The Stone - von Neumann theorem.

Math 212

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Contents

This theorem asserts the uniqueness of the "representation of the Heisenberg commutation relations in Weyl form". For a precise statement see Theorem 2.1 below. We will give three different proofs of this important theorem. The first proof will be in these notes. In these notes we follow the exposition given in Lions and Vergne *The Weil representation* ... .

1 The Heisenberg algebra and Group.

Let V be a symplectic vector space. So V comes equipped with a skew symmetric non-degenerate bilinear form ω. By the choice of a pair of transverse Lagrangian subspaces, and then dual bases in these subspaces, we obtain a basis

$$P_1, \dots, P_n, Q_1, \dots, Q_n$$

of V with

$$\begin{aligned} \omega(P_i, P_j) &= 0 \\ \omega(Q_i, Q_j) &= 0 \\ \omega(P_i, Q_j) &= \delta_{ij}. \end{aligned} \tag{1}$$

We make

$$\mathbf{h} := V \oplus R$$

into a Lie algebra by defining

$$[X, Y] := \omega(X, Y)E$$

where $E = 1 \in \mathbf{R}$ and

$$[E, E] = 0 = [E, X] \quad \forall X \in V.$$

The Lie algebra $\mathbf{h}$ is called the **Heisenberg algebra**. It is a nilpotent Lie algebra. In fact, the Lie bracket of any three elements is zero. If we write out the brackets in terms of the basis above we get

$$\begin{aligned} [P_i, Q_j] &= \delta_{ij}E \\ [P_i, P_j] &= 0 \\ [Q_i, Q_j] &= 0 \end{aligned}$$

which, together with

$$[E, P_j] = 0 = [E, Q_j]$$

are the "canonical commutation relations" up to inessential (or essential) factors such as $\hbar$ and i.

We will let N denote the simply connected Lie group with this Lie algebra. We may identify the $2n+1$ dimensional vector space $V + \mathbf{R}$ with N via the exponential map, and with this identification the multiplication law on N reads

$$\exp(v + tE)\exp(v' + t'E) = \exp\left(v + v' + (t + t' + \frac{1}{2}\omega(v, v'))E\right). \qquad (2)$$

Let dv be the Euclidean (Lebesgue) measure on V. Then the measure $dvdt$ is invariant under left and right multiplication.

If ℓ is a Lagrangian subspace of V, then $\ell \oplus \mathbf{R}$ is an Abelian subalgebra of $\mathbf{h}$, and in fact is maximal abelian. Similarly

$$L := \exp(\ell \oplus \mathbf{R})$$

is a maximal Abelian subgroup of N.

Define the function

$$f : N \to T^1$$
$$f(\exp(v + tE)) := e^{2\pi i t}.$$

We have

$$f\left((\exp(v + tE))(\exp(v' + t'E))\right) = e^{2\pi i(t + t' + \frac{1}{2}\omega(v, v'))}. \qquad (3)$$

Therefore

$$f(h_1 h_2) = f(h_1)f(h_2)$$

for

$$h_1, \quad h_2 \in L.$$

We say that the restriction of f to L is a character of L.

I want to consider the quotient space

$$N/L$$

which has a natural action of N (via left multiplication). In other words N/L is a homogeneous space for the Heisenberg group N. Let ℓ' be a Lagrangian subspace transverse to ℓ. Every element of N has a unique expression as

$$(\exp y)(\exp(x+sE)) \quad \text{where } y \in \ell' \quad x \in \ell.$$

This allows us to make the identification

$$N/L \sim \ell'$$

and the Euclidean measure dv' on ℓ' then becomes identified with the (unique up to scalar multiple) measure on N/L invariant under N.

