Summary of Lecture 13, The Riesz representation theorem.

Math 212a

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The purpose of this lecture is to give an alternative proof of the Riesz representation theorem which yields more information about the measure, information which will be used in the construction of Wiener measure. The details and proofs are found in the handout *Riesz*.

Contents

1 Regular Borel measures.

Let X be a locally compact Hausdorff space. Recall that $\mathcal{B}(X)$ is the smallest σ-field which contains the open sets.

Let $\mathcal{F}$ be a σ-field which contains $\mathcal{B}(X)$. A (non-negative valued) measure μ on $\mathcal{F}$ is called **regular** if

1. $\mu(K) < \infty$ for any compact subset $K \subset X$.
2. For any $A \in \mathcal{F}$
$$\mu(A) = \inf\{\mu(U) : \ A \subset U, \ U \text{ open}\}$$
3. If $U \subset X$ is open then
$$\mu(U) = \sup\{\mu(K) : K \subset U, \quad K \text{ compact }\}.$$

The second condition is called **outer regularity** and the third condition is called **inner regularity**.

2 Statement of the theorem.

Here is the improved version of the Riesz representation theorem:

Theorem 1 *Let X be a locally compact Hausdorff space, L the space of functions of compact support on X, and I a non-negative linear functional on L. Then there exists a σ-field $\mathcal{F}$ containing $\mathcal{B}(X)$ and a non-negative regular measure μ on $\mathcal{F}$ such that*

$$I(f) = \int f d\mu \tag{1}$$

for all $f \in L$. Furthermore, the restriction of μ to $\mathcal{B}(X)$ is unique.