

# Lebesgue measure

Math 212a

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## 1 Lebesgue outer measure.

We recall some results from the notes on topology: For any subset  $A \subset \mathbf{R}$  we defined its **Lebesgue outer measure** by

$$m^*(A) := \inf \sum \ell(I_n) : I_n \text{ are intervals with } A \subset \bigcup I_n. \quad (1)$$

Here the length  $\ell(I)$  of any interval  $I = [a, b]$  is  $b - a$  with the same definition for half open intervals  $(a, b]$  or  $[a, b)$ , or open intervals. Of course if  $a = -\infty$  and  $b$  is finite or  $+\infty$ , or if  $a$  is finite and  $b = +\infty$  the length is infinite. So the infimum in (1) is taken over all covers of  $A$  by intervals. By the usual  $\epsilon/2^n$  trick, i.e. by replacing each  $I_j = [a_j, b_j]$  by  $(a_j - \epsilon/2^{j+1}, b_j + \epsilon/2^{j+1})$  we may assume that the infimum is taken over open intervals. (Equally well, we could use half open intervals of the form  $[a, b)$ , for example.)

It is clear that if  $A \subset B$  then  $m^*(A) \leq m^*(B)$  since any cover of  $B$  by intervals is a cover of  $A$ . Also, if  $Z$  is any set of measure zero, then  $m^*(A \cup Z) = m^*(A)$ . In particular,  $m^*(Z) = 0$  if  $Z$  has measure zero. Also, if  $A = [a, b]$  is an interval, then we can cover it by itself, so

$$m^*([a, b]) \leq b - a,$$

and hence the same is true for  $(a, b]$ ,  $[a, b)$ , or  $(a, b)$ . If the interval is infinite, it clearly can not be covered by a set of intervals whose total length is finite, since

if we lined them up with end points touching they could not cover an infinite interval. We still must prove that

$$m^*(I) = \ell(I) \tag{2}$$

if  $I$  is a finite interval. We may assume that  $I = [c, d]$  is a closed interval by what we have already said, and that the minimization in (1) is with respect to a cover by open intervals. So what we must show is that if

$$[c, d] \subset \bigcup_i (a_i, b_i)$$

then

$$d - c \leq \sum_i (b_i - a_i).$$

We first applied Heine-Borel to replace the countable cover by a finite cover. (This only decreases the right hand side of preceding inequality.) So let  $n$  be the number of elements in the cover. We needed to prove that if

$$[c, d] \subset \bigcup_{i=1}^n (a_i, b_i) \text{ then } d - c \leq \sum_{i=1}^n (b_i - a_i),$$

and we did this this by induction on  $n$ . If  $n = 1$  then  $a_1 < c$  and  $b_1 > d$  so clearly  $b_1 - a_1 > d - c$ .

Suppose that  $n \geq 2$  and we know the result for all covers (of all intervals  $[c, d]$ ) with at most  $n - 1$  intervals in the cover. If some interval  $(a_i, b_i)$  is disjoint from  $[c, d]$  we may eliminate it from the cover, and then we are in the case of  $n - 1$  intervals. So every  $(a_i, b_i)$  has non-empty intersection with  $[c, d]$ . Among the the intervals  $(a_i, b_i)$  there will be one for which  $a_i$  takes on the minimum possible value. By relabeling, we may assume that this is  $(a_1, b_1)$ . Since  $c$  is covered, we must have  $a_1 < c$ . If  $b_1 > d$  then  $(a_1, b_1)$  covers  $[c, d]$  and there is nothing further to do. So assume  $b_1 \leq d$ . We must have  $b_1 > c$  since  $(a_1, b_1) \cap [c, d] \neq \emptyset$ . Since  $b_1 \in [c, d]$ , at least one of the intervals  $(a_i, b_i)$ ,  $i > 1$  contains the point  $b_1$ . By relabeling, we may assume that it is  $(a_2, b_2)$ . But now we have a cover of  $[c, d]$  by  $n - 1$  intervals:

$$[c, d] \subset (a_1, b_2) \cup \bigcup_{i=3}^n (a_i, b_i).$$

So by induction

$$d - c \leq (b_2 - a_1) + \sum_{i=3}^n (b_i - a_i).$$

But  $b_2 - a_1 \leq (b_2 - a_2) + (b_1 - a_1)$  since  $a_2 < b_1$ . QED

We repeat that the intervals used in (1) could be taken as open, closed or half open without changing the definition. If we take them all to be half open, of the form  $I_i = [a_i, b_i)$ , we can write each  $I_i$  as a disjoint union of finite or

countably many intervals each of length  $< \epsilon$ . So it makes no difference to the definition if we also require the

$$\ell(I_i) < \epsilon \tag{3}$$

in (1). We will see that when we pass to other types of measures this will make a difference.