For use in the next section we record the following "commutation calculation" at the group level: Let $y \in \ell'$ and $x \in \ell$. Then

$$\exp(-x)(\exp y) = \exp(y - x - \frac{1}{2}\omega(x,y)E)$$

while

$$\exp(y)\exp(-x) = \exp(y - x - \frac{1}{2}\omega(y,x)E)$$

so, since ω is antisymmetric, we get

$$(\exp(-x))(\exp y) = (\exp y)(\exp(-x))\exp(-\omega(x,y)E). \tag{4}$$

2 The Schrodinger representation.

We continue with the notation of the preceding section. In particular, we have chosen a Lagrangian subspace ℓ, have the corresponding subgroup L and the quotient space N/L. We are going to construct a unitary representation of N which is known in group theory language as the representation of N induced from the character f of L.

Its definition is as follows: Consider the space of continuous functions ϕ on N which satisfy

$$\phi(nh) = f(h)^{-1}\phi(n) \quad \forall\, n \in N \quad h \in L \tag{5}$$

and which in addition have the property that the function

$$n \mapsto |\phi(n)|$$

(which is well defined on N/L on account of (5)) is square integrable. We let $H(\ell)$ denote the Hilbert space which is the completion of this space of continuous functions relative to this L_2 norm. So $\phi \in H(\ell)$ is a "function" on N satisfying (5) with norm

$$\|\phi\|^2 = \int_{N/L} |\phi|^2 d\dot{n}$$

where $d\dot{n}$ is left invariant measure on N/L.

The representation ρ_ℓ of N on $H(\ell)$ is given by left translation:

$$(\rho_\ell(m)\phi)(n) := \phi\left(m^{-1}n\right). \tag{6}$$

For the rest of this section we will keep ℓ fixed, and so may write H for $H(\ell)$ and ρ for ρ_ℓ. The dependence on ℓ will become important for us later.

Since $\exp tE$ is in the center of N, we have

$$\rho(\exp tE)\phi(n) = \phi\left((\exp -tE)n\right) = \phi\left(n(\exp -tE)\right) = e^{2\pi i t}\phi(n).$$

In other words

$$\rho\left(\exp tE\right) = e^{2\pi i t}\mathrm{Id}_H. \tag{7}$$

The Stone - von Neumann (Theorem 2.1 below) characterizes all unitary representations of N which satisfy this condition.

Suppose we choose a complementary Lagrangian subspace ℓ' and then identify N/L with ℓ' as in the preceding section. Condition (5) becomes

$$\phi\left((\exp y)(\exp(x))(\exp tE)\right) = \phi(\exp y)e^{-2\pi i t}.$$

So $\phi \in H$ is completely determined by its restriction to $\exp \ell'$. In other words the map

$$\phi \mapsto \psi, \quad \psi(y) := \phi(\exp y)$$

defines a unitary isomorphism

$$R : H \to L_2(\ell')$$

and if we set

$$\sigma := R\rho R^{-1}$$

then

$$\begin{array}{lcll} [\sigma\left(\exp x\right)\psi](y) & = & e^{2\pi i \omega(x,y)}\psi(y) & x \in \ell,\ y \in \ell' \\ [\sigma(\exp u)\psi](y) & = & \psi(y-u) & y, u \in \ell' \\ \sigma(\exp(tE)) & = & e^{2\pi i t}\mathrm{Id}_{L_2}(\ell') & \end{array} \tag{8}$$

We define the infinitesimal version of the representation ρ by

$$\dot{\rho}(X) := \frac{d}{dt}\rho(\exp(tX))_{|t=0}$$

for $X \in \mathbf{h}$ with a similar notion and notation for σ. Under the P, Q basis (with $P_i \in \ell$ chosen above), we may identify $L_2(\ell')$ with $L_2(\mathbf{R}^n)$. Then it follows from (8) that

$$\begin{aligned} \dot\sigma(P_j) &= 2\pi i x_j \\ \dot\sigma(Q_j) &= -\frac{\partial}{\partial x_j} \\ \dot\sigma(E) &= 2\pi i \mathrm{Id} \end{aligned} \tag{9}$$

