Summary of Lectures 4 and 5.

Math 212a

October 4 and 11, 2001

The details of all the material presented here can be found in the notes on Hilbert Spaces and Compact Operators.

- **The Sobolev Spaces on the torus.** $\mathbf{T}$ stands for the n-dimensional torus. Let $\mathbf{P} = \mathbf{P}(\mathbf{T})$ denote the space of trigonometric polynomials. These are functions on the torus of the form

  $$u(x) = \sum a_\ell e^{i\ell\cdot x}$$

  where

  $$\ell = (\ell_1, \dots, \ell_n)$$

  is an n-tuplet of integers and the sum is finite. For each integer t (positive, zero or negative) we introduce the scalar product

  $$(u, v)_t := \sum_\ell (1 + \ell \cdot \ell)^t a_\ell \overline{b}_\ell. \tag{1}$$

  For $t = 0$ this is the scalar product

  $$(u, v)_0 = \frac{1}{(2\pi)^n} \int_{\mathbf{T}} u(x)\overline{v(x)}dx.$$

  (Contrary to what I did in lecture, I have decided to revert to our original notation, so this choice differs by a factor of $(2\pi)^{-n}$ from the scalar product that is used by Bers and Schecter.) The norm corresponding to the scalar product $(\ ,\ )_s$ is denoted by $\|\ \|_s$.

- We let

  $$\Delta := -\left(\frac{\partial^2}{\partial(x^1)^2} + \cdots + \frac{\partial^2}{\partial(x^n)^2}\right).$$

  Then

  $$\|(1+\Delta)^t u\|_s = \|u\|_{s+2t}. \tag{2}$$

- The "generalized Cauchy-Schwartz inequality"

  $$|(u, v)_s| \leq \|u\|_{s+t}\|v\|_{s-t} \tag{3}$$

  holds.

- The norms

$$u \mapsto \|u\|_t$$

  $t \geq 0$ and

$$u \mapsto \sum_{|p| \leq t} \|D^p\|_0$$

  are equivalent.

- $\mathbf{H}_t$ denotes the completion of the space $\mathbf{P}$ with respect to the norm $\|\ \|_t$. Each $\mathbf{H}_t$ is a Hilbert space, and we have natural embeddings

$$\mathbf{H}_t \hookrightarrow \mathbf{H}_s \quad \text{if } s < t.$$

  Equation (2) says that

$$(1+\Delta)^t : \mathbf{H}_{s+2t} \to \mathbf{H}_s$$

  and is an isometry.

- We have a natural pairing of $\mathbf{H}_t$ with $\mathbf{H}_{-t}$ given by the extension of $(\ ,\ )_0$, so

$$|(u,v)_0| \leq \|u\|_t \|v\|_{-t}. \tag{4}$$

  This pairing allows us to identify $\mathbf{H}_{-t}$ with the space of continuous linear functions on $\mathbf{H}_t$.

- **The delta function.**the series

$$\sum_\ell (1 + \ell \cdot \ell)^s$$

  converges for

$$s < -\frac{n}{2}.$$

  This means that if define v by taking

$$b_\ell \equiv 1$$

  then $v \in \mathbf{H}_s$ for $s < -\frac{n}{2}$. If u is given by $u(x) = \sum_\ell a_\ell e^{i\ell \cdot x}$ is any element of $\mathbf{H}_s$ for $s > n/2$ this functional is defined at u and sends $u \mapsto u(0)$.

- **Sobolev's lemma.** If $u \in \mathbf{H}_t$ and

$$t \geq \left[\frac{n}{2}\right] + k + 1$$

  then $u \in C^k(\mathbf{T})$ and

$$\sup_{x \in \mathbf{T}} |D^p u(x)| \leq \text{const.} \|u\|_t \quad \text{for } |p| \leq k. \tag{5}$$

- **Distributions, or generalized functions.** A **distribution** on $\mathbf{T}^n$ is a linear function T on $C^\infty(\mathbf{T}^n)$ with the continuity condition that

$$\langle T, \phi_k \rangle \to 0$$

whenever

$$D^p \phi_k \to 0$$

uniformly for each fixed p.

- $$\mathbf{H}_\infty := \bigcap \mathbf{H}_t, \qquad \mathbf{H}_{-\infty} := \bigcup \mathbf{H}_t.$$

- **Schwartz's theorem.** This says that $\mathbf{H}_\infty$ is the space of all distributions. In other words, any distribution belongs to $\mathbf{H}_{-t}$ for some t.