We have verified, or can easily verify the following properties:

1.

$$m^*(\emptyset) = 0.$$

2.

$$A \subset B \Rightarrow m^*(A) \leq m^*(B).$$

3.

$$m^*\left(\bigcup_i A_i\right) \leq \sum_i m^*(A_i).$$

4. If  $\text{dist}(A, B) > 0$  then

$$m^*(A \cup B) = m^*(A) + m^*(B).$$

5.

$$m^*(A) = \inf\{m^*(U) : U \supset A, U \text{ open}\}.$$

6. For an interval

$$m^*(I) = \ell(I).$$

The only items that we have not done already are items 4 and 5. But these are immediate: for 4 we may choose the intervals in (1) all to have length  $< \epsilon$  where  $2\epsilon < \text{dist}(A, B)$  so that there is no overlap. As for item 5, we know from 2 that  $m^*(A) \leq m^*(U)$  for any set  $U$ , in particular for any open set  $U$  which contains  $A$ . We must prove the reverse inequality: if  $m^*(A) = \infty$  this is trivial. Otherwise, we may take the intervals in (1) to be open and then the union on the right is an open set whose Lebesgue outer measure is less than  $m^*(A) + \delta$  for any  $\delta > 0$  if we choose a close enough approximation to the infimum.

I should also add that all the above works for  $\mathbf{R}^n$  instead of  $\mathbf{R}$  if we replace the word “interval” by “rectangle”, meaning a rectangular parallelepiped, i.e a set which is a product of one dimensional intervals. We also replace length by volume (or area in two dimensions). What is needed is the following

**Lemma 1** *Let  $\mathcal{C}$  be a finite non-overlapping collection of closed rectangles all contained in the closed rectangle  $J$ . Then*

$$\text{vol } J \geq \sum_{I \in \mathcal{C}} \text{vol } I.$$

If  $\mathcal{C}$  is any finite collection of rectangles such that

$$J \subset \bigcup_{I \in \mathcal{C}} I$$

then

$$\text{vol } J \leq \sum_{I \in \mathcal{C}} \text{vol } (I).$$

This lemma occurs on page 1 of Strook, *A concise introduction to the theory of integration* together with its proof. I will take this for granted. In the next few paragraphs I will talk as if we are in  $\mathbf{R}$ , but everything goes through unchanged if  $\mathbf{R}$  is replaced by  $\mathbf{R}^n$ .

## 2 Lebesgue inner measure.

Item 5. in the preceding paragraph says that the Lebesgue outer measure of any set is obtained by approximating it from the outside by open sets. The **Lebesgue inner measure** is defined as

$$m_*(A) = \sup\{m^*(K) : K \subset A, K \text{ compact}\}. \quad (4)$$

Clearly

$$m_*(A) \leq m^*(A)$$

since  $m^*(K) \leq m^*(A)$  for any  $K \subset A$ . We also have

**Proposition 1** *For any interval  $I$  we have*

$$m_*(I) = \ell(I). \quad (5)$$

**Proof.** If  $\ell(I) = \infty$  the result is obvious. So we may assume that  $I$  is a finite interval which we may assume to be open,  $I = (a, b)$ . If  $K \subset I$  is compact, then  $I$  is a cover of  $K$  and hence from the definition of outer measure  $m^*(K) \leq \ell(I)$ . So  $m_*(I) \leq \ell(I)$ . On the other hand, for any  $\epsilon > 0$ ,  $\epsilon < \frac{1}{2}(b - a)$  the interval  $[a + \epsilon, b - \epsilon]$  is compact and  $m^*([a + \epsilon, b - \epsilon]) = b - a - 2\epsilon \leq m_*(I)$ . Letting  $\epsilon \rightarrow 0$  proves the proposition. QED

## 3 Lebesgue's definition of measurability.

A set  $A$  with  $m^*(A) < \infty$  is said to be **measurable in the sense of Lebesgue** if

$$m_*(A) = m^*(A). \quad (6)$$

If  $A$  is measurable in the sense of Lebesgue, we write

$$m(A) = m_*(A) = m^*(A). \quad (7)$$

If  $K$  is a compact set, then  $m_*(K) = m^*(K)$  since  $K$  is a compact set contained in itself. Hence all compact sets are measurable in the sense of Lebesgue. If  $I$  is a bounded interval, then  $I$  is measurable in the sense of Lebesgue by Proposition 1.

