Constructing measures.

Math 212a

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Contents

1 σ-fields, measures, and outer measures.

We introduce some notation and results generalizing those of last time: X is a set (usually it will be a topological space or even a metric space). A collection $\mathcal{F}$ of subsets is called a σ field if:

- $X \in \mathcal{F}$,
- If $E \in \mathcal{F}$ then $E^c = X \setminus E \in \mathcal{F}$, and
- If $\{E_n\}$ is a sequence of elements in $\mathcal{F}$ then $\bigcup_n E_n \in \mathcal{F}$,

The intersection of any family of σ-fields is again a σ-field, and hence given any collection $\mathcal{C}$ of subsets of X, there is a smallest σ-field $\mathcal{F}$ which contains it. Then $\mathcal{F}$ is called the σ-field **generated** by $\mathcal{C}$.

If X is a metric space, the σ-field generated by the collection of open sets is called the **Borel** σ-field, usually denoted by $\mathcal{B}$ or $\mathcal{B}(X)$ and a set belonging to $\mathcal{B}$ is called a **Borel set**.

Given a σ-field $\mathcal{F}$ a (non-negative) **measure** is a function;

$$m : \mathcal{F} \to [0, \infty]$$

such that

- $m(\emptyset) = 0$ and
- **Countable additivity:** If F_n is a disjoint collection of sets in $\mathcal{F}$ then

$$m\left(\bigcup_n F_n\right) = \sum_n m(F_n).$$

In the countable additivity condition it is understood that both sides might be infinite.

An **outer measure** on a set X is a map m^* to $[0, \infty]$ defined on the collection of *all* subsets of X which satisfies

- $m(\emptyset) = 0$,
- **Monotonicity:** If $A \subset B$ then $m^*(A) \leq m^*(B)$, and
- **Countable subadditivity:** $m^*\left(\bigcup_n A_n\right) \leq \sum_n m^*(A_n)$.

Given an outer measure, m^*, we defined a set E to be **measurable** (relative to m^*) if

$$m^*(A) = m^*(A \cap E) + m^*(A \cap E^c)$$

for all sets A. Then Caratheodory's theorem that we proved last time asserts that the collection of measurable sets is a σ-field, and the restriction of m^* to the collection of measurable sets is a measure which we shall usually denote by m.

There is an unfortunate disagreement in terminology, in that many of the professionals, especially in geometric measure theory, use the term "measure" for what we have been calling "outer measure". However we will follow the above conventions which used to be the old fashioned standard.

An obvious task, given Caratheodory's theorem, is to look for ways of constructing outer measures.

2 Constructing outer measures, Method I.

Let $\mathcal{C}$ be a collection of sets which cover X. For any subset A of X let

$$\text{ccc}(A)$$

denote the set of (finite or) **c**ountable **c**overs of A by sets belonging to $\mathcal{C}$. In other words, an element of $\text{ccc}(A)$ is a finite or countable collection of elements of $\mathcal{C}$ whose union contains A.

Suppose we are given a function

$$\ell : \mathcal{C} \to [0, \infty].$$

Theorem 1 *There exists a unique outer measure m^* on X such that*

- *$m^*(A) \leq \ell(A)$ for all $A \in \mathcal{C}$ and*
- *If n^* is any outer measure satisfying the preceding condition then $n^*(A) \leq m^*(A)$ for all subsets A of X.*

This unique outer measure is given by

$$m^*(A) = \inf_{\mathcal{D} \in \mathrm{ccc}(A)} \sum_{D \in \mathcal{D}} \ell(D). \tag{1}$$

In other words, for each countable cover of A by elements of $\mathcal{C}$ we compute the sum above, and then minimize over all such covers of A.

If we had two outer measures satisfying both conditions then each would have to be $\leq$ the other, so the uniqueness is obvious.

