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1 The spectrum.

All algebras will be over the complex numbers. The **spectrum** of an element x in an algebra is the set of all $\lambda \in \mathbf{C}$ such that $(x - \lambda e)$ has no inverse. We denote the spectrum of x by $\mathrm{Spec}(x)$.

Proposition 1.1 *If P is a polynomial then*

$$P(\mathrm{Spec}(x)) = \mathrm{Spec}(P(x)). \tag{1}$$

Proof. The product of invertible elements is invertible. For any $\lambda \in \mathbf{C}$ write $P(t) - \lambda$ as a product of linear factors:

$$P(t) - \lambda = c \prod (t - \mu_i).$$

Thus

$$P(x) - \lambda e = c \prod (x - \mu_i e)$$

in A and hence $(P(x) - \lambda e)^{-1}$ fails to exist if and only if $(x - \mu_i e)^{-1}$ fails to exist for some i, i.e. $\mu_i \in \mathrm{Spec}(x)$. But these μ_i are precisely the solutions of

$$P(\mu) = \lambda.$$

Thus $\lambda \in \mathrm{Spec}(P(x))$ if and only if $\lambda = P(\mu)$ for some $\mu \in \mathrm{Spec}(x)$ which is precisely the assertion of the proposition. QED

We wish to apply this to the algebra $\mathcal{A}$ of all bounded operators on a Hilbert space $\mathbf{H}$.

2 The Neumann Series.

This is the geometric series

$$I + T + T^2 + T^3 + \cdots$$

which converges if $\|T\| < 1$ to $(I - T)^{-1}$ with the proof as usual:

$$(I - T)(I + T + T^2 + T^3 + \cdots + T^n) = I - T^{n+1} \to I.$$

So $I - T$ has an inverse if $\|T\| < 1$ given by the geometric series. As a miniscule extension of this, if $z \neq 0$ is a complex number, then

$$zI - T \quad \text{has an inverse if} \quad \|T\| < |z|$$

because then

$$\|z^{-1}T\| < 1$$

so

$$\left(I - z^{-1}T\right)^{-1}$$

exists and $zI - T = z(I - z^{-1}T)$ so

$$(zI - T)^{-1} = z^{-1}(I - z^{-1}T)^{-1}$$

exists.

3 $\|TT^*\|$ for $T \in \mathcal{A}$.

We wish to show that for any $T \in \mathcal{A}$

$$\|TT^*\| = \|T\|^2. \tag{2}$$

Proof.

$$\begin{aligned}
\|TT^*\| &= \sup_{\|\phi\|=1} \|TT^*\phi\| \\
&= \sup_{\|\phi\|=1, \|\psi\|=1} |(TT^*\phi, \psi)| \\
&= \sup_{\|\phi\|=1, \|\psi\|=1} |(T^*\phi, T^*\psi)| \\
&\geq \sup_{\|\phi\|=1} (T^*\phi, T^*\phi) \\
&= \|T^*\|^2
\end{aligned}$$

so

$$\|T^*\|^2 \leq \|TT^*\| \leq \|T\|\|T^*\|$$

so

$$\|T^*\| \leq \|T\|.$$

Reversing the role of T and T^* gives the reverse inequality so $\|T\| = \|T^*\|$. Inserting into the preceding inequality gives

$$\|T^2\| \leq \|TT^*\| \leq \|T\|^2$$

so we have the equality (2). QED

4 The spectrum of a self-adjoint operator is real.

We wish to show that if $T \in \mathcal{A} = \mathcal{A}(\mathbf{H})$ and

$$T = T^*$$

then

$$\text{Spec}(T) \subset \mathbf{R}.$$

Indeed, suppose that $a + ib \in \text{Spec}(T)$ with $b \neq 0$, and let

$$S := \frac{1}{b}(T - aI)$$

so that $S = S^*$ and $i \in \text{Spec}(S)$. So $I + iS$ is not invertible. So for any real number r,

$$(r+1)I - (rI - iS) = I + iS$$

is not invertible. This implies that

$$|r+1| \leq \|rI - iS\|$$

and so

$$(r+1)^2 \leq \|rI - iS\|^2 = \|(rI - iS)(rI + iS)\|^2.$$

by (2). Thus

$$(r+1)^2 \leq \|r^2 I + S^2\| \leq r^2 + \|S\|^2$$

which is not possible if $2r - 1 > \|S\|^2$.