If  $m^*(A) = \infty$ , we say that  $A$  is measurable in the sense of Lebesgue if all of the sets  $A \cap [-n, n]$  are measurable.

**Proposition 2** *If  $A = \bigcup A_i$  is a (finite or) countable disjoint union of sets which are measurable in the sense of Lebesgue, then  $A$  is measurable in the sense of Lebesgue and*

$$m(A) = \sum_i m(A_i).$$

**Proof.** We may assume that  $m(A) < \infty$  - otherwise apply the result to  $A \cap [-n, n]$  and  $A_i \cap [-n, n]$  for each  $n$ . We have

$$m^*(A) \leq \sum_n m^*(A_n) = \sum_n m(A_n).$$

Let  $\epsilon > 0$ , and for each  $n$  choose compact  $K_n \subset A_n$  with

$$m^*(K_n) \geq m_*(A_n) - \frac{\epsilon}{2^n} = m(A_n) - \frac{\epsilon}{2^n}$$

since  $A_n$  is measurable in the sense of Lebesgue. The sets  $K_n$  are pairwise disjoint, hence, being compact, at positive distances from one another. Hence

$$m^*(K_1 \cup \dots \cup K_n) = m^*(K_1) + \dots + m^*(K_n)$$

and  $K_1 \cup \dots \cup K_n$  is compact and contained in  $A$ . Hence

$$m_*(A) \geq m^*(K_1) + \dots + m^*(K_n),$$

and since this is true for all  $n$  we have

$$m_*(A) \geq \sum_n m(A_n) - \epsilon.$$

Since this is true for all  $\epsilon > 0$  we get

$$m_*(A) \geq \sum m(A_n).$$

But then  $m_*(A) \geq m^*(A)$  and so they are equal, so  $A$  is measurable in the sense of Lebesgue, and  $m(A) = \sum m(A_i)$ . QED

**Proposition 3** *Open sets and closed sets are measurable in the sense of Lebesgue.*

**Proof.** Any open set  $O$  can be written as the countable union of open intervals  $I_i$ , and

$$J_n := I_n \setminus \bigcup_{i=1}^{n-1} I_i$$

is a disjoint union of intervals (some open, some closed, some half open) and  $O$  is the disjoint union of the  $J_n$ . So every open set is a disjoint union of intervals hence measurable in the sense of Lebesgue.

If  $F$  is closed, and  $m^*(F) = \infty$ , then  $F \cap [-n, n]$  is compact. and so  $F$  is measurable in the sense of Lebesgue. Suppose that

$$m^*(F) < \infty.$$

For any  $\epsilon > 0$  consider the sets

$$\begin{aligned} G_{1,\epsilon} &:= [-1 + \frac{\epsilon}{2^2}, 1 + \frac{\epsilon}{2^2}] \cap F \\ G_{2,\epsilon} &:= ([-2 + \frac{\epsilon}{2^3}, -1] \cap F) \cup ([1, 2 - \frac{\epsilon}{2^3}] \cap F) \\ G_{3,\epsilon} &:= ([-3 + \frac{\epsilon}{2^4}, -2] \cap F) \cup ([2, 3 - \frac{\epsilon}{2^4}] \cap F) \\ &\vdots \end{aligned}$$

and set

$$G_\epsilon := \bigcup_i G_{i,\epsilon}.$$

The  $G_{i,\epsilon}$  are all compact, and hence measurable in the sense of Lebesgue, and the union in the definition of  $G_\epsilon$  is disjoint, so is measurable in the sense of Lebesgue. Furthermore, the sum of the lengths of the “gaps” between the intervals that went into the definition of the  $G_{i,\epsilon}$  is  $\epsilon$ . So

$$m(G_\epsilon) + \epsilon = m^*(G_\epsilon) + \epsilon \geq m^*(F) \geq m^*(G_\epsilon) = m(G_\epsilon) = \sum_i G_{i,\epsilon}.$$