To check that the m^* defined by (1) is an outer measure, observe that for the empty set we may take the empty cover, and the convention about an empty sum is that it is zero, so $m^*(\emptyset) = 0$. If $A \subset B$ then any cover of B is a cover of A, so that $m^*(A) \leq m^*(B)$. To check countable subadditivity we use the usual $\epsilon/2^n$ trick: If $m^*(A_n) = \infty$ for any A_n the subadditivity condition is obviously satisfied. Otherwise, we can find a $\mathcal{D}_n \in \mathrm{ccc}(A_n)$ with

$$\sum_{D \in \mathcal{D}_\backslash} \ell(D) \leq m^*(A_n) + \frac{\epsilon}{2^n}.$$

Then we can collect all the D together into a countable cover of A so

$$m^*(A) \leq \sum_n m^*(A_n) + \epsilon,$$

and since this is true for all $\epsilon > 0$ we conclude that m^* is countably subadditive. So we have verified that m^* defined (1) is an outer measure. We must check that it satisfies the two conditions in the theorem. If $A \in \mathcal{C}$ then the single element collection $\{A\} \in \mathrm{ccc}(A)$, so $m^*(A) \leq \ell(A)$, so the first condition is obvious. As to the second condition, suppose n^* is an outer measure with $n^*(D) \leq \ell(D)$ for all $D \in \mathcal{C}$. Then for any set A and any countable cover $\mathcal{D}$ of A by elements of $\mathcal{C}$ we have

$$\sum_{D \in \mathcal{D}} \ell(D) \geq \sum_{D \in \mathcal{D}} n^*(D) \geq n^* \left(\bigcup_{D \in \mathcal{D}} D \right) \geq n^*(A),$$

where in the second inequality we used the countable subadditivity of n^* and in the last inequality we used the monotonicity of n^*. Minimizing over all $\mathcal{D} \in \mathrm{ccc}(A)$ shows that $m^*(A) \geq n^*(A)$. QED

This theorem is basically a repeat performance of the construction of Lebesgue measure we did last time. However there is some trouble:

2.1 A pathological example.

Suppose we take $X = \mathbf{R}$, and let $\mathcal{C}$ consist of all *half open* intervals of the form $[a, b)$. However, instead of taking ℓ to be the length of the interval, we take it to be the square root of the length:

$$\ell([a,b)) := (b-a)^{\frac{1}{2}}.$$

I claim that any half open interval (say $[0,1)$) of length one has $m^*([a,b)) = 1$. (Since ℓ is translation invariant, it does not matter which interval we choose.) Indeed, $m^*([0,1)) \leq 1$ by the first condition in the theorem, since $\ell([0,1)) = 1$. On the other hand, if

$$[0,1) \subset \bigcup_i [a_i, b_i)$$

then we know from last lecture that

$$\sum (b_i - a_i) \geq 1,$$

so squaring gives

$$\left(\sum (b_i - a_i)^{\frac{1}{2}}\right)^2 = \sum_i (b_i - a_i) + \sum_{i \neq j} (b_i - a_i)^{\frac{1}{2}} (b_j - a_j)^{\frac{1}{2}} \geq 1.$$

So $m^*([0,1)) = 1$.

On the other hand, consider an interval $[a, b)$ of length 2. Since it covers itself, $m^*([a,b)) \leq \sqrt{2}$.

Consider the closed interval $I = [0,1]$. Then

$$I \cap [-1,1) = [0,1) \quad \text{and} \quad I^c \cap [-1,1) = [-1,0)$$

so

$$m^*(I \cap [-1,1)) + m^*(I^c \cap [-1,1) = 2 > \sqrt{2} \geq m^*([-1,1)).$$

In other words, the closed unit interval is *not measurable* relative to the outer measure m^* determined by the theorem. We would like Borel sets to be measurable, and the above computation shows that the measure produced by Method I as above does not have this desirable property. In fact, if we consider two half open intervals I_1 and I_2 of length one separated by a small distance of size ϵ, say, then their union $I_1 \cup I_2$ is covered by an interval of length $2 + \epsilon$, and hence

$$m^*(I_1 \cup I_2) \leq \sqrt{2+\epsilon} < m^*(I_1) + m^*(I_2).$$

In other words, m^* is not additive even on intervals separated by a finite distance. It turns out that this is the crucial property that is missing:

2.2 Metric outer measures.

Let X be a metric space. An outer measure on X is called a **metric outer measure** if

$$m^*(A \cup B) = m^*(A) + m^*(B) \quad \text{whenever } d(A,B) > 0. \tag{2}$$

The condition $d(A,B) > 0$ means that there is an $\epsilon > 0$ (depending on A and B) so that $d(x.y) > \epsilon$ for all $x \in A,\ y \in B$. The main result here is due to Caratheodory:

Theorem 2 *If m^* is a metric outer measure on a metric space X, then all Borel sets of X are m^* measurable.*

Proof. Since the σ-field of Borel sets is generated by the closed sets, it is enough to prove that every closed set F is measurable in the sense of Caratheodory, i.e. that for any set A

$$m^*(A) \geq m^*(A \cap F) + m^*(A \setminus F).$$

Let

$$A_j := \{x \in A | d(X,F) \geq \frac{1}{j}\}.$$

We have $d(A_j, A \cap F) \geq 1/j$ so, since m^* is a metric outer measure, we have

$$m^*(A \cap F) + m^*(A_j) = m^*((A \cap F) \cup A_j) \leq m^*(A) \tag{3}$$

since $(A \cap F) \cup A_j \subset A$. Now

$$A \setminus F = \bigcup A_j$$

since F is closed, and hence every point of A not belonging to F must be at a positive distance from F. We would like to be able to pass to the limit in (3). If the limit on the left is infinite, there is nothing to prove. So we may assume it is finite.

Now if $x \in A \setminus (F \cup A_{j+1})$ there is a $z \in F$ with $d(x,z) < 1/(j+1)$ while if $y \in A_j$ we have $d(x,z) \geq 1/j$ so

$$d(x,y) \geq d(y,z) - d(x,z) \geq \frac{1}{j} - \frac{1}{j+1} > 0.$$

Let $B_1 := A_1$ and $B_2 := A_2 \setminus A_1,\ B_3 = A_3 \setminus A_2$ etc. Thus if $i \geq j+2$, then $B_j \subset A_j$ and

$$B_i \subset A \setminus (F \cup A_{i-1}) \subset A \setminus (F \cup A_{j+1})$$

and so $d(B_i, B_j) > 0$. So m^* is additive on finite unions of even or odd B's:

$$m^*\left(\bigcup_{k=1}^n B_{2k-1}\right) = \sum_{k=1}^n m^*(B_{2k-1}), \quad m^*\left(\bigcup_{k=1}^n B_{2k}\right) = \sum_{k=1}^n m^*(B_{2k}).$$

Both of these are $\leq m^*(A_{2n})$ since the union of the sets involved are contained in A_{2n}. Since $m^*(A_{2n})$ is increasing, and assumed bounded, both of the above series converge. Thus

$$\begin{aligned} m^*(A/F) &= m^*\left(\bigcup A_i\right) \\ &= m^*\left(A_j \cup \bigcup_{k\geq j+1} B_j\right) \\ &\leq m^*(A_j) + \sum_{k=j+1}^{\infty} m^*(B_j) \\ &\leq \lim_{n\to\infty} m^*(A_n) + \sum_{k=j+1}^{\infty} m^*(B_j). \end{aligned}$$

But the sum on the right can be made as small as possible by choosing j large, since the series converges. Hence

$$m^*(A/F) \leq \lim_{n\to\infty} m^*(A_n).$$

So substituting into (3) and passing to the limit gives

$$m^*(A\cap F) + m^*(A \setminus F) \leq m^*(A\cap F) + \lim_{n\to\infty} m^*(A_n) \leq m^*(A).$$

So all closed sets, and hence all Borel sets are measurable. QED.

3 Constructing outer measures, Method II.

Let $\mathcal{C} \subset \mathcal{E}$ be two collections of sets which cover X. Suppose that ℓ is defined on $\mathcal{E}$, and hence, by restriction, on $\mathcal{C}$. In the definition (1) of the outer measure $m^*_{\ell,\mathcal{C}}$ associated to ℓ and $\mathcal{C}$, we are minimizing over a smaller collection of covers than in computing the metric outer measure $m^*_{\ell,\mathcal{E}}$ using all the sets of $\mathcal{E}$. Hence

$$m^*_{\ell,\mathcal{C}}(A) \geq m^*_{\ell,\mathcal{E}}(A)$$

for any set A.

