Problem Set One Answers

1. From the definitions of U_d and V_P, we see that

$$\begin{aligned}(U_s \circ V_P \circ U_t)c(x) &= (U_s \circ V_P)\left(e^{-\pi i/4}\frac{1}{\sqrt{2\pi t}}\int e^{i(x-y)^2/(2t)}c(y)\,dy\right)\\ &= U_s\left(e^{-\pi i/4}\frac{1}{\sqrt{2\pi t}}e^{-iPx^2/2}\int e^{i(x-y)^2/(2t)}c(y)\,dy\right)\\ &= e^{-\pi i/2}\frac{1}{2\pi\sqrt{st}}\int e^{i(x-z)^2/(2s)}e^{-iPz^2/2}\int e^{i(z-y)^2/(2t)}c(y)\,dy\,dz.\end{aligned}$$

We can switch the order of integration and evaluate the integral in z as the Fourier transform of a Gaussian,

$$\begin{aligned}\int e^{i(x-z)^2/(2s)}e^{-iPz^2/2}e^{i(z-y)^2/(2t)}\,dz &= e^{i\left(\frac{x^2}{s}+\frac{y^2}{t}\right)/2}\int e^{i\left(\frac{1}{s}+\frac{1}{t}-P\right)z^2/2}e^{-i\left(\frac{x}{s}+\frac{y}{t}\right)z}\,dz\\ &= \frac{\sqrt{2\pi st}}{\sqrt{-i(s+t-stP)}}e^{i\left(\frac{x^2}{s}+\frac{y^2}{t}\right)/2}e^{-i\left(\frac{st}{s+t-stP}\right)\left(\frac{(xt+ys)^2}{(st)^2}\right)/2}\\ &= \frac{\sqrt{2\pi st}}{\sqrt{-i(s+t-stP)}}e^{i\frac{x^2(st+t^2-st^2P)+y^2(s^2+st-s^2tP)-x^2t^2-y^2s^2-2xyst}{2st(s+t-stP)}}\\ &= \frac{\sqrt{2\pi st}}{\sqrt{-i(s+t-stP)}}e^{i\frac{x^2(1-tP)+y^2(1-sP)-2xy}{2(s+t-stP)}}\\ &= \frac{\sqrt{2\pi st}}{\sqrt{-iB}}e^{i\frac{Dx^2+Ay^2-2xy}{2B}}.\end{aligned}$$

Thus we have

$$(U_s \circ V_P \circ U_t)c(x) = e^{-\pi i/2}\frac{1}{\sqrt{-2\pi iB}}\int e^{iW(x,y)}c(y)\,dy,$$

which is the desired result since, because B is real, $\sqrt{-iB} = e^{-(\operatorname{sgn} B)\pi i/4}\sqrt{|B|}$.

2. First notice that

$$\begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}\begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix} = \begin{pmatrix} A_1A_2 + B_1C_2 & A_1B_2 + B_1D_2 \\ C_1A_2 + D_1C_2 & C_1B_2 + D_1D_2 \end{pmatrix} =: \begin{pmatrix} A_3 & B_3 \\ C_3 & D_3 \end{pmatrix},$$

and this gives the explicit formula for A_3, B_3, C_3, and D_3.

The composition of $O(A_1, B_1, D_1)$ and $O(A_2, B_2, D_2)$, $O(A_1, B_1, D_1) \circ O(A_2, B_2, D_2)$, is given by

$$e^{-\pi i}e^{(\operatorname{sgn} B_1 + \operatorname{sgn} B_2)\pi i/4}\frac{1}{2\pi}|B_1B_2|^{-\frac{1}{2}}\int\int e^{i\frac{D_1x^2+A_1y^2-2xy}{2B_1}}e^{i\frac{D_2y^2+A_2z^2-2yz}{2B_2}}\,dyc(z)\,dz,$$

where we have, as in problem 1, switched the order of integration. The inner integral can be evaluated as the Fourier transform of a Gaussian,

$$\begin{aligned}\int e^{i\frac{D_1x^2+A_1y^2-2xy}{2B_1}}e^{i\frac{D_2y^2+A_2z^2-2yz}{2B_2}}\,dy &= e^{i\left(\frac{D_1x^2}{B_1}+\frac{D_2y^2}{B_2}\right)/2}\int e^{i\frac{A_1B_2+B_1D_2}{B_1B_2}y^2/2}e^{-iy\left(\frac{x}{B_1}+\frac{z}{B_2}\right)}\,dy\\ &= e^{\sigma\frac{\pi i}{4}}\sqrt{\frac{2\pi|B_1B_2|}{|B_3|}}e^{i\left(\frac{D_1x^2}{B_1}+\frac{D_2y^2}{B_2}\right)/2}e^{-i\frac{B_1B_2}{B_3}\frac{(xB_2+zB_1)^2}{2B_1^2B_2^2}}\\ &= e^{\sigma\frac{\pi i}{4}}\sqrt{\frac{2\pi|B_1B_2|}{|B_3|}}e^{i\left(\frac{(B_2B_3D_1-B_2^2)x^2+(D_2B_1B_3-B_1^2)z^2-2B_1B_2xz}{2B_1B_2B_3}\right)},\end{aligned}$$

