Problem Set Three Answers

1. Since $d^2\omega = 0$ and $dy^j \wedge dy^j = 0$, we have

$$d(e^{i\tau x \cdot y}\omega) = \left(d(e^{i\tau x \cdot y})\right) \wedge \omega$$
$$= i\tau e^{i\tau x \cdot y}\left(x^1 dy^1 + \cdots + x^n dy^n\right) \wedge \omega$$
$$= i\tau |x|^2 e^{i\tau x \cdot y} dy.$$

So by Stoke's formula,

$$\hat{\chi}(\tau x) = \int_D e^{i\tau x \cdot y}\, dy = \frac{1}{i\tau |x|^2}\int_{\partial D} e^{i\tau x \cdot y}\omega.$$

This integral over ∂D, which is $(n-1)$ dimensional, will be big O of $\tau^{-\frac{n-1}{2}}$. Indeed, this follows from the paragraph following equation (15) in the problem set, where $M = \partial D$. The only thing we need to do in order to see that this formula holds is to check that the individual terms in the sum in equation (15) are uniformly bounded. But this is true since D is assumed to be such that the function $y \mapsto x \cdot y$ is non-degenerate on ∂D, and so the denominators in the sum in equation (15) are uniformly bounded away from zero.

Thus we see that

$$\hat{\chi}(\tau x) = O\left(\frac{1}{\tau}\tau^{-\frac{n-1}{2}}\right) = O\left(\tau^{-\frac{n+1}{2}}\right),$$

as desired.

2. As indicated in the hint, $X(e^{i\tau\psi}) = i\tau|d\psi|^2 e^{i\tau\psi}$, and so we see that

$$\int e^{i\tau\psi}a\, dy = \int \frac{a}{i\tau|d\psi|^2}X(e^{i\tau\psi})\, dy$$
$$= \frac{1}{\tau}\sum_{j=1}^n \int \frac{a}{i|d\psi|^2}\frac{\partial\psi}{\partial y^j}\frac{\partial}{\partial y^j}(e^{i\tau\psi})\, dy. \tag{$*$}$$

Integration by parts can now be used in each of these final integrals (integration by parts in y^j in the jth integral) in order to make the jth integral look like

$$\int \tilde{a}_j e^{i\tau\psi}\, dy,$$

where $\tilde{a}_j$ is another smooth function of compact support, depending on a and ψ.

From $(*)$ we see that $\int e^{i\tau\psi}a\, dy$ is $O(1/\tau)$, since the integrals are bounded (since a is of compact support). And, from the above integration by parts, we see that we can repeat this process indefinitely, to get that

$$\int e^{i\tau\psi}a\, dy = O(\tau^{-k}),$$

for any k.

3. First note that the square of $Q(y)$ in the exponents does not belong there. The problem should just have $Q(y)$ in the exponents, since this is the quadratic form in question.

The first integral we want to compute explicitly, which we may since it is just the product of Fourier transforms of imaginary Gaussians. We have

$$\int e^{i\tau Q(y)/2}a(0)\, dy = a(0)\left(\frac{2\pi}{\tau}\right)^{\frac{n}{2}}e^{i\,\mathrm{sgn}\,Q\pi/4},$$

where $\mathrm{sgn}\,Q$ is the signature of Q. (Since Q is diagonal, it's signature is simply the sum of the signs of its coefficients.)

In order to bound the second integral, we note that, since $\rho(y) - 1$ vanishes in a neighborhood of the origin in $\mathbf{R}^n$, we can write it as a sum

$$\rho(y) - 1 = \sum_{j=1}^n \rho_j(y),$$

1

where $\rho_j(y)$ vanishes identically in a neighborhood which contains an envelope of width ϵ about the $y_j = 0$ hyperplane, for some fixed ϵ. This lets us rewrite the second integral as a sum over j's,

$$\int e^{i\tau Q(y)/2}(\rho(y) - 1)a(0)\, dy = \sum_{j=1}^{n} \int e^{i\tau Q(y)/2} \rho_j(y) a(0)\, dy.$$

The jth integral can now be integrated by parts N times in the y_j variable to see that it is $O(\tau^{-N})$. This follows since $\rho_j(y)$ vanishes identically on the region about the $y_j = 0$, and so the results on page 26 can be used.

Finally, in order to bound the third integral, note that the function $y^j \rho(y) b_i(y)$ is smooth and compactly supported, and vanishes at $y^j = 0$. Thus we can integrate the jth integral by parts to obtain

$$\int e^{i\tau Q(y)/2} y^j \rho(y) b_j(y)\, dy = \pm \frac{1}{i\tau} \int e^{i\tau Q(y)/2} \frac{\partial}{\partial y^j} \left(\rho(y) b_j(y) \right)\, dy.$$

