = \frac{1}{r^2(\Delta\lambda)^2} \sum_{\text{all } k} (\lambda - \langle \lambda \rangle)^2 p_k = \frac{1}{r^2}.$$

Replacing λ_i by $\lambda_i + c$ does not change the variance.

Now suppose that A is a self-adjoint operator on V, that the λ_i are the eigenvalues of A with eigenvectors ϕ_i constituting an orthonormal basis, and that the $p_i = |(\phi, \phi_i|^2$ as above.

1. Show that $\langle \lambda \rangle = (A\phi, \phi)$ and that $(\Delta\lambda)^2 = (A^2\phi, \phi) - (A\phi, \phi)^2$.

We will write the expression $(A\phi, \phi)$ as $\langle A \rangle_\phi$, In quantum mechanics a unit vector is called a **state** and a self-adjoint operator is called an **observable** and the expression $\langle A \rangle_\phi$ is called the **expectation** of the observable A in the state ϕ. Similarly we denote $\left((A^2\phi, \phi) - (A\phi, \phi)^2\right)^{1/2}$ by $\Delta_\phi A$. It is called the **uncertainty** of the observable A in the state ϕ. Notice that

$$(\Delta_\phi A)^2 = \langle (A - \langle A \rangle I)^2 \rangle_\phi$$

where I denotes the identity operator. Indeed

$$((A - (A\phi, \phi)I)^2 = A^2 - 2(A\phi, \phi)A + (A\phi, \phi)^2 I$$

so

$$\langle (A - \langle A \rangle_\phi)^2 \rangle_\phi = (A^2\phi, \phi) - 2\langle A \rangle_\phi^2 + \langle A \rangle_\phi^2 = \langle A^2 \rangle_\phi - \langle A \rangle_\phi^2.$$

When the state ϕ is fixed in the course of discussion, we will drop the subscript ϕ and write $\langle A \rangle$ and ΔA instead of $\langle A \rangle_\phi$ and $\Delta_\phi A$. For example, we would write the previous result as

$$\Delta A = \langle (A - \langle A \rangle I)^2 \rangle.$$

If A and B are operators we let $[A, B]$ denote the commutator:

$$[A, B] := AB - BA.$$

Notice that $[A, B] = -[B, A]$ and $[I, B] = 0$ for any B. So if A and B are self adjoint, so is $i[A, B]$ and replacing A by $A - \langle A \rangle I$ and B by $B - \langle B \rangle I$ does not change ΔA, ΔB or $i[A, B]$.

The **uncertainty principle** says that for any two observables A and B we have

$$(\Delta A)(\Delta B) \geq \frac{1}{2} |\langle i[A, B] \rangle| q.$$

Proof. Set $A_1 := A - \langle A \rangle$, $B_1 := B - \langle B \rangle$ so that

$$[A_1, B_1] = [A, B].$$

Let

$$\psi := A_1\phi + ixB_1\phi.$$

Then

$$(\psi, \psi) = (\Delta A)^2 - x\langle i[A, B] \rangle + (\Delta B)^2.$$

Since $(\psi, \psi) \geq 0$ for all x this implies that $(b^2 \leq 4ac)$ that

$$\langle i[A, B] \rangle^2 \leq 4(\Delta A)^2(\Delta B)^2,$$

and taking square roots gives the result.

The purpose of the next few sections is to provide a vast generalization of the results we obtained for the operator D^2. We will prove the corresponding results for any "elliptic" differential operator (definitions below).

I plan to study differential operators acting on vector bundles over manifolds. But it requires some effort to set things up, and I want to get to the key analytic ideas which are essentially repeated applications of integration by parts. So I will start with elliptic operators L acting on functions on the torus $\mathbf{T} = \mathbf{T}^n$, where there are no boundary terms when we integrate by parts. Then an immediate extension gives the result for elliptic operators on functions on manifolds, and also for boundary value problems such as the Dirichlet problem.

The treatment here rather slavishly follows the treatment by Bers and Schechter in *Partial Differential Differential Equations* by Bers, John and Schechter AMS (1964).

6 The Sobolev Spaces.

Recall that $\mathbf{T}$ now stands for the n-dimensional torus. Let $\mathbf{P} = \mathbf{P}(\mathbf{T})$ denote the space of trigonometric polynomials. These are functions on the torus of the form

$$u(x) = \sum a_\ell e^{i\ell\cdot x}$$

where

$$\ell = (\ell_1, \ldots, \ell_n)$$

is an n-tuplet of integers and the sum is finite. For each integer t (positive, zero or negative) we introduce the scalar product

$$(u, v)_t := \sum_\ell (1 + \ell\cdot\ell)^t a_\ell \bar{b}_\ell. \tag{24}$$

For $t = 0$ this is the scalar product

$$(u, v)_0 = \frac{1}{(2\pi)^n}\int_{\mathbf{T}} u(x)\overline{v(x)}dx.$$

This differs by a factor of $(2\pi)^{-n}$ from the scalar product that is used by Bers and Schecter. We will denote the norm corresponding to the scalar product $(\ ,\ )_s$ by $\|\ \|_s$.

If

$$\Delta := -\left(\frac{\partial^2}{\partial(x^1)^2} + \cdots + \frac{\partial^2}{\partial(x^n)^2}\right)$$

the operator $(1 + \Delta)$ satisfies

$$(1 + \Delta)u = \sum (1 + \ell\cdot\ell)a_\ell e^{i\ell\cdot x}$$

and so

$$((1 + \Delta)^t u, v)_s = (u, (1 + \Delta)^t v)_s = (u, v)_{s+t}$$

and

$$\|(1 + \Delta)^t u\|_s = \|u\|_{s+2t}. \tag{25}$$

We then get the "generalized Schwartz inequality"

$$|(u,v)_s| \le \|u\|_{s+t}\|v\|_{s-t} \tag{26}$$

for any t, as a consequence of the usual Cauchy-Schwartz inequality. Indeed,

$$\begin{aligned}
(2\pi)^n \sum_\ell (1+\ell\cdot\ell)^s a_\ell \overline{b}_\ell &= (2\pi)^n \sum_\ell (1+\ell\cdot\ell)^{\frac{s+t}{2}} a_\ell (1+\ell\cdot\ell)^{\frac{s-t}{2}} \overline{b}_\ell \\
&= ((1+\Delta)^{\frac{s+t}{2}} u, (1+\Delta)^{\frac{s-t}{2}} v)_0 \\
&\le \|(1+\Delta)^{\frac{s+t}{2}} u\|_0 \|(1+\Delta)^{\frac{s-t}{2}} v\|_0 \\
&= \|u\|_{s+t}\|v\|_{s-t}.
\end{aligned}$$

The generalized inequality reduces to the usual Cauchy-Schwartz inequality when $t=0$.