This is the Schrodinger version of the Heisenberg commutation relations. So we can regard (8) as an "integrated version" of the Heisenberg commutation relations. The Stone-von Neumann theorem asserts that the representation σ, and hence the representation ρ is irreducible and is the unique irreducible representation (up to isomorphism) satisfying (8). In fact, to be more precise, the theorem asserts that any unitary representation of N such that

$$\exp(tE) \mapsto e^{2\pi i t}\mathrm{Id}$$

must be isomorphic to a **multiple** of ρ in the following sense:

Let H_1 and H_2 be Hilbert spaces. We can form their tensor product as vector spaces, and this tensor product inherits a scalar product determined by

$$(u \otimes v, x \otimes y) = (u, x)(v, y).$$

The completion of this (algebraic) tensor product with respect to this scalar product will be denoted by $H_1 \otimes H_2$ and will be called the (Hilbert space) tensor product of H_1 and H_2. If we have a representation τ of a group G on H_1 we get a representation

$$g \mapsto \tau(g) \otimes \mathrm{Id}_{H_2}$$

on $H_1 \otimes H_2$ which we call a multiple of the representation τ. We can now state:

Theorem 2.1 [The Stone-von-Neumann theorem.] *The representation $\rho(\ell)$ of N is irreducible, and any representation such that* $\exp(tE) \mapsto e^{2\pi i t}\mathrm{Id}$ *is isomorphic to a multiple of $\rho(\ell)$.*

We will spend the next few sections proving this basic theorem.

3 The group algebra.

If G is a topological group with a given choice of left invariant measure, we can define the convolution of two continuous functions of compact support on G by

$$(\phi_1 \star \phi_2)(g) := \int_G \phi_1(u)\phi_2(u^{-1}g)du.$$

If ψ is another continuous function on G we have

$$\int_G (\phi_1 \star \phi_2)(g)\psi(g)dg = \int_{G\times G} \phi_1(u)\phi_2(h)\psi(uh)dudh.$$

This right hand side makes sense if ϕ_1 and ϕ_2 are distributions of compact support and ψ is smooth. Also the left hand side makes sense if ϕ_1 and ϕ_2 belong to $L_1(G)$, etc.

If we have a continuous unitary representation τ of G on a Hilbert space H, we can define

$$\tau(\phi) := \int_G \phi(g)\tau(g)dg$$

which means that for u and $v \in H$

$$(\tau(\phi)u, v) = \int_G \phi(g)(\tau(g)u, v)dg.$$

This integral makes sense if ϕ is continuous and of compact support, or if G is a Lie group, if u is a C^∞ vector in the sense that $\tau(g)u$ is a C^∞ function of g and ϕ is a distribution. In either case we have

$$\tau(\phi_1 \star \phi_2) = \tau(\phi_1)\tau(\phi_2).$$

If the left invariant measure is also invariant under the map $g \mapsto g^{-1}$ and so right invariant, and if we define

$$\phi^*(g) := \overline{\phi(g^{-1})}$$

then

$$\tau(\phi^*) = \tau(\phi)^*.$$

4 The Weyl transform.

Let τ be a representation of N satisfying our condition

$$\tau(tE) = e^{2\pi it}\mathrm{Id}.$$

Then τ descends to a representation of

$$B := N/\exp(\mathbf{Z}E)$$

since $\tau(\exp(kE)) = \mathrm{Id}$ for $k \in \mathbf{Z}$.

