Summary of Lectures 11 and 12, The Daniell integral.

Math 212a

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Daniell's idea was to take the axiomatic properties of the integral as the starting point and develop integration for broader and broader classes of functions. Then derive measure theory as a consequence. Much of the presentation here is taken from the book *Abstract Harmonic Analysis* by Lynn Loomis. Some of the lemmas, propositions and theorems indicate the corresponding sections in Loomis's book. The details can be found in *Daniellintegration*

Contents

1 The Daniell Integral

Let L be a vector space of *bounded* real valued functions on a set S closed under $\wedge$ and $\vee$. For example, S might be a complete metric space, and L might be the space of continuous functions of compact support on S.

A map

$$I : L \to \mathbf{R}$$

is called an **Integral** if

1. I is linear: $I(af + bg) = aI(f) + bI(g)$
2. I is non-negative: $f \geq 0 \Rightarrow I(f) \geq 0$ or equivalently $f \geq g \Rightarrow I(f) \geq I(g)$.
3. $f_n \searrow 0 \Rightarrow I(f_n) \searrow 0$.

Define

$$U := \{\text{limits of monotone non-decreasing sequences of elements of } L\}.$$

Extend I to U by setting

$$I(f) := \lim I(f_n) \quad \text{for} \quad f_n \nearrow f.$$

This extension is well defined, and if $f \in L$, this coincides with our original I.

Define

$$-U := \{-f|\ f \in U\}$$

and

$$I(f) := -I(-f) \qquad f \in -U.$$

If $f \in U$ and $-f \in U$ then $I(f)+I(-f) = I(f-f) = I(0) = 0$ so $I(-f) = -I(f)$ in this case. So the definition is consistent.

A function f is called ***I*-summable** if for every $\epsilon > 0$, $\exists\ g \in -U,\ h \in U$ with

$$g \leq f \leq h, \quad |I(g)| < \infty,\ |I(h)| < \infty \ \text{ and } I(h - g) \leq \epsilon.$$

For such f define

$$I(f) = \text{glb } I(h) = \text{lub } I(g).$$

The space of summable functions is denoted by $\overline{L}_1$. It is a vector space, and I satisfies conditions 1) and 2) above, i.e. is linear and non-negative.

2 The monotone convergence theorem.

Theorem 1 *$f_n \in \overline{L}_1$, $f_n \nearrow f$ and $\lim I(f_n) < \infty \ \Rightarrow f \in \overline{L}_1$ and $I(f) = \lim I(f_n)$.*

3 Measure.

A set A **integrable** if $\mathbf{1}_A \in \mathcal{B}$, where $\mathcal{B}$ is defined to be the smallest monotone class containing L.

The monotone class properties of $\mathcal{B}$ imply that the integrable sets form a σ-field. Then define

$$\mu(A) := \int \mathbf{1}_A$$

and the monotone convergence theorem guarantees that μ is a measure.

Theorem 2 *If $f \in \mathcal{B}$ then $f \in L^1 \Leftrightarrow \exists g \in L^1$ with $|f| \leq g$.*

Add **Stone's axiom**

$$f \in L \Rightarrow f \wedge \mathbf{1} \in L.$$

Then the monotone class property implies that this is true with L replaced by $\mathcal{B}$.

Theorem 3 *$f \in \mathcal{B}$ and $a > 0 \Rightarrow$ then*

$$A_a := \{p | f(p) > a\}$$

is an integrable set If $f \in L^1$ then

$$\mu(A_a) < \infty.$$

If $f \geq 0$ and A_a is integrable for all $a > 0$ then $f \in \mathcal{B}$. Integration with respect to the measure μ is the same as I.

sectionHölder, Minkowski , L^p and L^q. The numbers $p, q > 1$ are called **conjugate** if

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Let L^p denote the space of functions such that $|f|^p \in L^1$. For $f \in L^p$ define

$$\|f\|_p := \left(\int |f|^p d\mu\right)^{\frac{1}{p}}.$$

If $f \in L^p$ and $g \in L^q$ we have Hölder's inequality

$$\left|\int fg d\mu\right| \leq \|f\|_p \|g\|_q \tag{1}$$

We also have

Proposition 1 [**Minkowski's inequality**] *If $f, g \in L^p$, $p \geq 1$ then $f + g \in L^p$ and*

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p.$$

Theorem 4 *L^p is complete and L is dense in L^p for any $1 \leq p < \infty$.*

4 $\|\cdot\|_\infty$ is the essential sup norm.

Theorem 5 *If $f \in L^p$ for some $p > 0$ then*

$$\|f\|_\infty = \lim_{q \to \infty} \|f\|_q. \tag{2}$$

5 The Radon-Nikodym Theorem.

Suppose we are given two integrals, I and J on the same space L. That is, both I and J satisfy the three conditions of linearity, positivity, and the monotone limit property that went into our definition of the term "integral". We say that J is **absolutely continuous** with respect to I if every set which is I null (i.e. has measure zero with respect to the measure associated to I) is J null.