Clearly we have

$$\|u\|_s \le \|u\|_t \quad \text{if } s \le t.$$

If D^p denotes a partial derivative,

$$D^p = \frac{\partial^{|p|}}{\partial(x^1)^{p_1}\cdots\partial(x^n)^{p_m}}$$

then

$$D^p u = \sum (i\ell)^p a_\ell e^{i\ell\cdot x}.$$

In these equations we are using the following notations:

- If $p = (p_1, dots, p_n)$ is a vector with non-negative integer entries we set

$$|p| := p_1 + \cdots + p_n.$$

- If $\xi = (\xi_1, \ldots, \xi_n)$ is a (row) vector we set

$$\xi^p := \xi_1^{p_1}\cdot\xi_2^{p_2}\cdots\xi_n^{p_n}$$

It is then clear that

$$\|D^p u\|_t \le \|u\|_{t+|p|} \tag{27}$$

and similarly

$$\|u\|_t \le (\text{constant depending on } t) \sum_{|p|\le t} \|D^p u\|_0 \quad \text{if } t \ge 0. \tag{28}$$

In particular,

Proposition 2 *The norms*

$$u \mapsto \|u\|_t$$

$t \geq 0$ *and*

$$u \mapsto \sum_{|p| \leq t} \|D^p\|_0$$

are equivalent.

We let $\mathbf{H}_t$ denote the completion of the space $\mathbf{P}$ with respect to the norm $\| \, \|_t$. Each $\mathbf{H}_t$ is a Hilbert space, and we have natural embeddings

$$\mathbf{H}_t \hookrightarrow \mathbf{H}_s \ \text{ if } s < t.$$

Equation (25) says that

$$(1+\Delta)^t : \mathbf{H}_{s+2t} \rightarrow \mathbf{H}_s$$

and is an isometry.

From the generalized Schwartz inequality we also have a natural pairing of $\mathbf{H}_t$ with $\mathbf{H}_{-t}$ given by the extension of $(\ , \ )_0$, so

$$|(u,v)_0| \leq \|u\|_t \|v\|_{-t}. \tag{29}$$

In fact, this pairing allows us to identify $\mathbf{H}_{-t}$ with the space of continuous linear functions on $\mathbf{H}_t$. Indeed, if ϕ is a continuous linear function on $\mathbf{H}_t$ the Riesz representation theorem tells us that there is a $w \in \mathbf{H}_t$ such that $\phi(u) = (u,w)_t$. Set

$$v := (1+\Delta)^t w.$$

Then

$$v \in \mathbf{H}_{-t}$$

and

$$(u,v)_0 = (u,(1+\Delta)^t w)_0 = (u,w)_t = \phi(u).$$

We record this fact as

$$\mathbf{H}_{-t} = (\mathbf{H}_t)^* . \tag{30}$$

As an illustration of (30), observe that the series

$$\sum_\ell (1+\ell\cdot\ell)^s$$

converges for

$$s < -\frac{n}{2}.$$

This means that if define v by taking

$$b_\ell \equiv 1$$

then $v \in \mathbf{H}_s$ for $s < -\frac{n}{2}$. If u is given by $u(x) = \sum_\ell a_\ell e^{i\ell\cdot x}$ is any trigonometric polynomial, then

$$(u, v)_0 = suma_\ell = u(0).$$

So the natural pairing (29) allows us to extend the linear function sending $u \mapsto u(0)$ to all of $\mathbf{H}_t$ if $t > \frac{n}{2}$. We can now give v its "true name": it is the Dirac "delta function" δ (on the torus) where

$$(u, \delta)_0 = u(0).$$

So $\delta \in H_{-t}$ for $t > \frac{n}{/}2$, and the preceding equation is usually written symbolically as

$$\frac{1}{(2\pi)^n}\int_{\mathbf{T}} u(x)\delta(x)dx = u(0);$$

but the true mathematical interpretation is as given above.

We set

$$\mathbf{H}_\infty := \bigcap \mathbf{H}_t, \quad \mathbf{H}_{-\infty} := \bigcup \mathbf{H}_t.$$

The space $\mathbf{H}_0$ is just $L_2(\mathbf{T})$, and we can think of the space $\mathbf{H}_t, \quad t > 0$ as consisting of those functions having "generalized L_2 derivatives up to order t". Certainly a function of class C^t belongs to $\mathbf{H}_t$. With a loss of degree of differentiability the converse is true:

Lemma 1 [Sobolev.] *If $u \in \mathbf{H}_t$ and*

$$t \geq \left[\frac{n}{2}\right] + k + 1$$

then $u \in C^k(\mathbf{T})$ and

$$\sup_{x\in\mathbf{T}} |D^p u(x)| \leq \text{const.}\|u\|_t \quad \textit{for } |p| \leq k. \tag{31}$$

By applying the lemma to $D^p u$ it is enough to prove the lemma for $k = 0$. So we assume that $u \in \mathbf{H}_t$ with $t \geq [n/2] + 1$. Then

$$(\sum |a_\ell|)^2 \leq \left(\sum (1 + \ell\cdot\ell)^t |a_\ell|^2\right) \sum (1 + \ell\cdot\ell)^{-t} < \infty,$$

since the series $\sum (1 + \ell\cdot\ell)^{-t}$ converges for $t \geq [n/2] + 1$. So for this range of t, the Fourier series for u converges absolutely and uniformly. The right hand side of the above inequality gives the desired bound. QED