We want to apply this remark to the case where X is a metric space, and we have a cover $\mathcal{C}$ with the property that for every $x \in X$ and every $\epsilon > 0$ there is a $C \in \mathcal{C}$ with $x \in C$ and $\text{diam}(C) < \epsilon$. In other words, we are assuming that the

$$\mathcal{C}_\epsilon := \{C \in \mathcal{C} |\ \text{diam }(C) < \epsilon\}$$

are covers of X for every $\epsilon > 0$. Then for every set A the

$$m^*_{\ell,\mathcal{C}_\epsilon}(A)$$

are increasing, so we can consider the function on set given by

$$m_{II}^*(A) := \sup_{\epsilon \to 0} m_{\ell, \mathcal{C}_\epsilon}(A).$$

The axioms for an outer measure are preserved by this limit operation, so m_{II}^* is an outer measure. If A and B are such that $d(A,B) > 2\epsilon$, then any set of $\mathcal{C}_\epsilon$ which intersects A does not intersect B and vice versa, so throwing away extraneous sets in a cover of $A \cup B$ which does not intersect either, we see that $m_{II}^*(A \cup B) = m_{II}^*(A) + m_{II}^*(B)$. The method II construction always yields a metric outer measure.

3.1 An example.

Let X be the set of all (one sided) infinite sequences of 0's and 1's. So a point of X is an expression of the form

$$a_1 a_2 a_3 \cdots$$

where each a_i is 0 or 1. For any finite sequence α of 0's or 1's, let

$$[\alpha] \text{ denote the set of all sequences which begin with } \alpha. \tag{4}$$

We also let

$$|\alpha|$$

denote the length of α, that is, the number of bits in α. For each

$$0 < r < 1$$

we define a metric d_r on X by: If

$$x = \alpha x', \qquad y = \alpha y'$$

where the first bit in x' is different from the first bit in y' then

$$d_r(x,y) = r^{|\alpha|}.$$

In other words, the distance between two sequence is r^k where k is the length of the longest initial segment where they agree. Clearly $d_r(x,y) \geq 0$ and $= 0$ if and only if $x = y$, and $d_r(y,x) = d_r(x,y)$. Also, for three x, y, and z we claim that

$$d_r(x,z) \leq \max\{d_r(x,y), d_r(y,z)\}.$$

Indeed, if two of the three points are equal this is obvious. Otherwise, let j denote the length of the longest common prefix of x and y, and let k denote the length of the longest common prefix of y and z. Let $m = \min(j,k)$. Then the first m bits of x agree with the first m bits of z and so $d_r(x,z) \leq r^m = \max(r^j, r^k)$. QED

A metric with this property (which is much stronger than the triangle inequality) is called an **ultrametric**.

Let X be a metric space. Recall that if A is any subset of X, the **diameter** of A is defined as

$$\operatorname{diam}(A) = \sup_{x,y\in A} d(x,y).$$

Notice that

$$\operatorname{diam}\,[\alpha] = r^{\alpha}. \tag{5}$$

The metrics for different r are different, and we will make use of this fact shortly. But

Proposition 1 *The spaces (X, d_r) are all homeomorphic under the identity map.*

It is enough to show that the identity map is a continuous map from (X, d_r) to (X, d_s) since it is one to one and we can interchange the role of r and s. So, given $\epsilon > 0$, we must find a $\delta > 0$ such that if $d_r(x,y) < \delta$ then $d_s(x,y) < \epsilon$. So choose k so that $s^k < \epsilon$. Then letting $r^k = \delta$ will do.

So although the metrics are different, the topologies are the same.

There is something special about the value $r = \frac{1}{2}$: Let $\mathcal{C}$ be the collection of all sets of the form $[\alpha]$ and let ℓ be defined on $\mathcal{C}$ by

$$\ell([\alpha]) = (\frac{1}{2})^{|\alpha|}.$$

We can construct the method II outer measure associated with this function, which will satisfy

$$m^*_{II}([\alpha]) \geq m_I([\alpha])$$

where m_I denotes the method I outer measure associated with ℓ. What is special about the value $\frac{1}{2}$ is that if $k = |\alpha|$ then

$$\ell([\alpha]) = \left(\frac{1}{2}\right)^k = \left(\frac{1}{2}\right)^{k+1} + \left(\frac{1}{2}\right)^{k+1} = \ell([\alpha 0]) + \ell([\alpha 1]).$$