The integral I is said to be **bounded** if

$$I(\mathbf{1}) < \infty,$$

or, what amounts to the same thing, that

$$\mu_I(S) < \infty$$

where μ_I is the measure associated to I.

Theorem 6 [**Radon-Nikodym**] *Let I and J be bounded integrals, and suppose that J is absolutely continuous with respect to I. Then there exists an element $f_0 \in \mathcal{L}^1(I)$ such that*

$$J(f) = I(ff_0) \quad \forall\ f \in \mathcal{L}^1(J). \tag{3}$$

The element f_0 is unique up to equality almost everywhere (with respect to μ_I).

6 The variations of a bounded functional.

7 Duality of L^p and L^q.

Theorem 7 *Suppose that $\mu(S) < \infty$ and that F is a bounded linear function on L^p with $1 \le p < \infty$. Then there exists a unique $g \in L^q$ such that*

$$F(f) = (f, g) = I(fg).$$

Here $q = p/(p-1)$ if $p > 1$ and $q = \infty$ if $p = 1$.

8 Integration on locally compact Hausdorff spaces.

Suppose that S is a locally compact Hausdorff space. As in the case of $\mathbf{R}^n$, we can (and will) take L to be the space of continuous functions of compact support. Dini's lemma then says that if $f_n \in L \searrow 0$ then $f_n \to 0$ in the uniform topology.

If A is any subset of S we will denote the set of $f \in L$ whose support is contained in A by L_A.

Lemma 1 *A non-negative linear function I is bounded in the uniform norm on L_C whenever C is compact.*

Proof. Choose $g \geq 0 \in L$ so that $g(x) \geq 1$ for $x \in C$. If $f \in L_C$ then

$$|f| \leq \|f\|_\infty g$$

so

$$|I(f)| \leq I(|f|) \leq I(g) \cdot \|f\|_\infty. \text{ QED.}$$

9 Riesz representation theorems.

Theorem 8 *Every non-negative linear functional I on L is an integral.*

Proof. This is Dini's lemma together with the preceding lemma. Indeed, by Dini we know that $f_n \in L \searrow 0$ implies that $\|f_n\|_\infty \searrow 0$. Since f_1 has compact support, let C be its support, a compact set. All the succeeding f_n are then also supported in C and so by the preceding lemma $I(f_n) \searrow 0$. QED

Theorem 9 *Let F be a bounded linear function on L (with respect to the uniform norm). Then there are two integrals I^+ and I^- such that*

$$F(f) = I^+(f) - I^-(f).$$

10 Fubini's theorem.

Theorem 10 *Let S_1 and S_2 be locally compact Hausdorff spaces and let I and J be non-negative linear functionals on $L(S_1)$ and $L(S_2)$ respectively. Then*

$$I_x(J_y h(x,y)) = J_y(I_x(h(x,y))$$

for every $h \in L(S_1 \times S_2)$ in the obvious notation, and this common value is an integral on $L(S_1 \times S_2)$.

Proof via Stone-Weierstrass. The equation in the theorem is clearly true if $h(x,y) = f(x)g(y)$ where $f \in L(S_1)$ and $g \in L(S_2)$ and so it is true for any h which can be written as a finite sum of such functions. Let h be a general element of $L(S_1 \times S_2)$. then we can find compact subsets $C_1 \subset S_1$ and $C_2 \subset S_2$ such that h is supported in the compact set $C_1 \times C_2$. The functions of the form

$$\sum f_i(x)g_i(y)$$

where the f_i are all supported in C_1 and the g_i in C_2, and the sum is finite, form an algebra which separates points. So for any $\epsilon > 0$ we can find a k of the above form with

$$\|h - k\|_\infty < \epsilon.$$

Let B_1 and B_2 be bounds for I on $L(C_1$ and J on $L(C_2$ as provided by Lemma 1. Then

$$|J_y h(x,y) - \sum J(g_i)f_i(x)| = |J_y(f-k)| < \epsilon B_2.$$

This shows that $J_y h(x,y)$ is the uniform limit of continuous functions supported in C_1 and so $J_y h(x,y)$ is itself continuous and supported in C_1. It then follows that $I_x(J_y(h)$ is defined, and that

$$|I_x(J_y h(x,y) - \sum I(f)_i J(g_i)| \leq \epsilon B_1 B_2.$$

Doing things in the reverse order shows that

$$I_x(J_y h(x,y)) - J_y(I_x(h(x,y))| \leq 2\epsilon B_1 B_2.$$

Since ϵ is arbitrary, this gives the equality in the theorem. Since this (same) functional is non-negative, it is an integral by the first of the Riesz representation theorems above. QED