A **distribution** on $\mathbf{T}^n$ is a linear function T on $C^\infty(\mathbf{T}^n)$ with the continuity condition that

$$\langle T, \phi_k\rangle \to 0$$

whenever

$$D^p \phi_k \to 0$$

uniformly for each fixed p. If $u \in \mathbf{H}_{-t}$ we may define

$$\langle u, \phi \rangle := (\phi, \overline{u})_0$$

and since $C^\infty(\mathbf{T})$ is dense in $\mathbf{H}_t$ we may conclude

Lemma 2 *$\mathbf{H}_{-t}$ is the space of those distributions T which are continuous in the $\| \ \|_t$ norm, i.e. which satisfy*

$$\|\phi_k\|_t \to 0 \quad \Rightarrow \langle T, \phi_k \rangle \to 0.$$

We then obtain

Theorem 4 [Laurent Schwartz.] *$\mathbf{H}_\infty$ is the space of all distributions. In other words, any distribution belongs to $\mathbf{H}_{-t}$ for some t.*

Proof. Suppose that T is a distribution that does not belong to any $\mathbf{H}_{-t}$. This means that for any $k > 0$ we can find a C^∞ function ϕ_k with

$$\|\phi_k\|_k < \frac{1}{k}$$

and

$$|\langle T, \phi_k \rangle| \geq 1.$$

But by Lemma 1 we know that $\|\phi_k\|_k < \frac{1}{k}$ implies that $D^p\phi_k \to 0$ uniformly for any fixed p contradicting the continuity property of T. QED

Suppose that ϕ is a C^∞ function on $\mathbf{T}$. Multiplication by ϕ is clearly a bounded operator on $\mathbf{H}_0 = L_2(\mathbf{T})$, and so it is also a bounded operator on $\mathbf{H}_t$, $t > 0$ since we can expand $D^p(\phi u)$ by applications of Leibnitz's rule.

For $t = -s < 0$ we know by the generalized Cauchy Schwartz inequality that

$$\|\phi u\|_t = \sup |(v, \phi u)_0| / \|v\|_s = \sup |(u, \overline{\phi} v)| / \|v\|_s \leq \|u\|_t \|\overline{\phi} v\|_s / \|v\|_s.$$

So in all cases we have

$$\|\phi u\|_t \leq (\text{const. depending on } \phi \text{ and } t) \|u\|_t. \tag{32}$$

Let

$$L = \sum_{|p| \leq m} \alpha_p(x) D^p$$

be a differential operator of degree m with C^∞ coefficients. Then it follows from the above that

$$\|Lu\|_{t-m} \leq \text{constant} \|u\|_t \tag{33}$$

where the constant depends on L and t.

Lemma 3 [Rellich's lemma.] *If $s < t$ the embedding $\mathbf{H_t} \hookrightarrow \mathbf{H}_s$ is compact.*

Proof. We must show that the image of the unit ball B of $\mathbf{H}_t$ in $\mathbf{H}_t$ can be covered by finitely many balls of radius ϵ. Choose N so large that

$$(1+\ell\cdot\ell)^{(s-t)/2} < \frac{\epsilon}{2}$$

when $\ell\cdot\ell > N$. Let Z be the subspace of $\mathbf{H}_t$ consisting of all u such that $a_\ell = 0$ when $\ell\cdot\ell \leq N$. This is a space of finite codimension, and hence the unit ball of $\mathbf{H}_t/Z$ can be covered by finitely many balls of radius $\frac{\epsilon}{2}$. On the other hand, for $u \in B\cap Z$ we have

$$\|u\|_s^2 \leq (1+N^2)^{s-t}\|u\|_t \leq \left(\frac{\epsilon}{2}\right)^2.$$

So the image of $B\cap Z$ is contained in a ball of radius $\frac{\epsilon}{2}$ and so the image of B is covered by finitely many balls of radius ϵ. QED

7 Gårding's inequality.

Let $x,\ a$, and b be positive numbers. Then

$$x^a + x^{-b} \geq 1$$

because if $x \geq 1$ the first summand is ≥ 1 and if $x \leq 1$ the second summand is ≥ 1. Setting $x = \epsilon^{1/a}A$ gives

$$1 \leq \epsilon A^a + \epsilon^{-b/a}A^{-b}$$

if ϵ and A are positive. Suppose that $t_1 > s > t_2$ and we set $a = t_1 - s,\ b = s - t_2$ and $A = 1+\ell\cdot\ell$. Then we get

$$(1+\ell\cdot\ell)^s \leq \epsilon(1+\ell\cdot\ell)^{t_1} + \epsilon^{-(s-t_2)/(t_1-s)}(1+\ell\cdot\ell)^{t_2}$$

and therefore

$$\|u\|_s \leq \epsilon\|u\|_{t_1} + \epsilon^{-(s-t_2)/(t_1-s)}\|u\|_{t_2} \quad \text{if } t_1 > s > t_2, \quad \epsilon > 0 \qquad (34)$$

for all $u \in \mathbf{H}_{t_1}$. This elementary inequality will be the key to several arguments in this section where we will combine (34) with integration by parts.

A differential operator $L = \sum_{|p|\leq m} a_p(x)D^p$ with real coefficients and m even is called **elliptic** if there is a constant $c > 0$ such that

$$(-1)^{m/2}\sum_{|p|=m} a_p(x)\xi^p \geq c(\xi\cdot\xi|^{m/2}. \qquad (35)$$

In this inequality, the vector ξ is a "dummy variable". (Its true invariant significance is that it is a covector, i.e. an element of the cotangent space at x.) The expression on the left of this inequality is called the **symbol** of the operator L. It is a homogeneous polynomial of degree m in the variable ξ whose coefficients are functions of x. The symbol of L is sometimes written as $\sigma(L)$ or $\sigma(L)(x,\xi)$. Another way of expressing condition (35) is to say that there is a positive constant c such that

$$\sigma(L)(x,\xi) \geq c \ \text{ for all } x \ \text{ and } \xi \text{ such that } \ \xi \cdot \xi = 1.$$

We will assume until further notice that the operator L is elliptic and that m is a positive even integer.

