Problem set 2

Math 212b

February 8, 2001 due Feb. 27

Contents

1 The L_2 Euler operator

This is a continuation of the preceding problem set. There are three types of one-parameter subgroups of $Sl(2,\mathbf{R})$, and we have only studied two out of the three. A one parameter subgroup of $Sl(2,\mathbf{R})$ is determined by its infinitesimal generator, that is by the two by two matrix A such that $t \mapsto \exp tA$ is the one parameter group in question. Since $\exp tM$ is to have determinant one, this means (by differentiation and setting $t = 0$) that tr $M = 0$. Multiplying M by a scalar only means that we are changing the parameter t by that scalar. So there are three possibilities (other than the trivial case where $M = 0$): the eigenvalues of M are both zero. Then M is a nilpotent matrix and is conjugate to

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

and we in fact studied the one-parameter groups generated by two such matrices,

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix},$$

the matrix M can have eigenvalues $\pm i$, and we studied one such one-parameter group - the rotations

$$\begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$$

(with rotation through ninety degrees corresponding to the Fourier transform). There is also the possibility that M has eigenvalues ± 1, for example

$$M = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

which generates the one parameter group

$$\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}.$$

1. What is the one parameter group of unitary transformations in $Mp(2)$ which corresponds to this one-parameter subgroup of $Sl(2)$? Notice that for all elements of this one parameter group the upper right hand corner is zero, so our "generic formula" valid for $B \neq 0$ will not apply. But there is a very simple expression for the elements of the one parameter group in $Mp(2)$ covering the above one parameter subgroup in $Sp(2)$. What is it? [Hint: Observe that

$$\left[\begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

to get the infinitesimal generator of the desired one parameter group, and then stare at this infinitesimal generator to guess and then prove what is the desired one parameter group in $Mp(2)$. Here the notation $[a, b]$ is the commutator of a and b, defined to be $[a, b] = ab - ba$.]

2 Symplectic vector spaces.

I want to pass from 2 dimensions to $2n$ dimensions, and study the groups $Sp(2n)$ and $Mp(2n)$. For this it is useful to begin with some general theory:

Let V be a (usually finite dimensional) vector space over the real numbers. A symplectic structure on V consists of an antisymmetric bilinear form

$$\omega : V \times V \to \mathbf{R}$$

which is non-degenerate. So we can think of ω as an element of $\wedge^2 V^*$ when V is finite dimensional, as we shall assume until further notice. A vector space equipped with a symplectic structure is called a symplectic vector space.

A basic example is $\mathbf{R}^2$ with

$$\omega_{\mathbf{R}^2}\left(\begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} c \\ d \end{pmatrix} \right) = \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc.$$

We will call this the standard symplectic structure on $\mathbf{R}^2$.

2.1 Special kinds of subspaces.

If W is a subspace of symplectic vector space V then $W^\perp$ denotes the symplectic orthocomplement of W:

$$W^\perp; = \{v \in V |\ \omega(v,w) = 0,\ \forall w \in W\}.$$

A subspace is called

1. **symplectic** if $W \cap W^\perp = \{0\}$,
2. **isotropic** if $W \subset W^\perp$,
3. **coisotropic** if $W^\perp \subset W$, and
4. **Lagrangian** if $W = W^\perp$.

Since $(W^\perp)^\perp = W$ by the non-degeneracy of ω it follows that W is symplectic if and only if $W^\perp$ is. Also, the restriction of ω to any symplectic subspace W is non-degenerate, making W into a symplectic vector space. Conversely, to say that the restriction of ω to W is non-degenerate means precisely that $W \cap W^\perp = \{0\}$.

2.2 Normal forms.

For any non-zero $e \in V$ we can find an $f \in V$ such that $\omega(e,f) = 1$ and so the subspace W spanned by e and f is a two dimensional symplectic subspace. Furthermore the map

$$e \mapsto \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad f \mapsto \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

gives a symplectic isomorphism of W with $\mathbf{R}^2$ with its standard symplectic structure. We can apply this same construction to $W^\perp$ if $W^\perp \neq 0$. Hence by induction, we can decompose any symplectic vector space into a direct sum of two dimensional symplectic subspaces:

$$V = W_1 \oplus \cdots \oplus W_d$$

where $\dim V = 2d$ (proving that every symplectic vector space is even dimensional) and where the W_i are pairwise (symplectically) orthogonal and where each W_i is spanned by e_i, f_i with $\omega(e_i, f_i) = 1$. In particular this shows that all $2d$ dimensional symplectic vector spaces are isomorphic, and isomorphic to a direct sum of d copies of $\mathbf{R}^2$ with its standard symplectic structure.