In particular, the sum on the right converges, and hence by considering a finite number of terms, we will have a finite sum whose value is at least  $m(G) - \epsilon$ . The corresponding union of sets will be a compact set  $K_\epsilon$  contained in  $F$  with

$$m(K_\epsilon) \geq m^*(F) - 2\epsilon.$$

Hence all closed sets are measurable in the sense of Lebesgue. QED

**Theorem 1** *A is measurable in the sense of Lebesgue if and only if for every  $\epsilon > 0$  there is an open set  $U \supset A$  and a closed set  $F \subset A$  such that*

$$m(U \setminus F) < \epsilon.$$

**Proof.** Suppose that  $A$  is measurable in the sense of Lebesgue with  $m(A) < \infty$ . Then there is an open set  $U \supset A$  with  $m(U) < m^*(A) + \epsilon/2 = m(A) + \epsilon/2$ , and there is a compact set  $F \subset A$  with  $m(F) \geq m_*(A) - \epsilon = m(A) - \epsilon/2$ . Since  $U \setminus F$  is open, it is measurable in the sense of Lebesgue, and so is  $F$  as it is compact. Also  $F$  and  $U \setminus F$  are disjoint. Hence by Proposition 2,

$$m(U \setminus F) = m(U) - m(F) < m(A) + \frac{\epsilon}{2} - \left(m(A) - \frac{\epsilon}{2}\right) = \epsilon.$$

If  $A$  is measurable in the sense of Lebesgue and  $m(A) = \infty$  we apply the above to each of the sets  $A \cap [-n, n]$  to obtain  $U_n \supset A \cap [-n, n]$  and  $F_n \subset A \cap [-n, n]$  with  $m(U_n \setminus F_n) < \epsilon/2^n$ . We may also enlarge each  $F_n$  if necessary so that  $F_n \cap [-n+1, n-1] \supset F_{n-1}$  keeping the  $F_n$  compact. We may also decrease the  $U_n$  if necessary so that  $U_n \cap (-n+1-\delta_n, n-1+\delta_n) \subset U_{n-1}$  for some sufficiently small  $\delta_n$ . We take  $U = \bigcup U_n$ . Then  $U$  is open. Suppose we take

$$F = \bigcup F_n \cap ([-n, -n+1] \cup [n-1, n]).$$

Then  $F$  is closed,  $U \supset A \supset F$  and

$$U \setminus F \subset \bigcup (U_n \setminus F_n) \cap ([-n, -n+1] \cup [n-1, n]) \subset \bigcup (U_n \setminus F_n)$$

so  $m(U \setminus F) < \epsilon$ .

In the other direction, suppose that for each  $\epsilon$ , there exist  $U \supset A \supset F$  with  $m(U \setminus F) < \epsilon$ . Suppose that  $m^*(A) < \infty$ . Then  $m(F) < \infty$  and  $m(U) \leq m(U \setminus F) + m(F) < \epsilon + m(F) < \infty$ . Then

$$m^*(A) \leq m(U) < m(F) + \epsilon = m_*(F) + \epsilon \leq m_*(A) + \epsilon.$$

Since this is true for every  $\epsilon > 0$  we conclude that  $m_*(A) \geq m^*(A)$  so they are equal and  $A$  is measurable in the sense of Lebesgue.

If  $m^*(A) = \infty$ , we have  $U \cap (-n - \epsilon, n + \epsilon) \supset A \cap [-n, n] \supset F \cap [-n, n]$  and

$$m((U \cap (-n - \epsilon, n + \epsilon)) \setminus (F \cap [-n, n])) < 2\epsilon + \epsilon = 3\epsilon$$

so we can proceed as before to conclude that  $m_*(A \cap [-n, n]) = m^*(A \cap [-n, n])$ .  
QED

Several facts emerge immediately from this theorem:

**Proposition 4** *If  $A$  is measurable in the sense of Lebesgue, so is its complement  $A^c = \mathbf{R} \setminus A$ .*

Indeed, if  $F \subset A \subset U$  with  $F$  closed and  $U$  open, then  $F^c \supset A^c \supset U^c$  with  $F^c$  open and  $U^c$  closed. Furthermore,  $F^c \setminus U^c = U \setminus F$  so if  $A$  satisfies the condition of the theorem so does  $A^c$ .