Theorem 5 **[Gårding's inequality.]** *For every $u \in C^\infty(\mathbf{T})$ we have*

$$(u, Lu)_0 \geq c_1 \|u\|_{m/2}^2 - c_2 \|u\|_0^2 \tag{36}$$

where c_1 and c_2 are constants depending on L.

Remark. If $u \in \mathbf{H}_{m/2}$, then both sides of the inequality make sense, and we can approximate u in the $\| \ \|_{m/2}$ norm by C^∞ functions. So once we prove the theorem, we conclude that it is also true for all elements of $\mathbf{H}_{m/2}$.

We will prove the theorem in stages:

1. When L is constant coefficient and homogeneous.
2. When L is homogeneous and approximately constant.
3. When the L can have lower order terms but the homogeneous part of L is approximately constant.
4. The general case.

Stage 1. $L = \sum_{|p|=m} \alpha_p D^p$ where the α_p are constants. Then

$$\begin{aligned}
(u, Lu)_0 &= \left(\sum a_\ell e^{i\ell\cdot x}, \sum_\ell \left(\sum_{|p|=m} \alpha_p (i\ell^p) \right) a_\ell e^{i\ell \cdot x} \right)_0 \\
&\geq c \sum_\ell (\ell \cdot \ell)^{m/2} |a_\ell|^2 \quad \text{by (35)} \\
&= c \sum [1 + (\ell \cdot \ell)^{m/2}] |a_\ell|^2 - c\|u\|_0^2 \\
&\geq cC \|u\|_{m/2}^2 - c\|u\|_0
\end{aligned}$$

where

$$C = \sup_{r \geq 0} \frac{1 + r^{m/2}}{(1+r)^{m/2}}.$$

This takes care of stage 1.

Stage 2. $L = L_0 + L_1$ where L_0 is as in stage 1 and $L_1 = \sum_{|p|=m} \beta_p(x) D^p$ and

$$\max_{p,x} |\beta_p(x)| < \eta,$$

where η sufficiently small. (How small will be determined very soon in the course of the discussion.) We have

$$(u, L_0 u)_0 \geq c' \|u\|^2_{m/2} - c\|u\|^2_0$$

from stage 1.

We integrate $(u, L_1 u)$ by parts $m/2$ times. There are no boundary terms since we are on the torus. In integrating by parts some of the derivatives will hit the coefficients. Let us collect all the these terms as I_2. The remain terms we collect as I_1, so

$$I_1 = \sum \int b_{p'+p''} D^{p'} u \overline{D^{p''} u} dx$$

where $|p'| = |p''| = m/2$. We can estimate this sum by

$$|I_1| \leq \eta \cdot \text{const.} \|u\|^2_{m/2}$$

and so will require that $\eta \cdot (\text{const.}) < c'$.

The remaining terms give a sum of the form

$$I_2 = \sum \int b_{p'q} D^{p'} u \overline{D^q u} dx$$

where $p' \leq m/2, q' < m/2$ so we have

$$|I_2| \leq \text{const.} \|u\|_{\frac{m}{2}} \|u\|_{\frac{m}{2}-1}.$$

Now let us take

$$s = \frac{m}{2} - 1, \; t_1 = \frac{m}{2}, \; t_2 = 0$$

in (34) which yields, for any $\epsilon > 0$,

$$\|u\|_{\frac{m}{2}-1} \leq \epsilon \|u\|_{\frac{m}{2}} + \epsilon^{-m/2} \|u\|_0.$$

Substituting this into the above estimate for I_2 gives

$$|I_2| \leq \epsilon \cdot \text{const.} \|u\|^2_{m/2} + \epsilon^{-m/2} \text{const.} \|u\|_{m/2} \|u\|_0.$$

For any positive numbers a, b and ζ the inequality $(\zeta a - \zeta^{-1} b)^2 \geq 0$ implies that $2ab \leq \zeta a^2 + \zeta^{-1} b^2$. Taking $\zeta^2 = \epsilon^{\frac{m}{2}+1}$ we can replace the second term on the right in the preceding estimate for $|I_2$ by

$$\epsilon(-m-1) \cdot \text{const.} \|u\|^2_0$$

at the cost of enlarging the constant in front of $\|u\|^2_{\frac{m}{2}}$. We have thus established that

$$|I_1| \leq \eta \cdot (\text{const.})_1 \|u\|^2_{m/2}$$

where the constant depends only on m, and

$$|I_2| \leq \epsilon (\text{const.})_2 \|u\|^2_{m/2} + \epsilon^{-m-1} \text{const.} \|u\|^2_0$$

where the constants depend on L_1 but ϵ is at our disposal. So if $\eta(\text{const.})_1 < c'$ and we then choose ϵ so that $\epsilon(\text{const.})_2 < c' - \eta \cdot (\text{const.})_1$ we obtain Gårding's inequality for this case.

Stage 3. $L = L_0 + L_1 + L_2$ where L_0 and L_1 are as in stage 2, and L_0 is a lower order operator. Here we integrate by parts and argue as in stage 2.

Stage 4, the general case. Choose an open covering of T such that the variation of each of the highest order coefficients in each open set is less than $\eta < c'$. (Recall that c' depended only on the c that entered into the definition of ellipticity.) Thus, if v is a smooth function supported in one of the sets of our cover, the action of L on v is the same as the action of an operator as in case 3) on v, and so we may apply Gårding's inequality. Choose a finite subcover and a partition of unity $\{\phi_i\}$ subordinate to this cover. Write $\phi_i = \psi_i^2$ (where we choose the ϕ so that the ψ are smooth). So $\sum \psi_i^2 \equiv 1$. Now

$$(\psi_i u, L(\psi_i u))_0 \geq c'' \|\psi_i u\|_{m/2}^2 - \text{const.}\|\psi_i u\|_0^2$$

where c'' is a positive constant depending only on c, η, and on the lower order terms in L. We have

$$(u, Lu)_0 = \int (\sum \psi_i^2 u)\overline{Lu} dx = \sum (\psi_i u, L\psi_i u)_0 + R$$

where R is an expression involving derivatives of the ψ_i and hence lower order derivatives of u. These can be estimated as in case 2) above, and so we get

