Problem Set Two Answers

1. As stated in the hint, the commutator of $\begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ is $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Thus the infinitesimal generator of the one parameter group we are looking for should be given by the commutator of $-\frac{1}{2}ix^2$ and $\frac{1}{2}i\frac{d^2}{dx^2}$, the infinitesimal generators of the V_P and U_t groups, respectively. This commutator is

$$\left[-\frac{1}{2}ix^2, \frac{1}{2}i\frac{d^2}{dx^2}\right] = -\frac{1}{2} - x\frac{d}{dx}.$$

Consider the group given by

$$W_s(f) = e^{-\frac{s}{2}} f\left(e^{-s}x\right).$$

Then

$$\left.\frac{dW_s(f)}{ds}\right|_{s=0} = -\frac{1}{2}f(x) - x\frac{df}{dx},$$

which means that the group W_s has the infinitesimal generator we want.

2. Direct multiplication out of the left hand side of the $T^tJT = J$ condition shows that

$$A^tC = C^tA, \qquad B^tD = D^tB, \qquad \text{and} \qquad A^tD - C^tB = I.$$

3. We have

$$\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} = \begin{pmatrix} I & I \\ 0 & I \end{pmatrix}\begin{pmatrix} I & 0 \\ -I & I \end{pmatrix}\begin{pmatrix} I & I \\ 0 & I \end{pmatrix}.$$

4. The "cube of the preceeding matrix" is

$$\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}^3 = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix},$$

and so

$$\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}\begin{pmatrix} I & 0 \\ -P & I \end{pmatrix}\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}^3 = \begin{pmatrix} I & P \\ 0 & I \end{pmatrix}.$$

5. Note that there should be a negative sign before the lower left P in the first matrix in this problem (it was a typo). Now, we have

$$\begin{pmatrix} I & P^{-1} \\ 0 & I \end{pmatrix}\begin{pmatrix} I & 0 \\ -P & I \end{pmatrix}\begin{pmatrix} I & P^{-1} \\ 0 & I \end{pmatrix} = \begin{pmatrix} 0 & P^{-1} \\ -P & 0 \end{pmatrix},$$

which is the first matrix we are after, and

$$\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}\begin{pmatrix} 0 & P^{-1} \\ -P & 0 \end{pmatrix} = \begin{pmatrix} P & 0 \\ 0 & P^{-1} \end{pmatrix},$$

which is the second matrix we are after.

6. If $bc \neq 0$, then $b, c \neq 0$, and so we can write

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & c/b \end{pmatrix}\begin{pmatrix} a & b \\ b & db/c \end{pmatrix},$$

which is the product of a diagonal matrix and a symmetric matrix. If $bc = 0$, then $ad \neq 0$ (since the matrix is non-singular), and so if we notice that

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} c & d \\ a & b \end{pmatrix},$$

and so we are reduced to the previous case. In particular, in this case,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ 0 & a/d \end{pmatrix} \begin{pmatrix} c & d \\ d & db/a \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & a/d \end{pmatrix} \begin{pmatrix} c & d \\ d & db/a \end{pmatrix}.$$

7. We know from the problem that $A = PO$ and $O = RAR^{-1}$, where P is symmetric, R is orthogonal, and A is block-diagonal with two by two blocks having $bc \neq 0$ or one by one blocks along its diagonal. Problem 6 therefore tells us that $A = SS'$, where S and S' are symmetric. Now notice that $O = (RSR^{-1})(RS'R^{-1})$, and recall that the conjugate of a symmetric matrix by an orthogonal matrix is symmetric. Therefore O can be written as a product of two symmetric matrices, and so A can be written as a product of three symmetric matrices.

8. Follow the hint to the problem with $E = CA^{-1}$. In this case,

$$\begin{pmatrix} I & 0 \\ -E & I \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A & B \\ 0 & -CA^{-1}B + D \end{pmatrix}.$$

Recall from problem 2 that, since $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is symplectic, $A^tD - C^tB = 1$ and $A^tC = C^tA$, and so

$$\begin{aligned} -CA^{-1}B + D &= (A^t)^{-1}\left(-A^tCA^{-1}B + A^tD\right) \\ &= (A^t)^{-1}\left(A^tD - C^tB\right) \\ &= (A^t)^{-1}. \end{aligned}$$

Moreover,

$$E^t = (A^t)^{-1}C^t = (A^t)^{-1}C^tAA^{-1} = (A^t)^{-1}A^tCA^{-1} = C^tA^{-1} = E.$$

Thus we can write

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} I & 0 \\ -E & I \end{pmatrix} \begin{pmatrix} A & B \\ 0 & (A^t)^{-1} \end{pmatrix},$$

where E is symmetric.

Now, if a matrix is symplectic, so is its inverse, as can be seen by considering the $T^tJT = J$ condition for being symplectic. Since the inverse of $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is $c\begin{pmatrix} D^t & -B^t \\ -C^t & A^t \end{pmatrix}$, where c is a constant, it follows from problem 2 that $-B^tA = -A^tB$, and so $F = A^{-1}B$ is symmetric. We can thus write

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} I & 0 \\ -E & I \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & (A^t)^{-1} \end{pmatrix} \begin{pmatrix} I & F \\ 0 & I \end{pmatrix},$$

where E and F are both symmetric. The preceeding problems show that we can get each of the matrices on the right.

9. From problem 2, we know that A^tC is symmetric, and so $C_2 = 0$ and C_1 symmetric.

10. If C_4 is singular, then, since $C_2 = 0$ (from problem 9), it is clear that the matrix

$$\begin{pmatrix} A' & B' \\ C & D \end{pmatrix} = \begin{pmatrix} I_r & 0 & \multirow{2}{*}{B'} \\ C_3 & C_4 & \\ C_1 & 0 & \multirow{2}{*}{D} \\ C_3 & C_4 & \end{pmatrix},$$

is singular by looking at its $(r+1)$st through nth columns. This is a contradiction, and so C_4 is not singular.