2.3 Existence of Lagrangian subspaces.

Let us collect the $e_1, \ldots, e_d$ in the above construction and let L be the subspace they span. It is clearly isotropic. Also, $e_1, \ldots, e_n, f_1, \ldots, f_d$ form a basis of V. If $v \in V$ has the expansion

$$v = a_1 e_1 + \cdots a_d e_d + b_1 f_1 + \cdots + b_d f_d$$

in terms of this basis, then $\omega(e_i, v) = b_i$. So $v \in L^\perp \Rightarrow v \in L$. Thus L is Lagrangian. So is the subspace M spanned by the f's.

Conversely, if L is a Lagrangian subspace of V and M is a complementary Lagrangian subspace. Then ω induces a non-degenerate linear pairing of L with M and hence any basis $e_1, \cdots e_d$ picks out a dual basis $f_1, \cdots .f_d$ of M giving a basis of the above form.

2.4 Consistent Hermitian structures.

In terms of the basis $e_1, \ldots, e_n, f_1, \ldots, f_d$ introduced above, consider the linear map

$$J: \quad e_i \mapsto -f_i, \quad f_i \mapsto e_i.$$

It satisfies

$$\begin{aligned} J^2 &= -I, & (1)\\ \omega(Ju, Jv) &= \omega(u, v), \quad \text{and} & (2)\\ \omega(Ju, v) &= \omega(Jv, u). & (3) \end{aligned}$$

Notice that any J which satisfies two of the three conditions above automatically satisfies the third. Condition (1) says that J makes V into a d-dimensional complex vector space. Condition (2) says that J is a symplectic transformation, i.e acts so as to preserve the symplectic form ω. Condition (3) says that $\omega(Ju, v)$ is a real symmetric bilinear form.

All three conditions (really any two out of the three) say that $(\ ,\ ) = (\ ,\ )_{\omega,J}$ defined by

$$(u, v) = \omega(Ju, v) + i\omega(u, v)$$

is a semi-Hermitian form whose imaginary part is ω. For the J chosen above this form is actually Hermitian, that is the real part of $(\ ,\ )$ is positive definite.

2.5 Choosing Lagrangian complements.

The results of this section are purely within the framework of symplectic linear algebra. Hence their logical place is here. However their main interest is that they serve as lemmas for more geometrical theorems, for example the Weinstein isotropic embedding theorem. The results here all have to do with making choices in a "consistent" way, so as to guarantee, for example, that the choices can be made to be invariant under the action of a group.

For any a Lagrangian subspace $L \subset V$ we will need to be able to choose a complementary Lagrangian subspace L', and do so in a consistent manner, depending, perhaps, on some auxiliary data. Here is one such way, depending on the datum of a symmetric positive definite bilinear form B on V. (Here B has nothing to do with with the symplectic form.)

Let L^B be the orthogonal complement of L relative to the form B. So

$$\dim L^B = \dim L = \frac{1}{2} \dim V$$

and any subspace $W \subset V$ with

$$\dim W = \frac{1}{2} \dim V \quad \text{and } W \cap L = \{0\}$$

can be written as graph (A) where $A : L^B \to L$ is a linear map. That is, under the vector space identification

$$V = L^B \oplus L$$

the elements of W are all of the form

$$w + Aw, \quad w \in L^B.$$

We have

$$\omega(u + Au, w + Aw) = \omega(u, w) + \omega(Au, w) + \omega(u, Aw)$$

since $\omega(Au, Aw) = 0$ as L is Lagrangian. Let C be the bilinear form on L^B given by

$$C(u, w) := \omega(Au, w).$$

Thus W is Lagrangian if and only if

$$C(u, w) - C(w, u) = -\omega(u, w).$$

Now

$$\mathrm{Hom}(L^B, L) \sim L \otimes L^{B*} \sim L^{B*} \otimes L^{B*}$$

under the identification of L with L^{B*} given by ω. Thus the assignment $A \leftrightarrow C$ is a bijection, and hence the space of all Lagrangian subspaces complementary to L is in one to one correspondence with the space of all bilinear forms C on L^B which satisfy $C(u, w) - C(w, u) = -\omega(u, w)$ for all $u, w \in L^B$. An obvious choice is to take C to be $-\frac{1}{2}\omega$ restricted to L^B. In short,