$$(u, Lu)_0 \geq c''' \sum \|\psi_i u\|_{m/2}^2 - \text{const.}\|u\|_0^2 \tag{37}$$

since $\|\psi_i u\|_0 \leq \|u\|_0$. Now $\|u\|_{m/2}$ is equivalent, as a norm, to $\sum_{p \leq m/2} \|D^p u\|_0$ as we verified in the preceding section. Also

$$\sum \|D^p(\psi_i u)\|_0 = \sum \|\psi_i D^p u\|_0 + R'$$

where R' involves terms differentiating the ψ and so lower order derivatives of u. Hence

$$\sum \|\psi_i u\|_{m/2}^2 \geq \text{pos. const.}\|u\|_{m/2}^2 - \text{const.}\|u\|_0^2$$

by the integration by parts argument again. Hence by (37)

$$\begin{aligned}(u, Lu)_0 &\geq c''' \sum \|\psi_i u\|_{m/2}^2 - \text{const.}\|u\|_0^2 \\ &\geq \text{pos. const.}\|u\|_{m/2}^2 - \text{const.}\|u\|_0^2\end{aligned}$$

which is Gårding's inequality. QED

For the time being we will continue to study the case of the torus. But a look ahead is in order. In this last step of the argument, where we applied the partition of unity argument, we have really freed ourselves of the restriction of being on the torus. Once we make the appropriate definitions, we will then get Gårding's inequality for elliptic operators on manifolds. Furthermore, the consequences we are about to draw from Gårding's inequality will be equally valid in the more general setting.

8 Consequences of Gårding's inequality.

Proposition 3 *For every integer t there is a constant $c(t) = c(t, L)$ and a positive number $\Lambda = \Lambda(t, L)$ such that*

$$\|u\|_t \leq c(t)\|Lu + \lambda u\|_{t-m} \tag{38}$$

when

$$\lambda > \Lambda$$

for all smooth u, and hence for all $u \in \mathbf{H}_t$.

Proof. Let s be some non-negative integer. We will first prove (38) for $t = s + \frac{m}{2}$. We have

$$\|u\|_t\|Lu + \lambda u\|_{t-m}\|u\|_t\|Lu + \lambda u\|_{s-\frac{m}{2}} = \|u\|_t\|(1+\Delta)^s Lu + \lambda(1+\Delta)^s u\|_{-s-\frac{m}{2}}$$

$$\geq (u, (1+\Delta)^s Lu + \lambda(1+\Delta)^s u)_0$$

by the generalized Cauchy - Schwartz inequality (26).

The operator $(1+\Delta)^s L$ is elliptic of order $m+2s$ so (25) and Gårding's inequality gives

$$(u, (1+\Delta)^s Lu + \lambda(1+\Delta)^s u)_0 \geq c_1\|u\|^2_{s+\frac{m}{2}} - c_2\|u\|^2_0 + \lambda\|u\|^2_s.$$

Since $\|u\|_s \geq \|u\|_0$ we can combine the two previous inequalities to get

$$\|u\|_t\|Lu + \lambda u|^2_{t-m} \geq c_1\|u\|^2_t + (\lambda - c_2)\|u\|^2_0.$$

If $\lambda > c_2$ we can drop the second term and divide by $\|u\|_t$ to obtain (38).

We now prove the proposition for the case $t = \frac{m}{2} - s$ by the same sort of argument: We have

$$\|u\|_t\|Lu + \lambda u\|_{-s-\frac{m}{2}} = \|(1+\Delta)^{-s}u\|_{s+\frac{m}{2}}\|Lu + \lambda u\|_{-s-\frac{m}{2}}$$

$$\geq ((1+\Delta)^{-s}u, L(1+\Delta)^s u + \lambda u)_0.$$

Now use the fact that $L(1+\Delta)^s$ is elliptic and Gårding's inequality to continue the above inequalities as

$$\geq c_1\|(1+\Delta)^{-s}u\|^2_{s+\frac{m}{2}} - c_2\|(1+\Delta)^{-s}u\|^2_0 + \lambda\|u\|^2_{-s}$$

$$= c_1\|u\|^2_t - c_2\|u\|_{-2s} + \lambda\|u\|^2_{-s} \geq c_1\|u\|^2_t$$

if $\lambda > c_2$. Again we may then divide by $\|u\|_t$ to get the result. QED

The operator $L + \lambda I$ is a bounded operator from $\mathbf{H}_t$ to $\mathbf{H}_{t-m}$ (for any t). Suppose we fix t and choose λ so large that (38) holds. Then (38) says that $(L+\lambda I)$ is invertible on its image, and bounded there with a bound independent of $\lambda > \Lambda$, and this image is a closed subspace of $\mathbf{H}_{t-m}$.

Let us show that this image is all of $\mathbf{H}_{t-m}$ for λ large enough. Suppose not, which means that there is some $w \in \mathbf{H}_{t-m}$ with

$$(w, Lu + \lambda u)_{t-m} = 0$$

for all $u \in \mathbf{H}_t$. We can write this last equation as

$$((1+\Delta)^{t-m} w, Lu + \lambda u)_0 = 0.$$

Integration by parts gives the adjoint differential operator L^* characterized by

$$(\phi, L\psi)_0 = (L^*\phi, \psi)_0$$

for all smooth functions ϕ and ψ, and by passing to the limit this holds for all element of $\mathbf{H}_r$ for $r \geq m$. The operator L^* has the same leading term as L and hence is elliptic. So let us choose λ sufficiently large that (38) holds for L^* as well as for L. Now

$$0 = \left((1+\Delta)^{t-m} w, Lu + \lambda u\right)_0 = \left(L^*(1+\Delta)^{t-m} w + \lambda(1+\Delta)^{t-m} w, u\right)_0$$

for all $u \in \mathbf{H}_t$ which is dense in $\mathbf{H}_0$ so

$$L^*(1+\Delta)^{t-m} w + \lambda(1+\Delta)^{t-m} w = 0$$

and hence (by (38)) $(1+\Delta)^{t-m} w = 0$ so $w = 0$. We have proved

Proposition 4 *For every t and for λ large enough (depending on t) the operator $L + \lambda I$ maps $\mathbf{H}_t$ bijectively onto $\mathbf{H}_{t-m}$ and $(L+\lambda I)^{-1}$ is bounded independently of λ.*