where $\sigma = \operatorname{sgn}(B_1B_2B_3)$, and where we have used the fact that $B_3 = A_1B_2 + B_1D_2$. Since $A_1D_1 - B_1C_1 = 1$ and $A_2D_2 - B_2C_2 = 1$, we have

$$\begin{aligned} B_2B_3D_1 - B_2^2 = B_2(B_3D_1 - B_2) &= B_2(A_1B_2D_1 + B_1D_2D_1 - B_2) \\ &= B_2(B_2(A_1D_1 - 1) + B_1D_2D_1) = B_2(B_2B_1C_1 + B_1D_2D_1) \\ &= B_1B_2D_3, \end{aligned}$$

and similarly

$$D_2B_1B_3 - B_1^2 = B_1B_2A_3.$$

Thus we see that

$$\int e^{i\frac{D_1x^2+A_1y^2-2xy}{2B_1}} e^{i\frac{D_2y^2+A_2z^2-2xz}{2B_2}}\,dy = e^{\sigma\pi i/4}\sqrt{\frac{2\pi|B_1B_2|}{|B_3|}}e^{i\left(\frac{D_3x^2+A_3z^2-2xz}{2B_3}\right)},$$

and so

$$O(A_1,B_1,D_1)\circ O(A_2,B_2,D_2) = e^{-\pi i}e^{\pi i/4(\operatorname{sgn}B_1+\operatorname{sgn}B_2+\operatorname{sgn}(B_1B_2B_3))}|2\pi B_3|^{-\frac{1}{2}}\int e^{i\left(\frac{D_3x^2+A_3y^2-2xy}{2B_3}\right)}c(y)\,dy.$$

Since

$$\operatorname{sgn}B_1 + \operatorname{sgn}B_2 + \operatorname{sgn}(B_1B_2B_3) \equiv 2 + \operatorname{sgn}B_3 \bmod 4,$$

this tells us that

$$O(A_1,B_1,D_1)\circ O(A_2,B_2,D_2) = \pm O(A_3,B_3,D_3),$$

as desired.

3. We first prove **Assertion 1**, that the composition of any number of U's and V's which has B in the corresponding matrix not equal to zero has the property that

$$(Xc)(x) = i^{\#}e^{-\pi i/4}|2\pi B|^{-\frac{1}{2}}\int e^{iW(x,y)}c(y)\,dy,$$

where $W(x,y) = \frac{1}{2B}(Dx^2 + Ay^2 - 2xy)$.

To prove Assertion 1, notice first that $U_s\circ U_t = U_{s+t}$ and $V_P\circ V_Q = V_{P+Q}$. Indeed, for the U's , it is enough to notice that the solution to problem one works if $P = 0$. For the V's, the result follows directly from multiplication. Thus any string of composed U's and V's can be reduced to one of alternating U's and V's.

Now suppose that Assertion 1 holds for some X, which is a composition of U's and V's. In this case, Assertion 1 is also true for $X\circ V_P$ and $V_P\circ X$. Indeed, on the matrix side,

$$\begin{pmatrix} 1 & 0 \\ -P & 1 \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A & B \\ -PA+C & -PB+D \end{pmatrix}$$

and

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -P & 1 \end{pmatrix} = \begin{pmatrix} A-PB & B \\ C-PD & D \end{pmatrix},$$

and so composition with V_P on either side does not change the value of B. Also,

$$\begin{aligned} (V_P\circ X)(x) &= i^{\#}e^{-\pi i/4}|2\pi B|^{-\frac{1}{2}}\int e^{i\frac{Dx^2+Ay^2-2xy}{2B}}e^{-iPx^2/2}c(y)\,dy \\ &= i^{\#}e^{-\pi i/4}|2\pi B|^{-\frac{1}{2}}\int e^{i\frac{(D-PB)x^2+Ay^2-2xy}{2B}}c(y)\,dy, \end{aligned}$$

and

$$(X \circ V_P)(x) = i^{\#} e^{-\pi i/4} |2\pi B|^{-\frac{1}{2}} \int e^{i\frac{Dx^2+Ay^2-2xy}{2B}} e^{-iPy^2/2} c(y)\, dy$$
$$= i^{\#} e^{-\pi i/4} |2\pi B|^{-\frac{1}{2}} \int e^{i\frac{Dx^2+(A-PB)y^2-2xy}{2B}} c(y)\, dy.$$

Thus it is clear that Assertion 1 holds for $X \circ V_P$ and $V_P \circ X$ if it holds for X.