Let Φ denote the collection of continuous functions on N which satisfy

$$\phi(n \exp tE) = e^{-2\pi it}\phi(n).$$

Every $\phi \in \Phi$ can be considered as a function on B, and every $n \in B$ has a unique expression as $n = (\exp v)(\exp tE)$ with $v \in V$ and $t \in \mathbf{R}/\mathbf{Z}$. We take as our left invariant measure on B the measure $dvdt$ where dv is Lebesgue measure on V and dt is the invariant measure on the circle with total measure one. The set of elements of Φ are then determined by their restriction to $\exp(V)$. Then for $\phi_1, \phi_2 \in \Phi$ of compact support we have (with $\star$ denoting convolution on B)

$$(\phi_1 \star \phi_2)(\exp v)$$

$$\begin{aligned}
&= \int_V \int_T \phi_1((\exp u)(\exp tE))\phi_2((-\exp u)(\exp(-tE))(\exp v))dudt \\
&= \int_V \phi_1(\exp u)\phi_2((\exp -u)(\exp v))du \\
&= \int_V \phi_1(\exp u)\phi_2(\exp(v-u)\exp(-\frac{1}{2}\omega(u,v)E))du \\
&= \int_V \phi_1(\exp u)\phi_2(\exp(v-u))e^{\pi i\omega(u,v)}du.
\end{aligned}$$

So if we use the notation

$$\psi(u) = \phi(\exp u)$$

and $\psi_1 \star \psi_2$ for the ψ corresponding to $\phi_1 \star \phi_2$ we have

$$(\psi_1 \star \psi_2)(v) = \int_V \psi_1(u)\psi_2(v-u)e^{\pi i\omega(u,v)}du. \tag{10}$$

We thus get a "twisted" convolution on V.

If $\phi \in \Phi$ then defining ϕ^* as above, then $\phi^* \in \Phi$ and the corresponding transformation on the ψ's is

$$\phi^*(v) = \overline{\psi(-v)}.$$

We now define

$$W_\tau(\psi) = \tau(\phi) = \int_B \phi(b)\tau(b)db = \int_V \psi(v)\tau(\exp v)dv.$$

The last equation holds because of the opposite transformation properties of τ and $\phi \in \Phi$.

If $\phi \in \Phi$ then $\delta_m \star \phi$ is given by

$$(\delta_m \star \phi)(n) = \phi(m^{-1}n)$$

which belongs to Φ if ϕ does and if $m = \exp(w)$ then

$$(\delta_m \star \phi)(\exp u) = e^{\pi i\omega(w,u)}\psi(u-w).$$

Similarly,

$$(\phi \star \delta_m)(\exp u) = e^{\pi i\omega(w,u)}\psi(u-w).$$

Let us write $w \star \psi$ for the function on V corresponding to $\delta_m \star \phi$ under our correspondence between elements of Φ and functions on V.

Then the facts that we have proved such as

$$\tau(\phi_1 \star \phi_2) = \tau(\phi_1)\tau(\phi_2)$$

translate into

$$\begin{aligned} W_\tau(\psi_1 \star \psi_2) &= W_\tau(\psi_1)W_\tau(\psi_2) & (11) \\ W_\tau(\psi^*) &= W_\tau(\psi)^* & (12) \\ W_\tau(w \star \psi) &= \tau(\exp w)W_\tau(\psi) & (13) \\ W_\tau(\psi \star w) &= W_\tau(\psi)\tau(\exp w). & (14) \end{aligned}$$

5 Hilbert-Schmidt Operators.

Let H be a Hilbert space. An operator A on H is called **Hilbert-Schmidt** if in terms of some orthonormal basis $\{e_i\}$ we have

$$\sum \|Ae_i\|^2 < \infty.$$

Since

$$Ae_i = \sum (Ae_i, e_j)e_j$$

this is the same as the condition

$$\sum_{ij} |(Ae_i, e_j)|^2 < \infty$$

or

$$\sum |a_{ij}|^2 < \infty$$

where

$$a_{ij} := (Ae_i, e_j)$$

is the matrix of A relative to the orthonormal basis. This condition and sum does not depend on the orthonormal basis and is denoted by

$$\|A\|^2.$$

This norm comes from the scalar product

$$(A, B)_{HS} = \operatorname{tr} B^* A = \sum (B^* Ae_i, e_i) = \sum (Ae_i, Be_i).$$

The rank one elements

$$E_{ij}, \quad E_{ij}(x) := (x, e_j)e_i$$

form an orthonormal basis of the space of Hilbert-Schmidt operators. We can identify the space of Hilbert-Schmidt operators with the tensor product $H \otimes \overline{H}$ where $\overline{H}$ is the space H with scalar multiplication and product given by the complex conjugate,e.g multiplication by $c \in \mathbf{C}$ is given by multiplication by $\overline{c}$ in H.