**Proposition 5** *If  $A$  and  $B$  are measurable in the sense of Lebesgue so is  $A \cap B$ .*

For  $\epsilon > 0$  choose  $U_A \supset A \supset F_A$  and  $U_B \supset B \supset F_B$  with  $m(U_A \setminus F_A) < \epsilon/2$  and  $m(U_B \setminus F_B) < \epsilon/2$ . Then  $(U_A \cap U_B) \supset (A \cap B) \supset (F_A \cap F_B)$  and  $(U_A \cap U_B) \setminus (F_A \cap F_B) \subset (U_A \setminus F_A) \cup (U_B \setminus F_B)$ . QED

Putting the previous two propositions together gives

**Proposition 6** *If  $A$  and  $B$  are measurable in the sense of Lebesgue then so is  $A \cup B$ .*

Indeed,  $A \cup B = (A^c \cap B^c)^c$ .

Since  $A \setminus B = A \cap B^c$  we also get

**Proposition 7** *If  $A$  and  $B$  are measurable in the sense of Lebesgue then so is  $A \setminus B$ .*

## 4 Caratheodory's definition of measurability.

A set  $E \subset \mathbf{R}$  is said to be **measurable according to Caratheodory** if for any set  $A \subset \mathbf{R}$  we have

$$m^*(A) = m^*(A \cap E) + m^*(A \cap E^c) \quad (8)$$

where we recall that  $E^c$  denotes the complement of  $E$ . In other words,  $A \cap E^c = A \setminus E$ . This definition has many advantages, as we shall see. Our first task is to show that it is equivalent to Lebesgue's:

**Theorem 2** *A set  $E$  is measurable in the sense of Caratheodory if and only if it is measurable in the sense of Lebesgue.*

**Proof.** We always have

$$m^*(A) \leq m^*(A \cap E) + m^*(A \setminus E)$$

so condition (8) is equivalent to

$$m^*(A \cap E) + m^*(A \setminus E) \leq m^*(A) \quad (9)$$

for all  $A$ .

Suppose  $E$  is measurable in the sense of Lebesgue. Let  $\epsilon > 0$ . Choose  $U \supset E \supset F$  with  $U$  open,  $F$  closed and  $m(U/F) < \epsilon$  which we can do by Theorem 1. Let  $V$  be an open set containing  $A$ . Then  $A \setminus E \subset V \setminus F$  and  $A \cap E \subset (V \cap U)$  so

$$\begin{aligned} m^*(A \setminus E) + m^*(A \cap E) &\leq m(V \setminus F) + m(V \cap U) \\ &\leq m(V \setminus U) + m(U \setminus F) + m(V \cap U) \\ &\leq m(V) + \epsilon. \end{aligned}$$

(We can pass from the second line to the third since both  $V \setminus U$  and  $V \cap U$  are measurable in the sense of Lebesgue and we can apply Proposition 2.) Taking the infimum over all open  $V$  containing  $A$ , the last term becomes  $m^*(A)$ , and as  $\epsilon$  is arbitrary, we have established (9) showing that  $E$  is measurable in the sense of Caratheodory.

In the other direction, suppose that  $E$  is measurable in the sense of Caratheodory. First suppose that

$$m^*(E) < \infty.$$

Then for any  $\epsilon > 0$  there exists an open set  $U \supset E$  with  $m(U) < m^*(E) + \epsilon$ . We may apply condition (8) to  $A = U$  to get

$$m(U) = m^*(U \cap E) + m^*(U \setminus E) = m^*(E) + m^*(U \setminus E)$$

so

$$m^*(U \setminus E) < \epsilon.$$

This means that there is an open set  $V \supset (U \setminus E)$  with  $m(V) < \epsilon$ . But we know that  $U \setminus V$  is measurable in the sense of Lebesgue, since  $U$  and  $V$  are, and

$$m(U) \leq m(V) + m(U \setminus V)$$

so

$$m(U \setminus V) > m(U) - \epsilon.$$

So there is a closed set  $F \subset U \setminus V$  with  $m(F) > m(U) - \epsilon$ . But since  $V \supset U \setminus E$ , we have  $U \setminus V \subset E$ . So  $F \subset E$ . So  $F \subset E \subset U$  and

$$m(U \setminus F) = m(U) - m(F) < \epsilon.$$

Hence  $E$  is measurable in the sense of Lebesgue.