So if we also use the metric $d_{\frac{1}{2}}$, we see, by repeating the above, that every $[\alpha]$ can be written as the disjoint union $C_1 \cup \cdots \cup C_n$ of sets in $\mathcal{C}_\epsilon$ with $\ell([\alpha])) = \sum \ell(C_i)$. Thus $m^*_{\ell,\mathcal{C}_\epsilon}([\alpha]) \leq \ell(\alpha)$ and so $m^*_{\ell,\mathcal{C}_\epsilon}([\alpha])(A) \leq m^*_I(A)$ or $m^*_{II} = m^*_I$. It also follows from the above computation that

$$m^*([\alpha]) = \ell([\alpha]).$$

There is also something special about the value $s = \frac{1}{3}$: Recall that one of the definitions of the Cantor set $\mathbf{C}$ is that it consists of all points $x \in [0,1]$ which have a base 3 expansion involving only the symbols 0 and 2. Let

$$h : X \to \mathbf{C}$$

where h sends the bit 1 into the symbol 2, e.g.

$$h(011001\ldots) = .022002\ldots.$$

In other words, for any sequence z

$$h(0z) = \frac{h(z)}{3}, \qquad h(1z) = \frac{h(z)+2}{3}. \tag{6}$$

I claim that:

$$\frac{1}{3} d_{\frac{1}{3}}(x,y) \le |h(x) - h(y)| \le d_{\frac{1}{3}}(x,y) \tag{7}$$

Proof. If x and y start with different bits, say $x = 0x'$ and $y = 1y'$ then $d_{\frac{1}{3}}(x,y) = 1$ while $h(x)$ lies in the interval $[0, \frac{1}{3}]$ and $h(y)$ lies in the interval $[\frac{2}{3}, 1]$ on the real line. So $h(x)$ and $h(y)$ are at least a distance $\frac{1}{3}$ and at most a distance 1 apart, which is what (7) says. So we proceed by induction. Suppose we know that (7) is true when $x = \alpha x'$ and $y = \alpha y'$ with x', y' starting with different digits, and $|\alpha| \le n$. (The above case was where $|\alpha| = 0$.) So if $|\alpha| = n+1$ then either and $\alpha = 0\beta$ or $\alpha = 1\beta$ and the argument for either case is similar: We know that (7) holds for $\beta x'$ and $\beta y'$ and

$$d_{\frac{1}{3}}(x,y) = \frac{1}{3} d_{\frac{1}{3}}(\beta x', \beta y')$$

while $|h(x) - h(y)| = \frac{1}{3}|h(\beta x') - h(\beta y')|$ by (6). Hence (7) holds by induction. QED

In other words, the map h is a Lipschitz map with Lipschitz inverse from $(X, d_{\frac{1}{3}})$ to the Cantor set $\mathbf{C}$.

In a short while, after making the appropriate definitions, these two computations, one with the measure associated to $\ell([\alpha]) = \left(\frac{1}{2}\right)^{|\alpha|}$ and the other associated with $d_{\frac{1}{3}}$ will show that the "Hausdorff dimension" of the Cantor set is $\log 2 / \log 3$.

4 Hausdorff measure.

Let X be a metric space. Recall once again that if A is any subset of X, the diameter of A is defined as

$$\operatorname{diam}(A) = \sup_{x,y \in A} d(x,y).$$

For any subset $A \subset X$ and any positive real number s define

$$\ell_s(A) = \operatorname{diam}(A)^s$$

(with $0^s = 0$). Take $\mathcal{C}$ to consist of *all* subsets of X. The method II outer measure is called the s-**dimensional Hausdorff outer measure**, and its restriction to the associated σ-field of (Caratheodory) measurable sets is called the s-**dimensional Hausdorff measure**. We will let $m^*_{s,\epsilon}$ denote the method I outer measure associated to ℓ_s and ϵ, and let $\mathcal{H}^*_s$ denote the Hausdorff outer measure of dimension s, so that

$$\mathcal{H}_s(A) = \lim_{\epsilon \to 0} m^*_{s,\epsilon}(A).$$

For example, we claim that for $X = \mathbf{R}$, $\mathcal{H}'_1$ is exactly Lebesgue outer measure, which we will denote here by L^*. Indeed, if A has diameter r, then A is contained in a closed interval of length r. Hence $L^*(A) \leq r$. The Method I construction theorem says that $m_{1,\epsilon}$ is the largest outer measure satisfying $m^*(A) \leq \operatorname{diam} A$ for sets of diameter less than ϵ. Hence $m^*_{1,\epsilon}(A) \geq L^*(A)$ for all sets A and all ϵ, and so

