Summary of Lec.3

Math 212a

Sept. 25, 2001

- **Completion.** Any metric space X can be completed to yield a complete metric space. More precisely, there is a metric space $X_{complete}$, unique up to isometry, which is complete, and a isometry $\phi : X \to X_{complete}$ such that $\phi(x)$ is dense in $X_{complete}$. See the notes on metric spaces for details. If X is a normed vector space, then $X_{complete}$ inherits the structure of a normed vector space. A pre-Hilbert space is characterized among all normed spaces by the fact that it satisfies the parallelogram law (see below). Hence, if X is a pre-Hilbert space, then $X_{complete}$ is a Hilbert space.

- **The Cauchy-Schwartz inequality.** This says that

$$|(f,g)| \leq (f,f)^{\frac{1}{2}}(g,g)^{\frac{1}{2}}$$

  for any elements f, g in a space with a positive semi-definite scalar product. See the *Hilbertspace* notes details.

- **The Pythagorean theorem.** This says that in a pre-Hilbert space

$$\|f+g\|^2 = \|f\|^2 + \|g\|^2 \qquad \text{if } (f,g) = 0.$$

  See *hilbertspace* in in the folder for Lecture 1 for details.

- **The parallelogram law.** This says that in a pre-Hilbert space

$$\|f+g\|^2 + \|f-g\|^2 = 2\left(\|f\|^2 + \|g\|^2\right).$$

  See the *Hilbertspace* notes for details.

- **Orthogonal projection.** If M is a complete subspace of a pre- Hilbert space H, then for any $v \in H$ there is a unique $w \in M$ such that $(v-w) \perp M$. This w is characterized as being the unique element of M which minimizes $\|v-x\|, x \in M$. The idea of the proof is to use the parallelogram law to conclude that if $\{x_n\}$ is a sequence of elements in M for which $\|v-x_m\|$ approaches the greatest lower bound of $\|v-x\|$, $x \in M$, then $\{x_m\}$ is a Cauchy sequence. Then the assumption that M is complete guarantees that this sequence has a limit $w \in M$ which minimizes $\|v-x\|, x \in M$. See the *Hilbertspace* notes for details. The unique $w \in M$ so obtained is denoted by $\pi_M(v)$.

- **The Riesz representation theorem.** This says that any bounded linear function on a Hilbert space H is given by scalar product with an element of H. That is, every bounded linear function ℓ is given by

$$f \mapsto (f, g)$$

where g is a (unique) element of H. If the linear function is identically zero, we may take $g = 0$. Otherwise, the codimension of the kernel of ℓ, the space of all f such that $\ell(f) = 0$, has codimension one. Let N denote this kernel which is a closed (hence complete) subspace. Via orthogonal projection, there is a $y \in H$ with $y \perp N$ and $\|y\| = 1$. So $(f - (f, y)y) \perp y$, and the set of all elements perpendicular to y is a subspace of codimension 1 which contains N, and so must coincide with N. Thus $(f - (f, y)y) \in N$. The decomposition

$$f = (f, y)y + (f - (f, y)y)$$

shows that $\ell(f) = (f, y)\ell(y)$ so

$$\ell(f) = (f, g) \quad \text{where } g = \overline{\ell(y)}y.$$

- **Description of $L_2(\mathbf{T})$ as a space of linear functions.** $L_2(\mathbf{T})$ is defined as the completion of the pre-Hilbert space $\mathcal{C}(\mathbf{T})$, the space of continuous functions on the circle $\mathbf{T}$ under the scalar product

$$(f, g) = \frac{1}{2\pi} \int_{\mathbf{T}} f(t)\overline{g(t)}dt.$$

Any bounded linear function on this pre-Hilbert space extends uniquely to a bounded linear function on its completion, $L_2(\mathbf{T})$. Hence by the Riesz representation theorem is given by scalar product with an element of $L_2(\mathbf{T})$. In other words, we should regard an element of $L_2(\mathbf{T})$ not as a function on the circle, but as a linear function on the space of continuous functions on the circle which is continuous relative to the norm $\| \cdot \|_2$.

- **Bessel's inequality** asserts that if $\{\phi_i\}$ is an orthonormal set of vectors and v an arbitrary vector in a Hilbert space, then

$$\sum |(v, \phi_i|^2 leq \|v\|^2.$$

We get equality for all v in the above equation if and only if the set of finite linear combinations of the ϕ_i is dense. We know this to be true for the orthonormal set $\{e^{ikx}\}$ in $L_2(\mathbf{T})$ by Fejer's theorem.

- **Self adjoint operators.** These are the ones which satisfy $(Tu, v) = (u, Tv)$ for all u, v in the Hilbert space. In particular (Tv, v) is real and any eigenvalue of T is real.

- **Non-negative operators** are those self adjoint operators which satisfy $(Tv, v) \geq 0$ for all v. If T is a non-negative operator, then

$$\|Tv\| \leq \|T\|^{\frac{1}{2}}(Tv, v)^{\frac{1}{2}}$$

for all v. In particular $(Tv_n, v_n) \to 0$ implies that $Tv_n \to 0$. For any self adjoint operator T, the operator $\|T\|I - T$ is non-negative.

- **Compact operators**. These satisfy the following condition: If v_n is any sequence of vectors with $\|v_n\| = 1$ for all n, then we can find a subsequence such that Tv_{n_j} converges. The main theorem about compact self adjoint operators is:

Let T be a compact self-adjoint operator. Then $R(T)$ has an orthonormal basis $\{\phi_i\}$ consisting of eigenvectors of T and if $R(T)$ is infinite dimensional then the corresponding sequence $\{r_n\}$ of eigenvalues converges to 0.

The idea of the proof is to look at a sequence of unit vectors $\{u_n\}$ such that $\|Tu_n\|$ approaches the maximum value $m_1 := \|T\|$ of $\|Tu\|$ for u on the unit sphere, and choose a subsequence such that Tu_{n_j} converges. Then one shows that the limit w is an eigenvector of T^2 with eigenvalue $\|T\|^2$. Either w is an eigenvector of T with eigenvalue m_1 or $(T - m_1)w$ is an eigenvector of T with eigenvalue $-m_1$. One then aplies the procedure over again to the subspace $w^\perp$ and repeats. Straightforward arguments show that the eigenvalues so obtained tend to zero and that the corresponding normalized eigenvectors form a basis of $R(T)$, See the *Hilbertspace* notes for details.