5 The spectral radius.

Define

$$|T|_{sp} = \max_{\lambda \in \text{Spec}(T)} |\lambda|.$$

I claim that for any $T \in \mathcal{A}$

$$|T|_{sp} = \lim_{n \to \infty} \|T^n\|^{\frac{1}{n}}. \tag{3}$$

Proof. If $|\lambda| > \|T\|$ then $e - T/\lambda$ is invertible, and therefore so is $T - \lambda e$ so $\lambda \notin \text{Spec}(T)$. Thus

$$|T|_{sp} \leq \|T\|.$$

We know from (1) that $\lambda \in \text{Spec}(T) \Rightarrow \lambda^n \in \text{Spec}(T^n)$, so the previous inequality applied to T^n gives

$$|T|_{sp} \leq \|T^n\|^{\frac{1}{n}}$$

and so

$$|T|_{sp} \leq \liminf \|T^n\|^{\frac{1}{n}}.$$

We must prove the reverse inequality with lim sup. Suppose that $|\mu| < 1/|T|_{sp}$ so that $\mu := 1/\lambda$ satisfies $|\lambda| > |T|_{sp}$ and hence $e - \mu T$ is invertible. The formula for the $(I-T)^{-1}$ gives

$$(I-T)^{-1} = \sum_0^\infty (\mu x)^n$$

where we know that this converges in the open disk of radius $1/\|x\|$. However, we know that $(I - \mu T)^{-1}$ exists for $|\mu| < 1/|T|_{sp}$. In particular, for any $\ell \in A^*$ the function $\lambda \mapsto \ell((I - \mu T)^{-1})$ is analytic and hence its Taylor series

$$\sum \ell(T^n)\mu^n$$

converges on this disk. Here we use the fact that the Taylor series of a function of a complex variable converges on any disk contained in the region where it is analytic. Thus

$$|\ell(\mu^n T^n)| \to 0$$

for each fixed $\ell \in A^*$ if $|\mu| < 1/|T|_{sp}$. Considered as a family of linear functions of ℓ, we see that

$$\ell \mapsto \ell(\mu^n T^n)$$

is bounded for each fixed ℓ, and hence by the *uniform boundedness principle*, there exists a constant K such that

$$\|\mu^n T^n\| < K$$

for each μ in this disk, in other words

$$\|T^n\|^{\frac{1}{n}} \leq K^{\frac{1}{n}}(1/|\mu|)$$

so

$$\lim\ \sup \|T^n\|^{\frac{1}{n}} \leq 1/|\mu| \ \text{ if } 1/|\mu| > |x|_{sp}.$$

QED

6 The Uniform Boundedness Principle.

For our convenience, I am reminding you of this result. I am merely copying this from the handout on point set topology.

Theorem 6.1 *Let* $\mathbf{B}$ *be a Banach space and* $\{F_n\}$ *be a sequence of elements in* B^* *such that for every fixed* $x \in \mathbf{B}$ *the sequence of numbers* $\{|F_n(x)|\}$ *is bounded. Then the sequence of norms* $\{\|F_n\|\}$ *is bounded.*

Proof. The proof will be by a Baire category style argument. We will prove

Proposition 6.1 *There exists some ball* $B = B(y, r), r > 0$ *and a constant* K *such that* $|F_n(z)| \leq K$ *for all* $z \in B$.

Proof that the proposition implies the theorem. For any z with $\|z\| < r$ we have

$$|F_n(z)| \leq |F_n(z-y)| + |F_n(y)| \leq 2K.$$

So

$$\|F_n\| \leq \frac{2K}{r}$$

for all n proving the theorem from the proposition.

Proof of the proposition. If the proposition is false, we can find n_1 such that $|F_{n_1}(x)| > 1$ at some $x \in B(0,1)$ and hence in some ball of radius $\epsilon < \frac{1}{2}$ about x. Then we can find an n_2 with $|F_{n_2}(z)| > 2$ in some non-empty closed ball of radius $< \frac{1}{3}$ lying inside the first ball. Continuing inductively, we choose a subsequence n_m and a family of nested non-empty balls B_m with $|F_{n_m}(z)| > m$ throughout B_m and the radii of the balls tending to zero. Since B is complete, there is a point x common to all these balls, and $\{|F_n(x)|\}$ is unbounded, contrary to hypothesis. QED

