Summary of Lecture 10, Lebesgue Integration.

Math 212a

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Contents

1 Real valued measurable functions. **1**

2 The integral of a non-negative function. **2**

3 Fatou's lemma. **3**

4 The monotone convergence theorem. **3**

5 The space $\mathcal{L}_1(X, \mathbf{R})$. **4**

6 The dominated convergence theorem. **4**

7 The Beppo-Levi theorem. **4**

8 $\mathcal{L}_1$ is complete. **4**

9 The Cantor-Lebesgue theorem. **5**

10 Fubini's theorem. **5**

1 Real valued measurable functions.

$(X, \mathcal{F}, m)$ is a space with a σ-field of sets, and m a measure on $\mathcal{F}$.

If $(X, \mathcal{F})$ and $(Y, \mathcal{G})$ are spaces with σ-fields, then

$$f : X \to Y$$

is called measurable if

$$f^{-1}(E) \in \mathcal{F} \quad \forall\, E \in \mathcal{G}. \tag{1}$$

Notice that the collection of subsets of Y for which (1) holds is a σ-field, and hence if it holds for some collection $\mathcal{C}$, it holds for the σ-field generated by $\mathcal{C}$. In what follows we will take $Y = \mathbf{R}$ and $\mathcal{G} = \mathcal{B}$, the Borel field. Since the collection of open intervals on the line generate the Borel field, a real valued function $f : X \to \mathbf{R}$ is measurable if and only if

$$f^{-1}(I) \in \mathcal{F} \quad \text{for all open intervals } I.$$

If f and g are measurable real valued functions then

- $f + g$ is measurable (since $(x, y) \mapsto x + y$ is continuous),
- fg is measurable (since $(x, y) \mapsto xy$ is continuous), hence
- $f\mathbf{1}_A$ is measurable for any $A \in \mathcal{F}$ hence
- f^+ is measurable since $f^{-1}([0, \infty]) \in \mathcal{F}$ and similarly for f^- so
- $|f|$ is measurable and so is $|f - g|$. Hence
- $f \wedge g$ and $f \vee g$ are measurable

and so on.

2 The integral of a non-negative function.

The purpose of the lecture is to develop the theory of the Lebesgue integral for functions defined on X. The theory starts with **simple** functions, that is functions which take on only finitely many non-zero values, say $\{a_1, \ldots, a_n\}$ and where

$$A_i := f^{-1}(a_i) \in \mathcal{F}.$$

In other words, we start with functions of the form

$$\phi(x) = \sum_{i=1}^{n} a_i \mathbf{1}_{A_i} \qquad A_i \in \mathcal{F}. \tag{2}$$

Then, for any $E \in \mathcal{F}$ we would like to define the integral of a simple function ϕ over E as

$$\int_E \phi dm = \sum_{i=1}^{n} a_i m(A_i \cap E) \tag{3}$$

and extend this definition by some sort of limiting process to a broader class of functions.

We extend the definition to an arbitrary ($[0, \infty]$ valued) function f by

$$\int_E f dm = \sup I(E, f) \tag{4}$$

where

$$I(E, f) = \left\{ \int_E \phi dm : 0 \leq \phi \leq f, \ \phi \text{ simple} \right\}. \tag{5}$$

In other words, we take all integrals of expressions of simple functions ϕ such that $\phi(x) \leq f(x)$ at all x. We then define the integral of f as the supremum of these values. This definition is consistent with the earlier one on simple functions and

$$f \leq g \quad \Rightarrow \quad \int_E f dm \leq \int_E g dm. \tag{6}$$

$$\int_E f dm \quad = \quad \int_X \mathbf{1}_E f dm \tag{7}$$

$$E \subset F \quad \Rightarrow \quad \int_E f dm \leq \int_F f dm. \tag{8}$$

$$\int_E a f dm \quad = \quad a \int_E f dm. \tag{9}$$

$$m(E) = 0 \quad \Rightarrow \quad \int_E f dm = 0. \tag{10}$$

$$E \cap F = \emptyset \quad \Rightarrow \quad \int_{E \cup F} f dm = \int_E f dm + \int_F f dm. \tag{11}$$

$$f = 0 \;\; \text{a.e.} \quad \Leftrightarrow \quad \int_X f dm = 0. \tag{12}$$

$$f \leq g \;\; \text{a.e.} \quad \Rightarrow \quad \int_X f dm \leq \int_X g dm. \tag{13}$$

3 Fatou's lemma.

This says:

Theorem 1 *If $\{f_n\}$ is a sequence of non-negative functions, then*

$$\lim_{n \to \infty} \inf_{k \geq n} \int f_k dm \geq \int \left(\lim_{n \to \infty} \inf_{k \geq n} f_k \right) dm. \tag{14}$$

