Summary of Lectures 1 and 2.

Math 212a

Sept. 13 and 20, 2001

- Because mathematicians at the beginning of the 19th century were unhappy with formulas such as Euler's famous formula

$$1 + 2 + 3 + 4 + \cdots = -\frac{1}{12}$$

they were concerned about how to interpret Fourier's claim that

$$f(x) = \sum a_n e^{inx}, \quad a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx$$

holds for "any" periodic function. For example for the "square wave"

$$s(x) := \begin{cases} -1, & -\pi < t < 0, \\ 1, & 0 < t < \pi \end{cases}$$

the series is

$$\frac{4}{\pi}\sum_{n\geq 1}^{\infty}\frac{\sin(2n-1)t}{2n-1}.$$

How could this series converge when the series of its coefficients

$$\frac{4}{\pi}\left(1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots\right)$$

diverges? This led to rethinking of what the concepts of "function" and "integral" really mean.

- After some false starts by Poisson, Cauchy, and others, Dirichlet proved in 1829 that the Fourier series of any piecewise differentiable function converges at all points, to $f(x)$ at points of continuity of f, and to the average of the limiting values, $\frac{1}{2}[f(x+) + f(x_)]$ at points of discontinuity. Thus the Fourier series of the square converges at all points.

- However at points of discontinuity we have the *Gibbs Phenomenon*: There is a blip with height about 9% of the jump which never disappears in the approximations, the blip just gets narrower and narrower. See the notes on this subject.

- Fejer proved in 1900 that the Cesaro means of the Fourier series of a piecewise continuous function always converge. See *Fejer.* Any continuous function on $\mathbf{T}$ can be uniformly approximated by trigonometrical series.

- As a consequence any continuous function on a closed bounded interval can be uniformly approximated by polynomials. This is the Weierstrass approximation theorem.

- A subset $A \subset \mathbf{R}$ is said to have measure zero if, for every $\epsilon > 0$ we can find countable many intervals I_i such that

  $$A \subset \bigcup I_i \quad \text{and} \sum |I_i| < \epsilon$$

  where $|I|$ denotes the length of the interval I.

- The union of a countable collection of sets of measure zero has measure zero. In particular, the set of rational numbers has measure zero.

- Although Fejer's theorem gives one interpretation of Fourier's claim that every periodic function can be expanded into a Fourier series, an entirely different approach to this assertion derives from the work of the astronomer Bessel, an interpretation derived from the method of "least squares". The modern (i.e. mid-20th century) setting for this approach is the concept of a Hilbert space and a pre-HIlbert space.

- **Completion.** Any metric space X can be completed to yield a complete metric space. More precisely, there is a metric space $X_{complete}$, unique up to isometry, which is complete, and a isometry $\phi : X \to X_{complete}$ such that $\phi(x)$ is dense in $X_{complete}$. See the notes on metric spaces for details. If X is a normed vector space, then $X_{complete}$ inherits the structure of a normed vector space. A pre-Hilbert space is characterized among all normed spaces by the fact that it satisfies the parallelogram law. This is the theorem of Jordan and von Neumann (see below). Hence, if X is a pre-Hilbert space, then $X_{complete}$ is a Hilbert space.

- **The Cauchy-Schwartz inequality.** This says that

  $$|(f, g)| \leq (f, f)^{\frac{1}{2}}(g, g)^{\frac{1}{2}}$$

  for any elements f, g in a space with a positive semi-definite scalar product. See the notes on Hilbert Space for details.

- **The Pythagorean theorem.** This says that in a pre-Hilbert space

  $$||f + g||^2 = ||f||^2 + ||g||^2 \qquad \text{if } (f, g) = 0.$$

  See the notes on Hilbert space for details.

- **The theorem of Apollonius, also known as the parallelogram law.** This says that in a pre-Hilbert space

$$\|f+g\|^2 + \|f-g\|^2 = 2\left(\|f\|^2 + \|g\|^2\right).$$

  See the notes on HIlbert space for details. The theorem of Jordan and von Neumann says that conversely, if a norm on a vector space satisfies this identity, then the norm derives from a scalar product.