In the previous section we will used this theorem in a “reversed form”. Recall that we have the norm preserving injection $B \to B^{**}$ sending $x \mapsto x^{**}$ where $x^{**}(F) = F(x)$. Since B^* is a Banach space (even if B is incomplete) we have

Corollary 6.1 *If $\{x_n\}$ is a sequence of elements in a normed linear space such that the numerical sequence $\{|F(x_n)|\}$ is bounded for each fixed $F \in B^*$ then the sequence of norms $\{\|x_n\|\}$ is bounded.*

7 The spectral radius of a normal operator.

We wish to show that if $TT^* = T^*T$ then

$$|T|_{sp} = \|T\|. \tag{4}$$

Proof. We know from (2) that $\|T\|^2 = \|TT^*\| \leq \|T\|\|T^*\|$ so $\|T\| \leq \|T^*\|$. Replacing T by T^* gives $\|T^*\| \leq \|T\|$. so

$$\|T\| = \|T^*\|$$

and

$$\|TT^*\| = \|T\|^2 = \|T\|\|T^*\|. \tag{5}$$

Now since T is normal, i.e.

$$TT^* = T^*T,$$

$$T^2(T^2)^* = T^2(T^*)^2 = (TT^*)(TT^*)^* \tag{6}$$

and so applying (5) to T^2 and then applying it once again to T we get

$$\|T^2\|\|(T^*)^2\| = \|TT^*(TT^*)^*\| = \|TT^*\|\|TT^*\| = \|T\|^2\|T^*\|^2$$

or

$$\|T^2\|^2 = \|T\|^4.$$

Thus

$$\|T^2\| = \|T\|^2,$$

and therefore

$$\|T^4\| = \|T^2\|^2 = \|T\|^4$$

and by induction

$$\|T^{2^k}\| = \|T\|^{2^k}$$

for all non-negative integers k. Hence taking $n = 2^k$ on the right hand side of (3) we get (4). QED

8 The functional calculus for a bounded self-adjoint operator.

Suppose that $T = T^*$ so in particular T is normal. Let P be a polynomial. Then (1) combined with the preceding equation says that

$$\|P(T)\| = \max_{\lambda \in \mathrm{Spec}(T)} |P(\lambda)|. \tag{7}$$

The norm on the right is the restriction to polynomials of the uniform norm $\|\cdot\|_\infty$ on the space $C(\mathrm{Spec}(T))$.

Now the map

$$P \mapsto P(T)$$

is a homomorphism of the ring of polynomials into bounded normal operators on our Hilbert space satisfying

$$\overline{P} \mapsto P(T)^*$$

and

$$\|P(T)\| = \|P\|_{\infty,\mathrm{Spec}(T)}.$$

The Weierstrass approximation theorem then allows us to conclude that this homomorphism extends to the ring of continuous functions on $\mathrm{Spec}(T)$. We denote this extended homomorphism by ϕ so that $\phi(f) \in \mathcal{A}$ for $f \in C(\mathrm{Spec}(T))$. Then ϕ is a ring homomorphism,

$$\|\phi(f)\| = \|f\|_{\mathrm{Spec}(T),\infty}$$

and

$$(\phi(f))^* = \phi(\overline{f}).$$

Furthermore,

$$T\psi = \lambda\psi, \quad \psi \in \mathbf{H} \ \Rightarrow \phi(f)\psi = f(\lambda)\psi \tag{8}$$

since this is true for polynomials.

The above assertions are the content of the spectral theorem for bounded self-adjoint operators.

In view of the fact that $\phi(P) = P(T)$ when P is a polynomial, it is more suggestive to introduce the notation

$$f(T) := \phi(f)$$

for any $f \in C(\mathrm{Spec}(T))$. We will use both notations.

9 Resolutions of the identity.

In this section we are going to show how to extend the homomorphism ϕ, which we defined on the space of continuous functions on $\mathrm{Spec}(T)$ to the space of all Borel functions. We will do this by making use of two of our Riesz representation theorems: the one about continuous linear functions on $C(X)$ where X is compact being given by a complex valued measure, and the earlier one about linear functions on a Hilbert space being given by scalar product with a vector.