As an immediate application we get the important

Theorem 6 *If u is a distribution and $Lu \in \mathbf{H}_s$ then $u \in \mathbf{H}_{s+m}$.*

Proof. Write $f = Lu$. By Schwartz' theorem, we know that $u \in \mathbf{H}_k$ for some k. So $f + \lambda u \in \mathbf{H}_{\min(k,s)}$ for any λ. Choosing λ large enough, we conclude that $u = (L+\lambda I)^{-1}(f + \lambda u) \in \mathbf{H}_{\min(k+m, s+m)}$. If $k + m < s + m$ we can repeat the argument to conclude that $u \in \mathbf{H}_{\min(k+2m, s+m)}$. we can keep going until we conclude that $u \in \mathbf{H}_{s+m}$. QED

Notice as an immediate corollary that any solution of the homogeneous equation $Lu = 0$ is C^∞.

We now obtain a second important consequence of Proposition 4. Choose λ so large that the operators

$$(L+\lambda I)^{-1} \quad \text{and} \quad (L^* + \lambda I)^{-1}$$

exist as operators from $\mathbf{H}_0 \to \mathbf{H}_m$. Follow these operators with the injection $\iota_m : \mathbf{H}_m \to \mathbf{H}_0$ and set

$$M := \iota_m \circ (L+\lambda I)^{-1}, \qquad M^* := \iota_m \circ (L^* + \lambda I)^{-1}.$$

Since ι_m is compact (Rellich's lemma) and the composite of a compact operator with a bounded operator is compact, we conclude

Theorem 7 *The operators M and M^* are compact.*

Suppose that $L = L^*$. (This is usually expressed by saying that L is "formally self-adjoint". More on this terminology will come later.) This implies that $M = M^*$. In other words, M is a compact self adjoint operator, and we can apply Theorem 3 to conclude that eigenvectors of M form a basis $R(M)$ and that the corresponding eigenvalues tend to zero. Prop 4 says that $R(M)$ is the same as $\iota_m(\mathbf{H}_m)$ which is dense in $\mathbf{H}_0 = L_2(\mathbf{T})$. We conclude that the eigenvectors of M form a basis of $L_2(\mathbf{T})$. If $Mu = ru$ then $u = (L + \lambda I)Mu = rLu + \lambda ru$ so u is an eigenvector of L with eigenvalue

$$\frac{1-\lambda}{r}.$$

We conclude that the eigenvectors of L are a basis of $\mathbf{H}_0$. We claim that only finitely many of these eigenvalues of L can be negative. Indeed, since we know that the eigenvalues r_n of M approach zero, the numerator in the above expression is positive, for large enough n, and hence if there were infinitely many negative eigenvalues μ_k, they would have to correspond to negative r_k and so these $\mu_k \to -\infty$. But taking $s_k = -\mu_k$ as the λ in (38) in Prop. 3 we conclude that $u = 0$, if $Lu = \mu_k u$ if k is large enough, contradicting the definition of an eigenvector. So all but a finite number of the r_n are positive, and these tend to zero. To summarize:

Theorem 8 *The eigenvectors of L are C^∞ functions which form a basis of $\mathbf{H}_0$. Only finitely many of the eigenvalues μ_k of L are negative and $\mu_n \to \infty$ as $n \to \infty$.*

It is easy to extend the results obtained above for the torus in two directions. One is to consider functions defined in a **domain** = bounded open set $\mathcal{G}$ of $\mathbf{R}^n$ and the other is to consider functions defined on a compact manifold. In both cases a few elementary tricks allow us to reduce to the torus case. We sketch what is involved for the manifold case.

9 Extension of the basic lemmas to manifolds.

Let $E \to M$ be a vector bundle over a manifold. We assume that M is equipped with a density which we shall denote by $|dx|$ and that E is equipped with a positive definite (smoothly varying) scalar product, so that we can define the L_2 norm of a smooth section s of E of compact support:

$$||s||_0^2 := \int_M |s|^2(x)|dx|.$$

Suppose for the rest of this section that M is compact. Let $\{U_i\}$ be a finite cover of M by coordinate neighborhoods over which E has a given trivialization, and ρ_i a partition of unity subordinate to this cover. Let ϕ_i be a diffeomorphism or

U_i with an open subset of $\mathbf{T}^n$ where n is the dimension of M. Then if s is a smooth section of E, we can think of $(\rho_i s)\circ\phi_i^{-1}$ as an $\mathbf{R}^m$ or $\mathbf{C}^m$ valued function on $\mathbf{T}^n$, and consider the sum of the $\|\cdot\|_k$ norms applied to each component. We shall continue to denote this sum by $\|\rho_i f\circ\phi_i^{-1}\|_k$ and then define

$$\|f\|_k := \sum_i \|\rho_i f\circ\phi_i^{-1}\|_k$$

where the norms on the right are in the norms on the torus. These norms depend on the trivializations and on the partitions of unity. But any two norms are equivalent, and the $\|\ \|_0$ norm is equivalent to the "intrinsic" L_2 norm defined above. We define the Sobolev spaces $\mathbf{W}_k$ to be the completion of the space of smooth sections of E relative to the norm $\|\ \|_k$ for $k\geq 0$, and these spaces are well defined as topological vector spaces independently of the choices. Since Sobolev's lemma holds locally, it goes through unchanged. Similarly Rellich's lemma: if s_n is a sequence of elements of $\mathbf{W}_\ell$ which is bounded in the $\|\ \|_\ell$ norm for $\ell>k$, then each of the elements $\rho_i s_n\circ\phi_i^{-1}$ belong to $\mathbf{H}_\ell$ on the torus, and are bounded in the $\|\ \|_\ell$ norm, hence we can select subsequence of $\rho_1 s_n\circ\phi_1^{-1}$ which converges in $\mathbf{H}_k$, then a subsubsequence such that $\rho_i s_n\circ\phi_i^{-1}$ for $i=1,2$ converge etc. arriving at a subsequence of s_n which converges in $\mathbf{W}_k$.