4 The monotone convergence theorem.

We assume that $\{f_n\}$ is a sequence of non-negative measurable functions, and that $f_n(x)$ is an increasing sequence for each x. Define $f(x)$ to be the limit (possibly $+\infty$) of this sequence. We describe this situation by $f_n \nearrow f$. The monotone convergence theorem asserts that:

$$f_n \geq 0, \quad f_n \nearrow f \;\; \text{a.e.} \quad \Rightarrow \quad \lim_{n \to \infty} \int f_n dm = \int f dm. \tag{15}$$

5 The space $\mathcal{L}_1(X, \mathbf{R})$.

We will say an $\mathbf{R}$ valued measurable function is **integrable** if both $\int f^+ dm < \infty$ and $\int f^- dm < \infty$. If this happens, we set

$$\int f dm := \int f^+ dm - \int f^- dm. \tag{16}$$

We will denote the set of all (real valued) integrable functions by $\mathcal{L}_1$ or $\mathcal{L}_1(X)$ or $\mathcal{L}_1(X, \mathbf{R})$.

Theorem 2 *The space $\mathcal{L}_1(X, \mathbf{R})$ is a real vector space and $f \mapsto \int f dm$ is a linear function on $\mathcal{L}_1(X, \mathbf{R})$.*

We also have

Proposition 1 *If $h \in \mathcal{L}_1$ and $\int_A h dm \geq 0$ for all $A \in \mathcal{F}$ then $h \geq 0$ a.e.*

6 The dominated convergence theorem.

This says that

Theorem 3 *Let f_n be a sequence of measurable functions such that*

$$|f_n| \leq g \;\; a.e., \quad g \in \mathcal{L}_1$$

$$f_n \to f \quad a.e. \;\; \Rightarrow \;\; f \in \mathcal{L}_1 \text{ and } \int f_n dm \to \int f dm.$$

7 The Beppo-Levi theorem.

Theorem 4 Beppo-Levi. *Let $f_n \in \mathcal{L}_1$ and suppose that*

$$\sum_{k=1}^{\infty} \int |f_k| dm < \infty.$$

Then $\sum f_k(x)$ converges to a finite limit for almost all x, the sum is integrable, and

$$\int \sum_{k=1}^{\infty} f_k \; dm = \sum_{k=1}^{\infty} \int f_k dm.$$

8 $\mathcal{L}_1$ is complete.

This is an immediate corollary of the Beppo-Levi theorem and Fatou's lemma.

Proposition 2 *The continuous functions of compact support are dense in $\mathcal{L}_1(\mathbf{R}, \mathbf{R})$.*

sectionThe Riemann-Lebesgue Lemma. Let h be a bounded measurable function on $\mathbf{R}$. We say that h satisfies the **averaging condition** if

$$\lim_{|c|\to\infty} \frac{1}{|c|} \int_0^c h dm \to 0. \tag{17}$$

Theorem 5 **[Generalized Riemann-Lebesgue Lemma].**
Let $f \in \mathcal{L}_1([c,d],\mathbf{R})$, $-\infty \le c < d \le \infty$. If h satisfies the averaging condition (17) then

$$\lim_{r\to\infty} \int_c^d f(t)h(rt)dt = 0. \tag{18}$$

9 The Cantor-Lebesgue theorem.

This says:

Theorem 6 *If a trigonometric series*

$$\frac{a_0}{2} + \sum_n d_n \cos(nt - \phi_n) \qquad d_n \in \mathbf{R}$$

converges on a set E of positive Lebesgue measure then

$$d_n \to 0.$$

10 Fubini's theorem.

Let $(X,\mathcal{F})$ and $(Y,\mathcal{G})$ be spaces with σ-fields. On $X \times Y$ we can consider the collection $\mathcal{P}$ of all sets of the form

$$A \times B, \quad A \in \mathcal{F},\ B \in \mathcal{G}.$$

The σ-field generated by $\mathcal{P}$ will, by abuse of language, be denoted by

$$\mathcal{F} \times \mathcal{G}.$$

If E is any subset of $X \times Y$, by an even more serious abuse of language we will let

$$E_x := \{y|(x,y) \in E\}$$

and (contradictorily) we will let

$$E_y := \{x|(x,y) \in E\}.$$

The set E_x will be called the x-**section** of E and the set E_y will be called the y-section of E.

Theorem 7 *Let* $(X, \mathcal{F}, m)$ *and* $(Y, \mathcal{G}, n)$ *be measure spaces with* $m(X) < \infty$ *and* $n(Y) < \infty$. *There exists a unique measure on* $\mathcal{F} \times \mathcal{G}$ *with the property that*

$$(m \times n)(A \times B) = m(A)n(B) \quad \forall\ A \times B \in \mathcal{P}.$$

For any bounded $\mathcal{F} \times \mathcal{G}$ *measurable function, the double integral is equal to the iterated integral in the sense that*

$$\int_{X \times Y} f(x,y)(m \times n) = \int_X \left(\int_Y f(x,y) n(dy) \right) m(dx) = \int_Y \left(\int_X f(x,y) m(dx) \right) n(dy).$$