If  $m(E) = \infty$ , we must show that  $E \cap [-n, n]$  is measurable in the sense of Caratheodory, for then it is measurable in the sense of Lebesgue from what we already know. We know that the interval  $[-n, n]$  itself, being measurable in the sense of Lebesgue, is measurable in the sense of Caratheodory. So we will have completed the proof of the theorem if we show that the intersection of  $E$  with  $[-n, n]$  is measurable in the sense of Caratheodory.

More generally, we will show that the union or intersection of two sets which are measurable in the sense of Caratheodory is again measurable in the sense of Caratheodory. Notice that the definition (8) is symmetric in  $E$  and  $E^c$  so if  $E$  is measurable in the sense of Caratheodory so is  $E^c$ . So it suffices to prove the next lemma to complete the proof.

**Lemma 2** *If  $E_1$  and  $E_2$  are measurable in the sense of Caratheodory so is  $E_1 \cup E_2$ .*

For any set  $A$  we have

$$m^*(A) = m^*(A \cap E_1) + m^*(A \cap E_1^c)$$

by (8) applied to  $E_1$ . Applying (8) to  $A \cap E_1^c$  and  $E_2$  gives

$$m^*(A \cap E_1^c) = m^*(A \cap E_1^c \cap E_2) + m^*(A \cap E_1^c \cap E_2^c).$$

Substituting this back into the preceding equation gives

$$m^*(A) = m^*(A \cap E_1) + m^*(A \cap E_1^c \cap E_2) + m^*(A \cap E_1^c \cap E_2^c). \quad (10)$$

Since  $E_1^c \cap E_2^c = (E_1 \cup E_2)^c$  we can write this as

$$m^*(A) = m^*(A \cap E_1) + m^*(A \cap E_1^c \cap E_2) + m^*(A \cap (E_1 \cup E_2)^c).$$

Now  $A \cap (E_1 \cup E_2) = (A \cap E_1) \cup (A \cap (E_1^c \cap E_2))$  so

$$m^*(A \cap E_1) + m^*(A \cap E_1^c \cap E_2) \geq m^*(A \cap (E_1 \cup E_2)).$$

Substituting this for the two terms on the right of the previous displayed equation gives

$$m^*(A) \geq m^*(A \cap (E_1 \cup E_2)) + m^*(A \cap (E_1 \cup E_2)^c)$$

which is just (9) for the set  $E_1 \cup E_2$ . This proves the lemma and the theorem.

We let  $\mathcal{M}$  denote the class of measurable subsets of  $\mathbf{R}$  -measurability in the sense of Lebesgue or Caratheodory these being equivalent. Notice by induction starting with two terms as in the lemma, that any finite union of sets in  $\mathcal{M}$  is again in  $\mathcal{M}$

## 5 Countable additivity.

The first main theorem in the subject is the following description of  $\mathcal{M}$  and the function  $m$  on it:

**Theorem 3**  *$\mathcal{M}$  and the function  $m : \mathcal{M} \rightarrow \mathbf{R}$  have the following properties:*

- $\mathbf{R} \in \mathcal{M}$ .
- $E \in \mathcal{M} \Rightarrow E^c \in \mathcal{M}$ .
- If  $E_n \in \mathcal{M}$  for  $n = 1, 2, 3, \dots$  then  $\bigcup_n E_n \in \mathcal{M}$ .
- If  $F_n \in \mathcal{M}$  and the  $F_n$  are pairwise disjoint, then  $F := \bigcup_n F_n \in \mathcal{M}$  and

$$m(F) = \sum_{n=1}^{\infty} m(F_n).$$

**Proof.** We already know the first two items on the list, and we know that a finite union of sets in  $\mathcal{M}$  is again in  $\mathcal{M}$ . We also know the last assertion which is Proposition 2. But it will be instructive and useful for us to have a proof starting directly from Caratheodory's definition of measurability:

If  $F_1 \in \mathcal{M}$ ,  $F_2 \in \mathcal{M}$  and  $F_1 \cap F_2 = \emptyset$  then taking

$$A = F_1 \cup F_2, \quad E_1 = F_1, \quad E_2 = F_2$$

in (10) gives

$$m(F_1 \cup F_2) = m(F_1) + m(F_2).$$

Induction then shows that if  $F_1, \dots, F_n$  are pairwise disjoint elements of  $\mathcal{M}$  then their union belongs to  $\mathcal{M}$  and

$$m(F_1 \cup F_2 \cup \dots \cup F_n) = m(F_1) + m(F_2) + \dots + m(F_n).$$

More generally, if we let  $A$  be arbitrary and take  $E_1 = F_1$ ,  $E_2 = F_2$  in (10) we get

$$m^*(A) = m^*(A \cap F_1) + m^*(A \cap F_2) + m^*(A \cap (F_1 \cup F_2)^c).$$

If  $F_3 \in \mathcal{M}$  is disjoint from  $F_1$  and  $F_2$  we may apply (8) with  $A$  replaced by  $A \cap (F_1 \cup F_2)^c$  and  $E$  by  $F_3$  to get

$$m^*(A \cap (F_1 \cup F_2)^c) = m^*(A \cap F_3) + m^*(A \cap (F_1 \cup F_2 \cup F_3)^c),$$

since

$$(F_1 \cup F_2)^c \cap F_3^c = F_1^c \cap F_2^c \cap F_3^c = (F_1 \cup F_2 \cup F_3)^c.$$

Substituting this back into the preceding equation gives

$$m^*(A) = m^*(A \cap F_1) + m^*(A \cap F_2) + m^*(A \cap F_3) + m^*(A \cap (F_1 \cup F_2 \cup F_3)^c).$$

Proceeding inductively, we conclude that if  $F_1, \dots, F_n$  are pairwise disjoint elements of  $\mathcal{M}$  then

$$m^*(A) = \sum_1^n m^*(A \cap F_i) + m^*(A \cap (F_1 \cup \dots \cup F_n)^c). \quad (11)$$

Now suppose that we have a countable family  $\{F_i\}$  of pairwise disjoint sets belonging to  $\mathcal{M}$ . Since

$$\left( \bigcup_{i=1}^n F_i \right)^c \supset \left( \bigcup_{i=1}^{\infty} F_i \right)^c$$

we conclude from (11) that

$$m^*(A) \geq \sum_1^n m^*(A \cap F_i) + m^* \left( A \cap \left( \bigcup_{i=1}^{\infty} F_i \right)^c \right)$$

and hence passing to the limit

$$m^*(A) \geq \sum_1^{\infty} m^*(A \cap F_i) + m^* \left( A \cap \left( \bigcup_{i=1}^{\infty} F_i \right)^c \right).$$

Now given any collection of sets  $B_k$  we can find intervals  $\{I_{k,j}\}$  with

$$B_k \subset \bigcup_j I_{k,j}$$

and

$$m(B_k) \leq \sum_j \ell(I_{k,j}) + \frac{\epsilon}{2^k}.$$

So

$$\bigcup_k B_k \subset \bigcup_{k,j} I_{k,j}$$

and hence

$$m^* \left( \bigcup B_k \right) \leq \sum m^*(B_k),$$

the inequality being trivially true if the sum on the right is infinite. So

$$\sum_{i=1}^{\infty} m^*(A \cap F_k) \geq m^* \left( A \cap \left( \bigcup_{i=1}^{\infty} F_i \right) \right).$$

Thus

$$\begin{aligned} m^*(A) &\geq \sum_1^{\infty} m^*(A \cap F_i) + m^* \left( A \cap \left( \bigcup_{i=1}^{\infty} F_i \right)^c \right) \geq \\ &\geq m^* \left( A \cap \left( \bigcup_{i=1}^{\infty} F_i \right) \right) + m^* \left( A \cap \left( \bigcup_{i=1}^{\infty} F_i \right)^c \right). \end{aligned}$$

The extreme right of this inequality is the left hand side of (9) applied to

$$E = \bigcup_i F_i,$$

and so  $E \in \mathcal{M}$  and the preceding string of inequalities must be equalities since the middle is trapped between both sides which must be equal. Hence we have proved that if  $F_n$  is a disjoint countable family of sets belonging to  $\mathcal{M}$  then their union belongs to  $\mathcal{M}$  and

$$m^*(A) = \sum_i m^*(A \cap F_i) + m^* \left( A \cap \left( \bigcup_{i=1}^{\infty} F_i \right)^c \right). \quad (12)$$