Proposition 1 *Given a positive definite symmetric form on a symplectic vector space V, there is a consistent way of assigning a Lagrangian complement L' to every Lagrangian subspace L.*

Here the word consistent means that the choice depends only on B. This has the following implication: Suppose that T is a linear automorphism of V which preserves both the symplectic form ω and B. In other words, suppose that

$$\omega(Tu, Tv) = \omega(u, v) \quad \text{and} \quad B(Tu, Tv) = B(u, v) \quad \forall\, u, v \in V.$$

Then if $L \mapsto L'$ is the correspondence given by the proposition, then

$$TL \mapsto TL'.$$

More generally, if $T : V \to W$ is a symplectic isomorphism which is an isometry for a choice of positive definite symmetric bilinear forms on each, the above equation holds.

Given L and B (and hence L') we determined the complex structure J by

$$J : L \to L', \quad \omega(u, Jv) = B(u, v) \quad u, v \in L$$

and then

$$J := -J^{-1} : L' \to L$$

and extending by linearity to all of V so that

$$J^2 = -I.$$

Then for $u, v \in L$ we have

$$\omega(u, Jv) = b(u, v) = b(v, u) = \omega(v, Ju)$$

while

$$\omega(u, JJv) = -\omega(u, v) = 0 = \omega(Jv, Ju)$$

and

$$\omega(Ju, JJv) = -\omega(Ju, v) = -\omega(Jv, u) = \omega(Jv, JJu)$$

so (3) holds for all $u, v \in V$. We should write $J_{B,L}$ for this complex stucture, or T_L when B is understood

Suppose that T preserves ω and B as above. We claim that

$$J_{TL} \circ T = T \circ J_L$$

so that T is complex linear for the complex structures J_L and J_{TL}. Indeed, for $u, v \in L$ we have

$$\omega(Tu, J_{TL}Tv) = B(Tu, Tv)$$

by the definition of J_{TL}. Since B is invariant under T the right hand side equals $B(u, v) = \omega(u, J_L v) = \omega(Tu, TJ_L v)$ since ω is invariant under T. Thus

$$\omega(Tu, J_{TL}Tv) = \omega(Tu, TJ_L v)$$

showing that

$$TJ_L = J_{TL}T$$

when applied to elements of L. This also holds for elements of L'. Indeed every element of L' is of the form $J_L u$ where $u \in L$ and $TJ_L u \in TL'$ so

$$J_{TL}TJ_L u = -J_{TL}^{-1}TJ_L u = -Tu = TJ_L(J_L u).$$

QED

3 The group $Sp(2n)$.

We may assume that our symplectic vector space is $\mathbf{R}^n \oplus \mathbf{R}^n$ and we write the typical vector as

$$u = \begin{pmatrix} q \\ p \end{pmatrix} \quad \text{where } q, p \in \mathbf{R}^n$$

and where the symplectic form ω is given by

$$\omega(u, u') = p \cdot q' - p' \cdot q$$

with $\cdot$ denoting the scalar product on $\mathbf{R}^n$. If we introduce the standard positive definite scalar product $\bullet$ on

$$\mathbf{R}^n \oplus \mathbf{R}^n = \mathbf{R}^{2n}$$

$$u \bullet u' = q \cdot q' + p \cdot p'$$

then

$$\omega(u, v) = u' \bullet Ju$$

where

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

where I is the $n \times n$ identity matrix. A $2n \times 2n$ matrix T is called symplectic if $\omega(Tu, Tu') = \omega(u, u')$ for all $u, u' \in \mathbf{R}^n$ which amounts to the condition

$$T^t JTu \bullet u' = Ju \bullet u' \quad \forall \;\; u, u' \in \mathbf{R}^n$$

where T^t denotes the transpose of T. So the condition on T is

$$T^t JT = J.$$

. **2.** Write T in "block form" as

$$T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where now A, B, C, and D are $n \times n$ matrices. Express the above condition on T in terms of A, B, C, D.

. The purpose of the next few problems is to prove that every element of the group $Sp(2n)$ of symplectic matrices can be written as a finite product of matrices of the following two types

$$\begin{pmatrix} I & dI \\ 0 & I \end{pmatrix} \quad d \geq 0$$

and

$$\begin{pmatrix} I & 0 \\ -P & I \end{pmatrix} \quad \text{where} \quad P = P^t$$

is a symmetric $n \times n$ matrix.