$$\mathcal{H}^*_1 \geq L^*.$$

On the other hand, any bounded half open interval $[a, b)$ can be broken up into a finite union of half open intervals of length $< \epsilon$, whose sum of diameters is $b - a$. So $m^*_{1,\epsilon}([a, b) \leq b - a$. But the method I construction theorem says that L^* is the largest outer measure satisfying

$$m^*([a, b)) \leq b - a.$$

Hence $\mathcal{H}^*_1 \leq L^*$ So they are equal.

In two or more dimensions, the Hausdorff measure $\mathcal{H}_k$ on $\mathbf{R}^k$ differs from Lebesgue measure by a constant. This is essentially because they assign different values to the ball of diameter one. In two dimensions for example, the Hausdorff measure $\mathcal{H}_2$ assigns the value one to the disk of diameter one, while its Lebesgue measure is $\pi/4$. For this reason, some authors prefer to put this "correction factor" into the definition of the Hausdorff measure, which would involve the Gamma function for non-integral s. I am following the convention that finds it simpler to drop this factor.

Theorem 3 *Let $F \subset X$ be a Borel set. Let $0 < s < t$. Then*

$$\mathcal{H}_s(F) < \infty \ \Rightarrow \mathcal{H}_t(F) = 0$$

and

$$\mathcal{H}_t(F) > 0 \quad \Rightarrow \mathcal{H}_s(F) = \infty.$$

Indeed, if $\operatorname{diam} A \leq \epsilon$, then

$$m^*_{t,\epsilon}(A) \leq (\operatorname{diam} A)^t \leq \epsilon^{t-s}(\operatorname{diam} A)^s$$

so by the method I construction theorem we have

$$m^*_{t,\epsilon}(B) \leq \epsilon^{t-s} m^*_{s,\epsilon}(B)$$

for all B. If we take $B = F$ in this equality, then the assumption $\mathcal{H}_s(F) < \infty$ implies that the limit of the right hand side tends to 0 as $\epsilon \to 0$, so $\mathcal{H}_t(F) = 0$. The second assertion in the theorem is the contrapositive of the first.

5 Hausdorff dimension.

This last theorem implies that for any Borel set, there is a unique value s_0 (which might be 0 or ∞) such that $\mathcal{H}_t(F) = 0$ for all $t < s_0$ and $\mathcal{H}_s(F) = \infty$ for all for all $s > s_0$. This value is called the **Hausdorff dimension** of F. It is one of many competing (and non-equivalent) definitions of dimension. Notice that it is a metric invariant, and in fact is the same for two spaces different by a Lipschitz homeomorphism with Lipschitz inverse. But it is not a topological invariant. In fact, we shall show that the space X of all sequences of zeros and one studied above has Hausdorff dimension 1 relative to the metric $d_{\frac{1}{2}}$ while it has Hausdorff dimension $\log 2/\log 3$ if we use the metric $d_{\frac{1}{3}}$. Since we have shown that $(X, d_{\frac{1}{3}})$ is Lipschitz equivalent to the Cantor set $\mathbf{C}$, this will also prove that $\mathbf{C}$ has Hausdorff dimension $\log 2/\log 3$.

We first discuss the $d_{\frac{1}{2}}$ case and use the following lemma

Lemma 1 *If* $\operatorname{diam}(A) > 0$, *then there is an* α *such that* $A \subset [\alpha]$ *and* $\operatorname{diam}([\alpha]) = \operatorname{diam} A$.