If we take  $A = \bigcup F_i$  we conclude that

$$m(F) = \sum_{n=1}^{\infty} m(F_n) \quad (13)$$

if the  $F_j$  are disjoint and

$$F = \bigcup F_j.$$

So we have reproved the last assertion of the theorem using Caratheodory's definition. For the third assertion, we need only observe that a countable union of sets in  $\mathcal{M}$  can be always written as a countable disjoint union of sets in  $\mathcal{M}$ . Indeed, set

$$\begin{aligned} F_1 &:= E_1, \quad F_2 := E_2 \setminus E_1 = E_1 \cap E_2^c \\ F_3 &:= E_3 \setminus (E_1 \cup E_2) \end{aligned}$$

etc. The right hand sides all belong to  $\mathcal{M}$  since  $\mathcal{M}$  is closed under taking complements and finite unions and hence intersections, and

$$\bigcup_j F_j = \bigcup E_j.$$

We have completed the proof of the theorem.

A number of easy consequences follow: The symmetric difference between two sets is the set of points belonging to one or the other but not both:

$$A \Delta B := (A \setminus B) \cup (B \setminus A).$$

**Proposition 8** *If  $A \in \mathcal{M}$  and  $m(A \Delta B) = 0$  then  $B \in \mathcal{M}$  and  $m(A) = m(B)$ .*

**Proof.** By assumption  $A \setminus B$  has measure zero (and hence is measurable) since it is contained in the set  $A \Delta B$  which is assumed to have measure zero. Similarly for  $B \setminus A$ . Also  $(A \cap B) \in \mathcal{M}$  since

$$A \cap B = A \setminus (A \setminus B).$$

Thus

$$B = (A \cap B) \cup (B \setminus A) \in \mathcal{M}.$$

Since  $B \setminus A$  and  $A \cap B$  are disjoint, we have

$$m(B) = m(A \cap B) + m(B \setminus A) = m(A \cap B) = m(A \cap B) + m(A \setminus B) = m(A).$$

QED

**Proposition 9** *Suppose that  $A_n \in \mathcal{M}$  and  $A_n \subset A_{n+1}$  for  $n = 1, 2, \dots$ . Then*

$$m\left(\bigcup A_n\right) = \lim_{n \rightarrow \infty} m(A_n).$$

Indeed, setting  $B_n := A_n \setminus A_{n-1}$  (with  $B_1 = A_1$ ) the  $B_i$  are pairwise disjoint and have the same union as the  $A_i$  so

$$m\left(\bigcup A_n\right) = \sum_{i=1}^{\infty} m(B_i) = \lim_{n \rightarrow \infty} \sum_{i=1}^n m(B_i) = \lim_{n \rightarrow \infty} m\left(\bigcup_{i=1}^n B_i\right) = \lim_{n \rightarrow \infty} m(A_n).$$

QED

**Proposition 10** *If  $C_n \supset C_{n+1}$  is a decreasing family of sets in  $\mathcal{M}$  and  $m(C_1) < \infty$  then*

$$m\left(\bigcap C_n\right) = \lim_{n \rightarrow \infty} m(C_n).$$

Indeed, set  $A_1 := \emptyset$ ,  $A_2 := C_1 \setminus C_2$ ,  $A_3 := C_1 \setminus C_3$  etc. The  $A$ 's are increasing so

$$m\left(\bigcup (C_1 \setminus C_i)\right) = \lim_{n \rightarrow \infty} m(C_1 \setminus C_n) = m(C_1) - \lim_{n \rightarrow \infty} m(C_n)$$

by the preceding proposition. Since  $m(C_1) < \infty$  we have

$$m(C_1 \setminus C_n) = m(C_1) - m(C_n).$$

Also

$$\bigcup_n (C_1 \setminus C_n) = C_1 \setminus \left(\bigcap_n C_n\right).$$

So

$$m\left(\bigcup_n (C_1 \setminus C_n)\right) = m(C_1) - m\left(\bigcap_n C_n\right) = m(C_1) - \lim_{n \rightarrow \infty} m(C_n).$$

Subtracting  $m(C_1)$  from both sides of the last equation gives the equality in the proposition. QED