3. Obtain the matrix

$$\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

as the product of three such matrices.

4. Obtain the matrix

$$\begin{pmatrix} I & P \\ 0 & I \end{pmatrix}$$

by conjugating

$$\begin{pmatrix} I & 0 \\ -P & I \end{pmatrix}$$

by the cube of the preceding matrix.

5. If P is symmetric and non-singular, show how to obtain

$$\begin{pmatrix} 0 & P^{-1} \\ P & 0 \end{pmatrix}$$

and then

$$\begin{pmatrix} P & 0 \\ 0 & P^{-1} \end{pmatrix}$$

as products of the matrices already constructed.

We now need a little lemma which says that *any non-singular $n \times n$ matrix A can be written as the product of three symmetric matrices.*

6. Prove this directly for the case of a 2×2 matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

by first showing that A can be written as the product of a diagonal matrix and a symmetric matrix if $bc \neq 0$ and then show that if $bc = 0$, so $ad \neq 0$ multiplying by

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

reduces to the previous case.

7. Recall from linear algebra that any non-singular $n \times n$ matrix can be written as $A = PO$ where P is symmetric positive definite and O is orthogonal. So we must show that every orthogonal matrix can be written as the product of two symmetric matrices. Recall that any orthogonal matrix can be "block diagonalized", i.e can be written as $O = RAR^{-1}$ where R is orthogonal and A is block diagonal where the blocks are either two by two orthogonal matrices with non-zero off diagonal matrices or are one-by-one blocks. Conclude the lemma in general.

We have now obtained all matrices of the form

$$\begin{pmatrix} A & 0 \\ 0 & (A^t)^{-1} \end{pmatrix}$$

with A non-singular.

8. Show that we can obtain any symplectic matrix of the form

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

with A non-singular. [Hint: multiply on the left by

$$\begin{pmatrix} I & 0 \\ -E & I \end{pmatrix}$$

and choose E appropriately.]

We now complete the proof: If A is singular, we can (by row and column reduction) find invertible $n \times n$ matrices L and M such that

$$LAM = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$$

where I_r is the $r \times r$ identity matrix with r = rank of A. So pre- and post multiplying by

$$\begin{pmatrix} L & 0 \\ 0 & (L^t)^{-1} \end{pmatrix} \text{ and } \begin{pmatrix} M & 0 \\ 0 & (M^t)^{-1} \end{pmatrix}$$

we may assume that our matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is of the form

$$A = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}.$$

Write

$$C = \begin{pmatrix} C_1 & C_2 \\ C_3 & C_4 \end{pmatrix}$$

where C_1 is the upper left $r \times r$ block of C etc. Then

$$A^t C = \begin{pmatrix} C_1 & C_2 \\ 0 & 0 \end{pmatrix}.$$

9. Conclude from one of the conditions that T be symplectic that $C_2 = 0$ and C_1 is symmetric.

Multiply $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ on the left by

$$\begin{pmatrix} I & E \\ 0 & I \end{pmatrix}$$

where

$$E = \begin{pmatrix} 0 & 0 \\ 0 & I_{n-r} \end{pmatrix}$$

to obtain a matrix

$$\begin{pmatrix} A' & B' \\ C & D \end{pmatrix}$$

where

$$A' = \begin{pmatrix} I_r & 0 \\ C_3 & C_4 \end{pmatrix}.$$

10. Show that C_4 and hence A' is non-singular, completing the proof that the matrices

$$\begin{pmatrix} I & dI \\ 0 & I \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} I & 0 \\ -P & I \end{pmatrix}$$

generate $Sp(2n)$.

Whew! A lot of algebra! But we are now in position to generalize our construction of $Mp(2)$ to $Mp(2n)$. Just define the operators U_d and V_P on $L_2(\mathbf{R}^n)$ by

$$V_P \text{ is multiplication by } \exp(-iPx \cdot x/2)$$

and

$$(U_d f)(x) = \exp(-\pi i n/4) d^{-n/2} (2\pi)^{-n/2} \int \exp[i(x-y)\cdot(x-y)/2d] f(y)dy.$$

We take the product of these operators and prove just as before that there is a two to one map of the group generated by these operators onto the symplectic group. You are probably wiped out by now so I won't press the details, as I will give a much more abstract description of $Mp(2n)$ in terms of the Stone-von-Neumann theorem in class.