Proof. Given any set A, it has a "longest common prefix". Indeed, consider the set of lengths of common prefixes of elements of A. This is finite set of non-negative integers since A has at least two distinct elements. Let n be the largest of these, and let α be a common prefix of this length. Then it is clearly the longest common prefix of A. Hence $A \subset [\alpha]$ and $\operatorname{diam}([\alpha] = \operatorname{diam} A$.QED

Let $\mathcal{C}$ denote the collection of all sets of the form $[\alpha]$ and let ℓ be the function on $\mathcal{C}$ given by

$$\ell([\alpha]) = (\frac{1}{2})^{|\alpha|},$$

and let ℓ^* be the associated method I outer measure, and m the associated measure; all these as we introduced above. We have

$$\ell^*(A) \leq \ell^*([\alpha]) = \operatorname{diam}([\alpha]) = \operatorname{diam}(A).$$

By the method I construction theorem, $m^*_{1,\epsilon}$ is the largest outer measure with the property that $n^*(A) \leq \operatorname{diam} A$ for sets of diameter $< \epsilon$. Hence $\ell^* \leq m^*_{1,\epsilon}$, and since this is true for all $\epsilon > 0$, we conclude that

$$\ell^* \leq \mathcal{H}^*_1.$$

On the other hand, for any α and any $\epsilon > 0$, there is an n such that $2^{-n} < \epsilon$ and $n \geq |\alpha|$. The set $[\alpha]$ is the disjoint union of all sets $[\beta] \subset [\alpha]$ with $|\beta| \geq n$, and there are $2^{n-|\alpha|}$ of these subsets, each having diameter 2^{-n}. So

$$m^*_{1,\epsilon}([\alpha]) \leq 2^{-|\alpha|}.$$

However ℓ^* is the largest outer measure satisfying this inequality for all $[\alpha]$. Hence $m^*_{1,\epsilon} \leq \ell^*$ for all ϵ so $\mathcal{H}^*_1 \leq \ell^*$. In other words

$$\mathcal{H}_1 = m.$$

But since we computed that $m(X) = 1$, we conclude that

The Hausdorff dimension of $(X, d_{\frac{1}{2}})$ *is* 1.

Now let us turn to $(X, d_{\frac{1}{3}})$. Then the diameter $\mathrm{diam}_{\frac{1}{2}}$ relative to the metric $d_{\frac{1}{2}}$ and the diameter $\mathrm{diam}_{\frac{1}{3}}$ relative to the metric $d_{\frac{1}{3}}$ are given by

$$\mathrm{diam}_{\frac{1}{2}}([\alpha]) = \left(\frac{1}{2}\right)^k, \quad \mathrm{diam}_{\frac{1}{3}}([\alpha]) = \left(\frac{1}{3}\right)^k, \quad k = |\alpha|.$$

If we choose s so that $2^{-k} = (3^{-k})^s$ then

$$\mathrm{diam}_{\frac{1}{2}}([\alpha]) = (\mathrm{diam}_{\frac{1}{3}}[\alpha])^s.$$

This says that relative to the metric $d_{\frac{1}{3}}$, the previous computation yields

$$\mathcal{H}_s(X) = 1.$$

Hence $s = \log 2/\log 3$ is the Hausdorff dimension of X.

Since we have shown that there is a Lipschitz map h from $(X, d_{\frac{1}{3}})$ to the Cantor set, with Lipschitz inverse, we conclude that the Hausdorff dimension of the Cantor set is $\log 2/\log 3$.

The material above (with some slight changes in notation) was taken from the book *Measure, Topology, and Fractal Geometry* by Gerald Edgar, (1990) Springer where a thorough and delightfully clear discussion can be found of the subjects listed in the title.

6 Push forward.

The above discussion is a sampling of introductory material to what is known as "geometric measure theory". However the construction of measures that we will be mainly working with will be an abstraction of the "simulation" approach that we have been developed in Problem set 2. The setup is as follows: Let $(X, \mathcal{F}, m)$ be a set with a σ-field and a measure on it, and let $(Y, \mathcal{G})$ be some other set with a σ-field on it. A map

$$f : X \to Y$$

is called **measurable** if

$$f^{-1}(B) \in \mathcal{F} \quad \forall\, B \in \mathcal{G}.$$

We may then define a measure $f_*(m)$ on $(Y, \mathcal{G})$ by

$$[(f_*)m](B) = m(f^{-1}(B)).$$

For example, if $f = Y_\lambda$ is the Poisson random variable from the exercises, and u is the uniform measure (the restriction of Lebesgue measure to) on $[0,1]$, then $f_*(u)$ is the measure on the non-negative integers given by

$$f_*(u)(\{k\}) = e^{-\lambda}\frac{\lambda^k}{k!}.$$

It will be this construction of measures and variants on it which will occupy us over then next few weeks.