Fix

$$x, y \in H.$$

The map

$$f \mapsto (\phi(f)x, y) = (f(T)x, y)$$

is a linear function on $C(\mathcal{M})$ with

$$|(\phi(f)x, y)| \leq \|\phi(f)\|\|x\|\|y\| = \|f\|_\infty \|x\|\|y\|.$$

In particular it is a continuous linear function on $C(\mathcal{M})$. Hence, by the Riesz representation theorem, there exists a unique complex valued bounded measure

$$\mu_{x,y}$$

such that

$$(f(T)x, y) = \int_{\mathrm{Spec}(T)} d\mu_{x,y} \quad \forall \ f \in C(\mathrm{Spec}(T)).$$

When f is real,

$$f(T)^* = f(T),$$

so

$$(f(T)x, y) = (x, f(T)y) = \overline{(f(T)y, x)}.$$

The uniqueness of the measure implies that

$$\mu_{y,x} = \overline{\mu_{x,y}}.$$

Thus, for each fixed Borel set $U \subset \mathrm{Spec}(T)$ its measure $\mu_{x,y}(U)$ depends linearly on x and anti-linearly on y. We have

$$\mu_{x,y}(\mathrm{Spec}(T)) = \int_{\mathrm{Spec}(T)} \mathbf{1} d\mu_{x,y} = (Ix, y) = (x, y)$$

so

$$|\mu_{x,y}(\mathrm{Spec}(T))| \leq \|x\|\|y\|.$$

This implies that if f is any bounded Borel function on $\mathrm{Spec}(T)$, the integral

$$\int_{\mathrm{Spec}(T)} f d\mu_{x,y}$$

is well defined, and is bounded in absolute value by $\|f\|_\infty \|x\|\|y\|$. If we hold f and x fixed, this integral is a bounded anti-linear function of y, and hence by the Riesz representation theorem there exists a $w \in \mathbf{H}$ such that this integral is given by (w, y). The w in question depends linearly on f and on x because the integral does, and so we have defined a linear map O from bounded Borel functions on $\mathrm{Spec}(T)$ to bounded operators on $\mathbf{H}$ such that

$$(O(f)x, y) = \int_{\mathrm{Spec}(T)} f d\mu_{x,y}$$

and

$$\|O(f)\| \leq \|f\|_\infty.$$

On continuous functions we have

$$O(f) = \phi(f) = f(T).$$

So we know that O is multiplicative and takes complex conjugation into adjoint when restricted to continuous functions. Let us prove these facts for all Borel functions. If f is real we know that $(O(f)y, x)$ is the complex conjugate of $(O(f)x, y)$ since $\mu_{y,x} = \overline{\mu_{x,y}}$. Hence $O(f)$ is self-adjoint if f is real from which we deduce that

$$O(\overline{f}) = O(f)^*.$$

Now to the multiplicativity: If f and g are continuous functions on $\mathrm{Spec}(T)$, we have

$$\int_{\mathrm{Spec}(T)} O(f)O(g) d\mu_{x,y} = (O(f)O(g)x, y) = (O(f)(O(g)x), y) = \int_{\mathrm{Spec}(T)} f d\mu_{O(g)x,y}.$$

Since this holds for all $f \in C(\mathrm{Spec}(T))$ (for fixed g, x, y) we conclude by the uniqueness of the measure that

$$\mu_{O(g)x,y} = g\mu_{x,y}.$$

Therefore, for any bounded Borel function f we have

$$(O(g)x, O(f)^* y) = (O(f)O(g)x, y) = \int_{\mathrm{Spec}(T)} f d\mu_{O(g)x,y} = \int_{\mathrm{Spec}(T)} fg d\mu_{x,y}.$$

In particular this holds for all continuous g, and so by the uniqueness of the measure again, we conclude that

$$\mu_{x,O(f)^*y} = f\mu_{x,y}$$

and hence

$$(O(fg)x,y) = \int_{\mathrm{Spec}(T)} gf d\mu_{x,y} = \int_{\mathrm{Spec}(T)} g d\mu_{x,O(f)^*y}$$
$$= (O(g)x, O(f)^*y) = (O(f)O(g)x,y)$$

or

$$O(fg) = O(f)O(g)$$

as desired.

We have now extended the homomorphism ϕ from $C(\mathrm{Spec}(T))$ to bounded operators on $\mathbf{H}$ to a homomorphism O from the bounded Borel functions on $\mathrm{Spec}(T)$ to bounded operators on $\mathbf{H}$. I will now drop the letter ϕ. So I will use either $O(f)$ or $f(T)$ for the operator corresponding to the bounded Borel function f on $\mathrm{Spec}(T)$, Also, as I am running out of letters, P will no longer stand for a polynomial, but rather (as we will see) for a projection.