- **Rellich's lemma.** If $s < t$ the embedding $\mathbf{H_t} \hookrightarrow \mathbf{H}_s$ is compact.

- **Differential operators.** Let

$$L = \sum_{|p| \le m} \alpha_p(x) D^p$$

be a differential operator of degree m with C^∞ coefficients. Then

$$\|Lu\|_{t-m} \le \text{constant} \|u\|_t \tag{6}$$

where the constant depends on L and t.

All of the above were results of a general nature, describing the Sobolev spaces and relations among them and between them and the theory of distributions. The remaining results consisted of the proof that if L is an elliptic differential operator of degree m, then for λ sufficiently large, the operator

$$L + \lambda I : \quad \mathbf{H}_m \to \mathbf{H_0}$$

is surjective with bounded inverse. Combined with Rellich's lemma, this implies that

$$\iota_m \circ (L + \lambda I)^{-1} : \mathbf{H}_0 \to \mathbf{H}_0$$

is compact, where ι_m denotes the injection of $\mathbf{H}_m$ into $\mathbf{H}_0$. If in addition $L = L^*$ so that the operator $\iota_m \circ (L + \lambda I)^{-1}$ is self adjoint, we can apply the theory of self-adjoint compact operators to conclude that we can find an orthonormal basis of $\mathbf{H}_0$ consisting of eigenvectors of $\iota_m \circ (L + \lambda I)^{-1}$, and hence of eigenvectors of L, and that these eigenvectors are C^∞ functions.

For this we introduce the notations

- If $p = (p_1, dots, p_n)$ is a vector with non-negative integer entries we set

$$|p| := p_1 + \cdots + p_n.$$

- If $\xi = (\xi_1, \ldots, \xi_n)$ is a (row) vector we set

$$\xi^p := \xi_1^{p_1} \cdot \xi_2^{p_2} \cdots \xi_n^{p_n}$$

- A differential operator $L = \sum_{|p| \leq m} \alpha_p(x) D^p$ with real coefficients and m even is called **elliptic** if there is a constant $c > 0$ such that

$$(-1)^{m/2} \sum_{|p|=m} a_p(x) \xi^p \geq c(\xi \cdot \xi|^{m/2}. \tag{7}$$

  In this inequality, the vector ξ is a "dummy variable".

- **Gårding's inequality** says that if L is an elliptic differential operator (of even degree $2m$) then

$$(u, Lu)_0 \geq c_1 \|u\|_{m/2}^2 - c_2 \|u\|_0^2 \tag{8}$$

  where c_1 and c_2 are constants depending on L. The proof is a combination of working with the definition and using integration by parts, together with a general inequality of the form

$$\|u\|_s \leq \epsilon \|u\|_{t_1} + \epsilon^{-(s-t_2)/(t_1-s)} \|u\|_{t_2} \quad \text{if } t_1 > s > t_2, \quad \epsilon > 0 \tag{9}$$

  for all $u \in \mathbf{H}_{t_1}$ when $t_1 > s > t_2$.

- **A priori estimates.** This is the assertion that For every integer t there is a constant $c(t) = c(t, L)$ and a positive number $\Lambda = \Lambda(t, L)$ such that

$$\|u\|_t \leq c(t) \|Lu + \lambda u\|_{t-m} \tag{10}$$

  when

$$\lambda > \Lambda$$

  for all smooth u, and hence for all $u \in \mathbf{H}_t$. The proof is a direct consequence of Gårding's inequality and the generalized Cauchy - Schwartz inequality.

- From the a priori estimates applied to L and to L^* it follows that for every t and for λ large enough (depending on t) the operator $L + \lambda I$ maps $\mathbf{H}_t$ bijectively onto $\mathbf{H}_{t-m}$ and $(L + \lambda I)^{-1}$ is bounded independently of λ.

- This last result is enough to complete our goal.

- **The resolvent.** For any operator A on a Hilbert space, if $zI - A$ is surjective with a bounded inverse, then $z \in \mathbf{C}$ is said to belong to the resolvent set of A and

$$R(z, A) := (zI - A)^{-1}$$

  is called the resolvent of A at the point z. If we take $A = -L$ then the eigenvalues of A tend to $-\infty$ and if z does not equal any of these eigenvalues, the z belongs to the resolvent set. If Re z is $>$ than all the eigenvalues of A we have the formula

$$R(z, A) = \int_0^\infty e^{-zt} e^{tA} dt.$$