Now we notice that $\frac{\partial}{\partial y^j}\left(\rho(y) b_j(y) \right)\, dy$ is the same sort of function that $a(y)$ was to begin with (smooth, compactly supported). Thus we can iterate the procedure we have carried out on a so far to see that

$$\int e^{i\tau Q(y)/2} a(y)\, dy = a(0) \left(\frac{2\pi}{\tau} \right)^{\frac{n}{2}} e^{i \operatorname{sgn} Q \pi/4} + O\left(\frac{1}{\tau} \left(\left(\frac{2\pi}{\tau} \right)^{\frac{n}{2}} + \frac{1}{\tau} \left(\left(\frac{2\pi}{\tau} \right)^{\frac{n}{2}} + \cdots \right) \right) \right),$$

where we're using the fact that at any finite step of this iteration, we can just put in a big O term, since the integral $\int e^{i\tau Q(y)/2} c(y)\, dy$ is bounded for any of the c's in question (since c is smooth and compactly supported). We have also used the fact that the second integral in the original expansion is $O(\tau^{-N})$ for any N we like. By carrying out this iteration to at least $n/2 + 1$ steps, we see that the big O term is actually just $O\left(\tau^{-\frac{n}{2} - 1} \right)$, and so

$$\int e^{i\tau Q(y)/2} a(y)\, dy = a(0) \left(\frac{2\pi}{\tau} \right)^{\frac{n}{2}} e^{i \operatorname{sgn} Q \pi/4} + O\left(\tau^{-\frac{n}{2} - 1} \right),$$

as desired.

4. Make the change of variables $x \mapsto x/\tau$ to change

$$\frac{1}{\tau^n} f(0) g(\eta) = \frac{1}{(2\pi)^n} \int \int e^{i\tau x \cdot (\xi - \eta)} f(x) g(\xi)\, dx d\xi + O(\tau^{-(n+1)})$$

into

$$f(0) g(\eta) = \frac{1}{(2\pi)^n} \int \int e^{i x \cdot (\xi - \eta)} f(x/\tau) g(\xi)\, dx d\xi + O(1/\tau).$$

If we let $\tau \to \infty$, this is seen to give us

$$f(0) g(\eta) = f(0) \frac{1}{(2\pi)^n} \int \int e^{i x \cdot (\xi - \eta)} g(\xi)\, dx d\xi,$$

or just

$$g(\eta) = \frac{1}{(2\pi)^n} \int \int e^{i x \cdot (\xi - \eta)} g(\xi)\, dx d\xi,$$

as desired.

5. Note that we need to assume that $\Delta u = -k^2 u$ for this problem.

We have, by Green's formula and the equation $\Delta v + k^2 v = \delta_P$ that

$$-\frac{1}{4\pi} \int \int_{\partial D} \left[\frac{e^{ikr}}{r} * du - u * d\left(\frac{e^{ikr}}{r} \right) \right] = -\frac{1}{4\pi} \int \int \int_D \left(\frac{e^{ikr}}{r} \Delta u - u \Delta \left(\frac{e^{ikr}}{r} \right) \right)$$

$$= -\frac{1}{4\pi} \int \int \int_D \left(\frac{e^{ikr}}{r} (-k^2) u - u \left(-k^2 \frac{e^{ikr}}{r} + 4\pi \delta_P \right) \right)$$

$$= \int \int \int_D u \delta_P$$

$$= \begin{cases} u(P) & \text{if } P \in D, \\ 0 & \text{if } P \notin D. \end{cases}$$

2

6. We have

$$\left|\int\int_{S_R}\left(\frac{e^{ikr}}{r}*du-u*d\left(\frac{e^{ikr}}{r}\right)\right)\right|=\left|\int\int_{S_R}\left(\frac{e^{ikr}}{r}\frac{\partial u}{\partial n}-iku\frac{e^{ikr}}{r}\frac{\partial r}{\partial n}+u\frac{e^{ikr}}{r^2}\frac{\partial r}{\partial n}\right)\right|$$

$$=\left|\int\int_{S_R}\left(e^{ikr}\frac{\partial u}{\partial r}-ikue^{ikr}+u\frac{e^{ikr}}{r}\right)\right|$$

$$\leq\int\int_{S_R}\left|\frac{\partial u}{\partial r}-iku\right|+\frac{1}{R}\int\int_{S_R}|u|.$$

The Sommerfeld radiation conditions then show that this last line goes to zero as $R\to\infty$. Problem 5 and this result combine to give equation 51, as desired.

7. The critical points of $\phi+r$ on W are those points where $\nabla(\phi+r)$ is parallel to the outward unit normal to W.

By assumption, $\|\nabla\phi\|=1$. We also have $\|\nabla r\|=1$, by explicit calculation. At any particular point on the surface W, therefore, what we require is that $\nabla\phi+\nabla r$ be parallel to n. Since $nabla\phi$ and ∇r have length 1, this means that there are only two possible solutions, that for which $\nabla\phi=-\nabla r$, and that for which

$$\nabla\phi+\nabla r=2\langle\nabla\phi,n\rangle n.$$

Since

$$*d\phi=\frac{\partial\phi}{\partial n}=\langle\nabla\phi,n\rangle,$$

we thus see that either $*d\phi=-*dr$ or that $*d\phi=*dr$. In the second case, the factor

$$(*d\phi-*dr)$$

vanishes, and so the top order terms in the stationary phase formula vanish as well.