Define:

$$P(U) := O(\mathbf{1}_U)$$

for any Borel set U. The following facts are immediate:

1. $P(\emptyset) = 0$
2. $P(\mathrm{Spec}(T)) = I$ the identity
3. $P(U \cap V) = P(U)P(V)$ and $P(U)^* = P(U)$. In particular, $P(U)$ is a self-adjoint projection operator.
4. If $U \cap V = \emptyset$ then $P(U \cup V) = P(U) + P(V)$.
5. For each fixed $x, y \in H$ the set function $P_{x,y} : U \mapsto (P(U)x, y)$ is a complex valued measure.

Such a P is called a **resolution of the identity**. It follows from the last item that for any fixed $x \in \mathbf{H}$, the map $U \mapsto P(U)x$ is an $\mathbf{H}$ valued measure.

We have shown that any bounded self-adjoint operator T gives rise to a unique resolution of the identity on $\mathrm{Spec}(T)$. Furthermore, if λ denotes the "identity function" $\lambda(z) = z$ for all $z \in \mathrm{Spec}(T)$ then

$$T = \int_{\mathrm{Spec}(T)} \lambda dP \tag{9}$$

in the "weak" sense that

$$(Tx, y) = \int_{\mathrm{Spec}(T)} \lambda d\mu_{x,y} \quad \mu_{x,y}(U) = (P(U)x, y).$$

Actually, given any resolution of the identity on a compact subset $\mathcal{M} \subset \mathbf{R}$, or, more generally on any locally compact Hausdorff space $\mathcal{M}$, we can give a meaning to the integral

$$\int_{\mathcal{M}} f dP$$

for any bounded Borel function f in the strong sense as follows: if

$$s = \sum \alpha_i \mathbf{1}_{U_i}$$

is a simple function where

$$\mathcal{M} = U_1 \cup \cdots \cup U_n, \quad U_i \cap U_j = \emptyset, \quad i \neq j$$

and $\alpha_1, \ldots, \alpha_n \in \mathbf{C}$, define

$$O(s) := \sum \alpha_i P(U_i) =: \int_{\mathcal{M}} s dP.$$

This is well defined on simple functions (is independent of the expression) and is multiplicative

$$O(st) = O(s)O(t).$$

Also, since the $P(U)$ are self adjoint,

$$O(\overline{s}) = O(s)^*.$$

It is also clear that O is linear and

$$(O(s)x, y) = \int_{\mathcal{M}} s dP_{x,y}.$$

As a consequence, we get

$$\|O(s)x\|^2 = (O(s)^* O(s)x, x) = \int_{\mathcal{M}} |s|^2 dP_{x,x}$$

so

$$\|O(s)x)\|^2 \leq |s\|_\infty \|x\|^2.$$

If we choose i such that $|\alpha_i| = \|s\|_\infty$ and take $x = P(U_i)y \neq 0$, then we see that

$$\|O(s)\| = \|s\|_\infty$$

provided we now take $\|f\|_\infty$ to denote the **essential supremum** which means the following:

It follows from the properties of a resolution of the identity that if U_n is a sequence of Borel sets such that $P(U_n) = 0$, then $P(U) = 0$ if $U = \bigcup U_n$. So if f is any complex valued Borel function on $\mathcal{M}$, there will exist a largest open subset $V \subset \mathbf{C}$ such that $P(f^{-1}(V)) = 0$. We define the **essential range** of f to be the complement of V, say that f is **essentially bounded** if its essential range is compact, and then define its essential supremum $\|f\|_\infty$ to be the supremum of $|\lambda|$ for λ in the essential range of f. Furthermore we identify two essentially bounded functions f and g if $\|f-g\|_\infty = 0$ and call the corresponding space $L^\infty(P)$.

Every element of $L^\infty(P)$ can be approximated in the $\| \ \|_\infty$ norm by simple functions, and hence the integral

$$O(f) = \int_{\mathcal{M}} f dp$$

is defined as the strong limit of the integrals of the corresponding simple functions. The map $f \mapsto O(f)$ is linear, multiplicative, and satisfies

$$O(\overline{f}) = O(f)^*$$

and

$$\|O(f)\| = \|f\|_\infty$$

as before.