The problem of proving Assertion 1 is thus reduced to proving that it holds when X is a string of alternating U's and V's, beginning and ending with U's, with the B corresponding to X not equal to zero. We proceed by induction on the number of U's. The case of one U is trivial, and the case of two U's is handled by problem one. Now suppose that we know the result for n U's. We can write

$$X = X' \circ (U_s \circ V_P \circ U_t),$$

where X' is a product of $(n-1)$ U's and $(n-1)$ V's, beginning with a U and ending with a V. We can write $s = s' + s''$, with $s', s'' > 0$, in such a way that $Y = X' \circ U_{s'}$ has corresponding $B \neq 0$ and $U_{s''} \circ V_P \circ U_t$ has corresponding $B \neq 0$. In this way, we have

$$X = Y \circ (U_{s''} \circ V_P \circ U_t),$$

and by induction, we have that Assertion 1 holds for Y and for $U_{s''} \circ V_P \circ U_t$. Taking the composition of these two operators and applying the result of problem two, we see that Assertion 1 holds for X. Thus Assertion 1 follows for all X which are compositions of U's and V's with the property that the corresponding $B \neq 0$.

From Assertion 1 it follows that if X is decomposed into a product of U's and V's in two different ways and the B corresponding to X is not equal to zero, then these two factorizations give rise to the same matrix $\rho(X)$. Indeed, since the operators are given by equation (5) in the text, up to sign, and they hold for all c's, it is clear that B, A, and D must be the same for both operators. Since $B \neq 0$, this means that C is determined by A, B, and D. Thus the map ρ is well-defined.

If we multiply X on the right by U_t, we get

$$\rho(X \circ U_t) = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} A & At + B \\ C & Ct + D \end{pmatrix}.$$

If $B = 0$, then by multiplying by U_t with $t > 0$ small, $X \circ U_t$ will thus have a corresponding B which is not equal to zero. Since ρ is well defined for operators with $B \neq 0$, and since $X \circ U_t \to X$ as $t \to 0$, this shows that ρ is in fact well defined for operators with $B = 0$.

The fact that X is determined by $\rho(X)$ up to a sign follows from the $\pm$ factor at the end of problem two.

4. If $C \neq 0$, we have

$$\begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ C & 1 \end{pmatrix} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 + sC & C \\ s + t + stC & 1 + tC \end{pmatrix}.$$

We can let $s = \frac{A-1}{C}$ and $t = \frac{D-1}{C}$, and this will give $s + t + stC = B$, since $AD - BC = 1$. Thus this choice of s and t yields

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ C & 1 \end{pmatrix} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}.$$

If $C = 0$, notice that

$$\begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} A - B & B \\ -D & D \end{pmatrix},$$

and so multiplication by $\rho(V_1)$ changes the matrix into one for which $C \neq 0$. We have used the fact that the determinant is 1 and $C = 0$ in order to deduce that $D \neq 0$. Thus the previous paragraph tell us that this new matrix can be written as a product of three matrices, in the way described there. Multiplication on the right by $\rho(V_{-1})$ now represents our original matrix as a product of four matrices of the desired type.

Now notice that

$$U_1V_1U_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

and that

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Thus we can get

$$\begin{pmatrix} 1 & -s \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & 0 \\ P & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & -P \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

for $s, P > 0$. This means that U's and V's with negative subscripts can be obtained from the U's and V's with positive subscripts. Thus, in light of the results of the previous paragraph, we can get all of the two by two matrices of determinant one by taking products of U's and V's with only positive subscripts.

We now see that products of U's and V's with positive subscripts give rise to all U's and V's with negative subscripts, and hence to the inverses of U's and V's with positive subscripts. Thus the U's and V's with positive subscripts form a group, and ρ is clearly a group homomorphism of this group to $SL_2(\mathbf{R})$, and the homomorphism is onto since we've already shown that any element of SL_2 can be written as a product of $\rho(U)$'s and $\rho(V)$'s. The homomorphism is two to one because the $\pm$ factor at the end of problem two shows that the product of two matrices only determines the product of the corresponding operators up to sign.

