Summary of Lectures 8 and 9, Abstract measure theory.

Math 212a

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Contents

1 Outer measures and measures.

An **outer measure** on a set X is a map m^* to $[0, \infty]$ defined on the collection of *all* subsets of X which satisfies

- $m(\emptyset) = 0$,
- **Monotonicity:** If $A \subset B$ then $m^*(A) \leq m^*(B)$, and
- **Countable subadditivity:** $m^*\left(\bigcup_n A_n\right) \leq \sum_n m^*(A_n)$.

An outer measure m determines a σ-field of sets measurable according to he definition of Caratheodory, and the restriction of m to this σ-field is a measure according to the following definition:

Given a σ-field $\mathcal{F}$ a (non-negative) **measure** is a function;

$$m : \mathcal{F} \to [0, \infty]$$

such that

- $m(\emptyset) = 0$ and
- **Countable additivity:** If F_n is a disjoint collection of sets in $\mathcal{F}$ then

$$m\left(\bigcup_n F_n\right) = \sum_n m(F_n).$$

In the countable additivity condition it is understood that both sides might be infinite.

2 Constructing outer measures, Method I.

Let $\mathcal{C}$ be a collection of sets which cover X. For any subset A of X let

$$\mathrm{ccc}(A)$$

denote the set of (finite or) **c**ountable **c**overs of A by sets belonging to $\mathcal{C}$. In other words, an element of $\mathrm{ccc}(A)$ is a finite or countable collection of elements of $\mathcal{C}$ whose union contains A.

Suppose we are given a function

$$\ell : \mathcal{C} \to [0, \infty].$$

Theorem 1 *There exists a unique outer measure m^* on X such that*

- $m^*(A) \leq \ell(A)$ *for all* $A \in \mathcal{C}$ *and*
- *If n^* is any outer measure satisfying the preceding condition then $n^*(A) \leq m^*(A)$ for all subsets A of X.*

This unique outer measure is given by

$$m^*(A) = \inf_{\mathcal{D} \in \mathrm{ccc}(A)} \sum_{D \in \mathcal{D}} \ell(D). \tag{1}$$

In other words, for each countable cover of A by elements of $\mathcal{C}$ we compute the sum above, and then minimize over all such covers of A.

This procedure is very useful, but can lead to pathological examples when applied to a metric space - for example, the Borel sets need not be measurable.

3 Metric outer measures and the Method II construction.

Let X be a metric space. An outer measure on X is called a **metric outer measure** if

$$m^*(A \cup B) = m^*(A) + m^*(B) \quad \text{whenever } d(A, B) > 0. \tag{2}$$

Theorem 2 [**Caratheodory**] *If m^* is a metric outer measure on a metric space X, then all Borel sets of X are m^* measurable.*

The method II construction for metric spaces works as follows: X is a metric space, and we have a cover $\mathcal{C}$ with the property that for every $x \in X$ and every $\epsilon > 0$ there is a $C \in \mathcal{C}$ with $x \in C$ and $\mathrm{diam}(C) < \epsilon$. In other words, we are assuming that the

$$\mathcal{C}_\epsilon := \{C \in \mathcal{C} | \text{ diam } (C) < \epsilon\}$$

are covers of X for every $\epsilon > 0$. Then for every set A the

$$m^*_{\ell,\mathcal{C}_\epsilon}(A)$$

are increasing, so we can consider the function on set given by

$$m^*_{II}(A) := \sup_{\epsilon \to 0} m_{\ell,\mathcal{C}_\epsilon}(A).$$

The axioms for an outer measure are preserved by this limit operation, so m^*_{II} is an outer measure. If A and B are such that $d(A, B) > 2\epsilon$, then any set of $\mathcal{C}_\epsilon$ which intersects A does not intersect B and vice versa, so throwing away extraneous sets in a cover of $A \cup B$ which does not intersect either, we see that $m^*_{II}(A \cup B) = m^*_{II}(A) + m^*_{II}(B)$. The method II construction always yields a metric outer measure.

4 Hausdorff measure.

Let X be a metric space. Recall once again that if A is any subset of X, the diameter of A is defined as

$$\mathrm{diam}(A) = \sup_{x,y \in A} d(x, y).$$

For any subset $A \subset X$ and any positive real number s define

$$\ell_s(A) = \mathrm{diam}(A)^s$$

(with $0^s = 0$). Take $\mathcal{C}$ to consist of *all* subsets of X. The method II outer measure is called the **s-dimensional Hausdorff outer measure**, and its restriction to the associated σ-field of (Caratheodory) measurable sets is called the **s-dimensional Hausdorff measure**. We will let $m^*_{s,\epsilon}$ denote the method