A differential operator L mapping sections of E into sections of E is an operator whose local expression (in terms of a trivialization and a coordinate chart) has the form

$$Ls = \sum_{|p|\leq m}\alpha_p(x)D^p s$$

Here the a_p are linear maps (or matrices if our trivializations are in terms of $\mathbf{R}^m$).

Under changes of coordinates and trivializations the change in the coefficients are rather complicated, but the **symbol** of the differential operator

$$\sigma(L)(\xi) := \sum_{|p|=m} a_p(x)\xi^p \qquad \xi\in T^*M_x$$

is well defined.

If we put a Riemann metric on the manifold, we can talk about the length $|\xi|$ of any cotangent vector.

If L is a differential operator from E to itself (i.e. F=E) we shall call L **even elliptic** if m is even and there exists some constant C such that

$$\langle v,\sigma(L)(\xi)v\rangle \geq C|\xi|^m|v|^2$$

for all $x\in M$, $v\in E_x$, $\xi\in T^*M_x$ and $\langle,\ \rangle$ denote the scalar product on E_x. Gårding's inequality holds. Indeed, locally, this is just a restatement of the (vector valued version) of Gårding's inequality that we have already proved for the torus. But Stage 4 in the proof extends unchanged (other than the replacement of scalar valued functions by vector valued functions) to the more general case.

10 Example: Hodge theory.

We assume knowledge of the basic facts about differentiable manifolds, in particular the existence of an operator $d : \Omega^k \to \Omega^{k+1}$ with its usual properties, where Ω^k denotes the space of exterior k-forms. Also, if M is orientable and carries a Riemann metric then the Riemann metric induces a scalar product on the exterior powers of T^*M and also picks out a volume form. So there is an induced scalar product $(\ ,\ ) = (\ ,\ )_k$ on Ω^k and a formal adjoint δ of d

$$\delta : \ \Omega^k \to \Omega^{k-1}$$

and satisfies

$$(d\psi, \phi) = (\phi, \delta\phi)$$

where ϕ is a $(k+1)$-form and ψ is a k-form. Then

$$\Delta := d\delta + \delta d$$

is a second order differential operator on Ω^k and satisfies

$$(\Delta\phi, \phi) = ||d\phi||^2 + ||\delta\phi||^2$$

where $||\phi||^2 = (\phi, \phi)$ is the intrinsic L_2 norm (so $||\ || = ||\ ||_0$ in terms of the notation of the preceding section). Furthermore, if

$$\phi = \sum_I \phi_I dx^I$$

is a local expression for the differential form ϕ, where

$$dx^I = dx_{i_1} \wedge \cdots \wedge dx_{i_k} \quad I = (i_1, \ldots, i_k)$$

then a local expression for Δ is

$$\Delta\phi = -\sum g^{ij} \frac{\partial\phi_I}{\partial x^i \partial x^j} + \cdots$$

where

$$g^{ij} = \langle dx^i, dx^j \rangle$$

and the $\cdots$ are lower order derivatives. In particular Δ is elliptic.

Let $\phi \in \Omega^k$ and suppose that

$$d\phi = 0.$$

Let $\mathcal{C}(\phi)$, the **cohomology class** of ϕ be the set of all $\psi \in \Omega^k$ which satisfy

$$\phi - \psi = d\alpha, \quad \alpha \in \Omega^{k-1}$$

and let

$$\overline{\mathcal{C}(\phi)}$$

denote the closure of $\mathcal{C}$ in the L_2 norm. It is a closed subspace of the Hilbert space obtained by completing Ω^k relative to its L_2 norm. Let us denote this space by L_2^k, so $\overline{\mathcal{C}(\phi)}$ is a closed subspace of L_2^k.

Proposition 5 *If $\phi \in \Omega^k$ and $d\phi = 0$, there exists a unique $\tau \in \overline{\mathcal{C}(\phi)}$ such that*

$$\|\tau\| \leq \|\psi\| \quad \forall\ \psi \in \mathcal{C}(\phi).$$

Furthermore, τ is smooth,and

$$d\tau = 0 \quad \textit{and} \quad \delta\tau = 0.$$

If choose a minimizing sequence for $\|\psi\|$ in $\mathcal{C}(\phi)$.

If we choose a minimizing sequence for $\|\psi\|$ in $\mathcal{C}(\phi)$ we know it is Cauchy, cf. the proof of the existence of orthogonal projections in a Hilbert space. So we know that τ exists and is unique. For any $\alpha \in \Omega^{k+1}$ we have

$$(\tau, \delta\alpha) = \lim(\psi, \delta\alpha) = \lim(d\psi, \alpha) = 0$$

as ψ ranges over a minimizing sequence. The equation $(\tau, \delta\alpha) = 0$ for all $\alpha \in \Omega^{k+1}$ says that τ is a weak solution of the equation $d\tau = 0$.

We claim that

$$(\tau, d\beta) = 0 \quad \forall\ \beta \in \Omega^{k-1}$$

which says that τ is a weak solution of $\delta\tau = 0$. Indeed, for any $t \in \mathbf{R}$,

$$\|\tau\|^2 \leq \|\tau + td\beta\|^2 = \|\tau\|^2 + t^2\|d\beta\|^2 + 2t(\tau, d\beta)$$

so

$$-2t(\tau, d\beta) \leq t^2\|d\beta\|^2.$$

If $(\tau, d\beta) \neq 0$, we can choose

$$t = -\epsilon\frac{(\tau, d\beta)}{|(\tau, d\beta)|}, \quad \epsilon > 0$$

so

$$|(\tau, d\beta)| \leq \epsilon|d\beta|^2.$$

As ϵ is arbitrary, this implies that $(\tau, d\beta) = 0$.