If S is a bounded operator on $\mathbf{H}$ which commutes with all the $O(f)$ then it commutes with all the $P(U) = O(\mathbf{1}_U)$. Conversely, if S commutes with all the $P(U)$ it commutes with all the $O(s)$ for s simple and hence with all the $O(f)$.

Finally, getting back to the projection valued measure we constructed on $\mathrm{Spec}(T)$, we can consider it as projection valued measure on all of $\mathbf{R}$, by simply assigning the operator 0 to any Borel set in the complement of $\mathrm{Spec}(T)$.

Putting it all together we have:

Theorem 9.1 *Let T be a bounded self-adjoint operator on a Hilbert space $\mathbf{H}$. Then there exists a resolution of the identity $U \mapsto P(U)$ defined on $\mathbf{R}$ and a homomorphism O from the space of essentially bounded Borel functions (with respect to P) to the algebra of bounded operators on $\mathbf{H}$ such that*

$$O(f) = \int_{\mathbf{R}} f dP,$$

and, in particular,

$$T = \int_{\mathbf{R}} \lambda dP(\lambda).$$

The projection valued measure P is supported on $\mathrm{Spec}(T)$. *An operator S commutes with all the $P(U)$ if an only if it commutes with T.*

The only assertion we have not proved is the last one. Clearly any operator which commutes with all the $P(U)$ commutes with all the $O(f)$ and hence with T since T is of this form. On the other hand, if S commutes with T, it commutes with $f(T)$ when f is a polynomial, hence with all the $O(f)$ when f is continuous, hence the measures $\mu_{Sx,y}$ and μ_{x,S^*y} are equal, hence

$$(P(U)Sx, y) = (P(U)x, S^*y) = (SP(U)x, y)$$

for all x and y which means that

$$SP(U) = P(U)S$$

for all U.

10 Stone's formula.

Let T be a bounded self-adjoint operator. We know that

$$T = \int_{\mathbf{R}} \lambda dP(\lambda)$$

for some projection valued measure P on $\mathbf{R}$. We also know that every bounded Borel function on $\mathbf{R}$ gives rise to an operator. In particular, if z is a complex number which is not real, the function

$$\lambda \mapsto \frac{1}{\lambda - z}$$

is bounded, and hence corresponds to a bounded operator

$$R(z,T) = \int_{\mathbf{R}} (z-\lambda)^{-1} dP(\lambda).$$

Since

$$(zI - T) = \int_{\mathbf{R}} (z-\lambda) dP(\lambda)$$

and our homomorphism is multiplicative, we have

$$R(z,T) = (zI-T)^{-1}.$$

A conclusion of the above argument is that this inverse does indeed exist for all non-real z. The operator (valued function) $R(z,T)$ is called the **resolvent** of T.

Stone's formula turns the table around and gives an expression for the projection valued measure in terms of the resolvent. It says that for any real numbers $a < b$ we have

$$\text{s-}\lim_{\epsilon \to 0} \frac{1}{2\pi i} \int_a^b \left[R(\lambda - i\epsilon, T) - R(\lambda + i\epsilon, T)\right] d\lambda = \frac{1}{2} \left(P((a,b)) + P([a,b])\right). \quad (10)$$

Although this formula cries out for a "complex variables" proof, and I plan to give one later, we can give a direct "real variables" proof in terms of what we already know. Indeed, let

$$f_\epsilon(x) := \frac{1}{2\pi i} \int_a^b \left(\frac{1}{x - \lambda - i\epsilon} - \frac{1}{x - \lambda + i\epsilon} \right) d\lambda.$$

We have

$$f_\epsilon(x) = \frac{1}{\pi} \int_a^b \frac{\epsilon}{(x-\lambda)^2 + \epsilon^2} d\lambda = \frac{1}{\pi} \left(\arctan\left[\frac{x-a}{\epsilon}\right] - \arctan\left[\frac{x-b}{\epsilon}\right] \right).$$

The expression on the right is uniformly bounded, and approaches zero if $x \notin [a,b]$, approaches $\frac{1}{2}$ if $x = a$ or $x = b$, and approaches 1 if $x \in (a,b)$. In short,

$$\lim_{\epsilon \to 0} f_\epsilon(x) = \frac{1}{2}(\mathbf{1}_{(a,b)} + \mathbf{1}_{[a,b]}).$$

We may apply the dominated convergence theorem to conclude Stone's formula. QED