I outer measure associated to ℓ_s and ϵ, and let $\mathcal{H}_s^*$ denote the Hausdorff outer measure of dimension s, so that

$$\mathcal{H}_s(A) = \lim_{\epsilon \to 0} m_{s,\epsilon}^*(A).$$

Theorem 3 *Let $F \subset X$ be a Borel set. Let $0 < s < t$. Then*

$$\mathcal{H}_s(F) < \infty \ \Rightarrow \mathcal{H}_t(F) = 0$$

and

$$\mathcal{H}_t(F) > 0 \ \Rightarrow \mathcal{H}_s(F) = \infty.$$

5 Hausdorff dimension.

This last theorem implies that for any Borel set, there is a unique value s_0 (which might be 0 or ∞) such that $\mathcal{H}_t(F) = 0$ for all $t < s_0$ and $\mathcal{H}_s(F) = \infty$ for all for all $s > s_0$. This value is called the **Hausdorff dimension** of F. It is one of many competing (and non-equivalent) definitions of dimension. Notice that it is a metric invariant, and in fact is the same for two spaces different by a Lipschitz homeomorphism with Lipschitz inverse. But it is not a topological invariant.

6 Examples.

The space X of all sequences of zeros and one studied has Hausdorff dimension 1 relative to the metric $d_{\frac{1}{2}}$ while it has Hausdorff dimension $\log 2/\log 3$ if we use the metric $d_{\frac{1}{3}}$. Using the $d_{\frac{1}{3}}$ metric the space X is Lipschitz equivalent to the Cantor set, so the Cantor set has Hausdorff dimension $\log 2/\log 3$.

7 Hutchinson's theorem and the Hausdorff dimension of fractals.

7.1 Contracting ratio lists.

A finite collection of real numbers

$$(r_1, \ldots, r_n)$$

is called a **contracting ratio list** if

$$0 < r_i < 1 \quad \forall\, i = 1, \ldots, n.$$

Proposition 1 *Let $(r_1, \ldots, r_n)$ be a contracting ratio list. There exists a unique non-negative real number s such that*

$$\sum_{i=1}^{n} r_i^s = 1. \tag{3}$$

The number s is 0 if and only if $n = 1$.

The number s in (3) is called the bf similarity dimension of the ratio list $(r_1, \ldots, r_n)$.

7.2 Hutchinson's theorem.

Theorem 4 *Let $K_1, \ldots, K_n$ be contractions on a complete metric space X and let c be the maximum of their Lipschitz contants. Define the Hutchinson operator, K, on $\mathcal{H}(X)$ by*

$$K(A) := K_1(A) \cup \cdots \cup K_n(A).$$

Then K is a contraction with Lipschtz constant c on the space $\mathcal{H}(X)$ of compact subsets of X in the Hausdorff metric. In particullar there is a unique compact set F fixed by K.

A map $f : X \to Y$ between two metric spaces is called a **similarity** with similarity ratio r if

$$d_Y(f(x_1), f(x_2)) = r d_X(x_1, x_2) \ \ \forall \ x_1, \ x_2 \in X.$$

(Recall that a map is called **Lipschitz** with LIpschitz constant r if we only had an inequality, $\leq$, instead of an equality in the above.

Let X be a complete metric space, and let $(r_1, \ldots, r_n)$ be a contracting ratio list. A collection

$$(f_1, \ldots, f_n), \quad f_i : \ X \to X$$

is called an **iterated function system** which **realizes** the contracting ratio list if

$$f_i : X \to X, \quad i = 1, \ldots, n$$

is a similarity with ratio r_i. We also say that $(f_1, \ldots, f_n)$ is a **realization** of the ratio list $(r_1, \ldots, r_n)$.

It is a consequence of *Hutchinson's theorem*, see below, that

Proposition 2 *If $(f_1, \ldots, f_n)$ is a realization of the contracting ratio list $(r_1, \ldots, r_n)$ on a complete metric space, X, then there exists a unique non-empty compact subset $K \subset X$ such that*

$$K = f_1(K) \cup \cdots \cup f_n(K).$$

In fact, Hutchinson's theorem asserts the corresponding result where the f_i are merely assumed to be LIpschitz maps with Lipschitz constants $(r_1, \ldots, r_n)$.

The set K is sometimes called the fractal associated with the realization $(f_1, \ldots, f_n)$ of the contracting ratio list $(r_1, \ldots, r_n)$. The facts we want to establish are: First,

$$\dim(K) \leq s \tag{4}$$

where dim denotes Hausdorff dimension, and s is the similarity dimension of $(r_1, \ldots, r_n)$. In general, we can only assert an inequality here, for the the set

K does not fix $(r_1, \ldots, r_n)$ or its realization. For example, we can repeat some of the r_i and the corresponding f_i. This will give us a longer list, and hence a larger s, but will not change K. But we can demand a rather strong form of non-redundancy known as **Moran's condition**: There exists an open set U such that

$$U \supset f_i(U) \ \forall\, i \quad \text{and} \quad f_i(U) \cap f_j(U) = \emptyset \quad \forall\, i \neq j. \tag{5}$$

Then

Theorem 5 *If* $(f_1, \ldots, f_n)$ *is a realization of* $(r_1, \ldots, r_n)$ *on* $\mathbf{R}^d$ *and if Moran's condition holds then*

$$\dim K = s.$$