So $(\tau, \Delta\psi) = (\tau, [d\delta + \delta d]\psi) = 0$ for any $\psi \in \Omega^k$. Hence τ is a weak solution of $\Delta\tau = 0$ and so is smooth. The space $\mathcal{H}^k$ of weak, and hence smooth solutions of $\Delta\tau = 0$ is finite dimensional by the general theory. It is called the space of Harmonic forms. We have seen that there is a unique harmonic form in the cohomology class of any closed form, s the cohomology groups are finite dimensional. In fact, the general theory tells us that

$$L_2^k \bigoplus_\lambda E_\lambda^k$$

(Hilbert space direct sum) where E_λ^k is the eigenspace with eigenvalue λ of Δ. Each E_λ is finite dimensional and consists of smooth forms, and the $\lambda \to \infty$. The eigenspace E_0^k is just $\mathcal{H}^k$, the space of harmonic forms. Also, since

$$(\Delta\phi, \phi) = \|d\phi\|^2 + \|\delta\phi\|^2$$

we know that all the eigenvalues λ are non-negative.

Since $d\Delta = d(d\delta + \delta d) = d\delta d = \Delta d$, we see that

$$d : E^k_\lambda \to E^{k+1}_\lambda$$

and similarly

$$\delta : E^k_\lambda \to E^{k-1}_\lambda.$$

For $\lambda \neq 0$, if $\phi \in E^k_\lambda$ and $d\phi = 0$, then $\lambda\phi = \Delta\phi = d\delta\phi$ so $\phi = d(1/\lambda)\delta\phi$ so d restricted to the E_λ is exact, and similarly so is δ. Furthermore, on $\bigoplus_k E^k_\lambda$ we have

$$\lambda I = \Delta = (d + \delta)^2$$

so we have

$$E^k_\lambda = dE^{k-1}_\lambda \oplus \delta E^{k+1}_\lambda$$

and this decomposition is orthogonal since $(d\alpha, \delta\beta) = (d^2\alpha, \beta) = 0$.

As a first consequence we see that

$$L^k_2 = \mathcal{H}^k \oplus \overline{d\Omega^{k-1}} \oplus \overline{\delta\Omega^{k-1}}$$

(the Hodge decomposition). If H denotes projection onto the first component, then Δ is invertible on the image of $I - H$ with a an inverse there which is compact. So if we let N denote this inverse on im $I - H$ and set $N = 0$ on $\mathcal{H}^k$ we get

$$\begin{aligned} \Delta N &= I - H \\ Nd &= dN \\ \delta N &= N\delta \\ \Delta N &= N\Delta \\ NH &= 0 \end{aligned}$$

which are the fundamental assertions of Hodge theory, together with the assertion proved above that $H\phi$ is the unique minimizing element in its cohomology class.

We have seen that

$$d + \delta : \bigoplus_k E^{2k}_\lambda \to \bigoplus_k E^{2k+1}_\lambda \text{ is an isomorphism for } \lambda \neq 0 \tag{39}$$

which of course implies that

$$\sum_k (-1)^k \dim E^k_\lambda = 0$$

This shows that the index of the operator $d + \delta$ acting on $\bigoplus L^k_2$ is the Euler characteristic of the manifold. (The index of any operator is the difference between the dimensions of the kernel and cokernel).

Let $P_{k,\lambda}$ denote the projection of L_2^k onto E_λ^k. So

$$e^{-t\Delta} = \sum e^{-\lambda t} P_{k,\lambda}$$

is the solution of the heat equation on L_2^k. As $t \to \infty$ this approaches the operator H projecting L_2^k onto $\mathcal{H}_k$. Letting Δ_k denote the operator Δ on L_2^k we see that

$$\operatorname{tr} e^{-t\Delta_k} = \sum e^{-\lambda_k}$$

where the sum is over all eigenvalues λ_k of Δ_k counted with multiplicity. It follows from (39) that the alternating sum over k of the corresponding sum over non-zero eigenvalues vanishes. Hence

$$\sum (-1)^k \operatorname{tr} e^{-t\Delta_k} = \chi(M)$$

is independent of t. The index theorem computes this trace for small values of t in terms of local geometric invariants.

The operator $d + \delta$ is an example of a Dirac operator whose general definition we will not give here. The corresponding assertion and local evaluation is the content of the celebrated Atiyah-Singer index theorem, one of the most important theorems discovered in the twentieth century.

11 The resolvent.

In order to connect what we have done here notation that will come later, it is convenient to let $A = -L$ so that now the operator

$$(zI - A)^{-1}$$

is compact as an operator on $\mathbf{H}_0$ for z sufficiently negative. (I have dropped the ι_m which should come in front of this expression.) The operator A now has only finitely many positive eigenvalues, with the corresponding spaces of eigenvectors being finite dimensional. In fact, the eigenvectors $\lambda_n = \lambda_n(A)$ (counted with multiplicity) approach $-\infty$ as $n \to \infty$ and the operator $(zI - A)^{-1}$ exists and is a bounded (in fact compact) operator so long as $z \neq \lambda_n$ for any n. Indeed, we can write any $u \in \mathbf{H}_0$ as

$$u = \sum_n a_n \phi_n$$

where ϕ_n is an eigenvector of A with eigenvalue λ_n and the ϕ form an orthonormal basis of $\mathbf{H}_0$. Then

$$(zI - A)^{-1} u = \sum \frac{1}{z - \lambda_n} a_n \phi_n.$$

The operator $(zI - A)^{-1}$ is called the **resolvent** of A at the point z and denoted by

$$R(z, A)$$

or simply by $R(z)$ if A is fixed. So

$$R(z, A) := (zI - A)^{-1}$$

for those values of $z \in \mathbf{C}$ for which the right hand side is defined.

If z and are complex numbers with $\mathrm{Re}z > \mathrm{Re}a$, then the integral

$$\int_0^\infty e^{-zt} e^{at} dt$$

converges, and we can evaluate it as

$$\frac{1}{z - a} = \int_0^\infty e^{-zt} e^{at} dt.$$

If $\mathrm{Re}z$ is greater than the largest of the eigenvalues of A we can write

$$R(z, A) = \int_0^\infty e^{-zt} e^{tA} dt$$

where we may interpret this equation as a shorthand for doing the integral for the coefficient of each eigenvector, as above, or as an actual operator valued integral. We will spend a lot of time later on in this course generalizing this formula and deriving many consequences from it.