5. Direct computation shows that

$$A(x^n e^{-x^2/2}) = (n+1/2)x^n e^{-x^2/2} - \frac{n(n-1)}{2}x^{n-2}e^{-x^2/2},$$

and so the first assertion of the problem holds with $p(x) = -n(n-1)/2x^{n-2}$.

We now want to find a sequence of polynomials $H_n(x)$ such that $H_n(x)e^{-x^2/2}$ is an eigenvector of A with eigenvalue $n + \frac{1}{2}$. Let $H_0(x) = 1$. Then we see that

$$A(H_0(x)e^{-x^2/2}) = \frac{1}{2}e^{-x^2/2},$$

and so we have the case $n = 0$. Now proceed by induction, assuming that $H_n(x)e^{-x^2/2}$ is an eigenvector with eigenvalue $n + 1/2$. Set

$$H_{n+1}(x)e^{-x^2/2} = \left(x - \frac{d}{dx}\right)(H_n(x)e^{-x^2/2}).$$

From direct computation,

$$A\left(x - \frac{d}{dx}\right) = \left(x - \frac{d}{dx}\right)(A+1).$$

Thus we have

$$\begin{aligned} A\left(H_{n+1}(x)e^{-x^2/2}\right) &= A\left(x - \frac{d}{dx}\right)\left(H_n(x)e^{-x^2/2}\right) \\ &= \left(x - \frac{d}{dx}\right)(A+1)\left(H_n(x)e^{-x^2/2}\right) \\ &= \left(x - \frac{d}{dx}\right)\left(\left(n + \frac{3}{2}\right)H_n(x)e^{-x^2/2}\right) \\ &= \left(n + \frac{3}{2}\right)H_{n+1}(x)e^{-x^2/2}, \end{aligned}$$

and so we see by induction that the polynomial $H_n(x)$ has the eigenvalue $n+1/2$.

Note that

$$\mathcal{F}(H_0(x)e^{-x^2/2}) = \mathcal{F}(e^{-x^2/2}) = e^{-x^2/2} = H_0(x)e^{-x^2/2},$$

and so $H_0(x)e^{-x^2/2}$ is an eigenvalue of $\mathcal{F}$ with eigenvalue 1. Now suppose that $H_n(x)e^{-x^2/2}$ is an eigenvalue of $\mathcal{F}$ with eigenvalue $\left(\frac{1}{i}\right)^n$. Note that

$$\mathcal{F}\left(x - \frac{d}{dx}\right) = \frac{1}{i}\left(x - \frac{d}{dx}\right),$$

and so

$$\begin{aligned}
\mathcal{F}(H_{n+1}(x)e^{-x^2/2}) &= \mathcal{F}\left(x - \frac{d}{dx}\right)(H_n(x)e^{-x^2/2}) \\
&= \frac{1}{i}\left(x - \frac{d}{dx}\right)\mathcal{F}(H_n(x)e^{-x^2/2}) \\
&= \left(\frac{1}{i}\right)^{n+1}\left(x - \frac{d}{dx}\right)(H_n(x)e^{-x^2/2}) \\
&= \left(\frac{1}{i}\right)^{n+1} H_{n+1}(x)e^{-x^2/2}.
\end{aligned}$$

Thus $H_n(x)e^{-x^2/2}$ is an eigenvector for $\mathcal{F}$ with eigenvalue $\left(\frac{1}{i}\right)^n$.

6. We have $p_1 = \frac{q_2 - Aq_1}{B}$ and

$$-\frac{\partial W(q_2, q_1)}{\partial q_1} = -\frac{\partial}{\partial q_1}\left(\frac{Dq_2^2 + Aq_1^2 - 2q_1q_2}{2B}\right) = \frac{q_2 - Aq_1}{B}.$$

Also,

$$p_2 = Cq_1 + Dp_1 = Cq_1 + D\left(\frac{q_2 - Aq_1}{B}\right) = \frac{-q_1 + Dq_2}{B},$$

and

$$\frac{\partial W(q_2, q_1)}{\partial q_2} = \frac{\partial}{\partial q_2}\left(\frac{Dq_2^2 + Aq_1^2 - 2q_1q_2}{2B}\right) = \frac{Dq_1 - q_2}{B}.$$

Thus $p_1 = -\frac{\partial W(q_2,q_1)}{\partial q_1}$ and $p_2 = \frac{\partial W(q_2,q_1)}{\partial q_2}$.

Finally, using the above computations, we see that

$$\frac{1}{2}(p_2q_2 - p_1q_1) = \frac{-q_1 + Dq_2}{B}q_2 - \frac{q_2 - Aq_1}{B}q_1 = \frac{Dq_2^2 + Aq_1^2 - 2q_1q_2}{B} = W(q_2, q_1).$$