If $H = L_2(V, dy)$ (where V can be any measure space with measure dy, but we will be interested in our case) we can describe the space of Hilbert-Schmidt operators as follows: Let $\{e_i\}$ be an orthonormal basis of $H = L_2(V)$ and consider the rank one operators E_{ij} introduced above. Then

$$\begin{aligned}(E_{ij}\psi)(x) = (\psi, e_j)e_i(x) &= \int_V \psi(y)\overline{e_j(y)}e_i(x)dy \\ &= \int_Y K_{ij}(x,y)\psi(y)dy\end{aligned}$$

where

$$K_{ij}(x,y) = e_i(x)\overline{e_j(y)}.$$

This has norm one in $L_2(V \times V)$ and hence the most general Hilbert-Schmidt operator A is given by the $L_2(V \times V)$ kernel

$$K = \sum a_{ij} K_{ij}$$

with a_{ij} the matrix of A as above.

Let us consider that case where $\tau = \rho = \rho(\ell)$. I claim that the map W_ρ defined on the elements of Φ of compact support extends to an isomorphism from $L_2(V)$ to the space of Hilbert-Schmidt operators on $H(\ell)$. Indeed, write

$$W_\rho(\psi) = \int_V \psi(v)\rho(\exp v)dV$$

and decompose

$$V = \ell \oplus \ell'$$

$$v = y + x, \quad s \in \ell, \ y \in \ell'$$

so

$$\exp(y+x) = \exp(y)\exp(x)\exp(-\frac{1}{2}\omega(y,x))$$

so

$$\rho(\exp(y+x)) = \rho(y)\rho(x)e^{-i\pi\omega(y,x)}$$

and hence

$$W_\rho(\psi) = \int\int \psi(y+x)\rho(\exp y)\rho(\exp x)e^{-\pi i\omega(y,x)}dxdy.$$

So far the above would be true for any τ, not necessarily ρ. Now let us use the explicit realization of ρ as σ on $L_2(\mathbf{R}^n)$ in the form given in (8).

We obtain

$$[W_\sigma(\psi)(f)](\xi) = \int\int e^{-\pi i\omega(y,x)}\psi(y+x)e^{2\pi i\omega(x,\xi-y)}f(\xi-y)dxdy.$$

Making the change of variables $y \mapsto \xi - y$ this becomes

$$\int \int e^{-\pi i \omega(\xi - y, x)} e^{2\pi i \omega(x,y)} \psi(\xi - y + x) f(y) dy.$$

so if we define

$$K_\psi(\xi, y) := \int e^{\pi i \omega(x, y+\xi)} \psi(\xi - y + x) dx$$

we have

$$[W_\sigma(\psi) f](\xi) = \int K_{\psi(\xi,y)} f(y) dy.$$

Here we have identified ℓ' with $\mathbf{R}^n$ and $V = \ell' + \ell$ where ℓ is the dual space of ℓ' under ω. So if we consider the partial Fourier transform

$$\mathcal{F}_x : L_2(\ell' \oplus \ell) \to L_2(\ell' \oplus \ell')$$

$$(\mathcal{F}_x \psi)(y, \xi) = \int e^{-2\pi i \omega(x,\xi)} \psi(y + x) dx$$

(which is an isomorphism) we have

$$K_\psi(\xi, y) = (\mathcal{F}_x \psi)(\xi - y, -\frac{1}{2}(y + \xi)).$$

We thus see that the set of all K_ϕ is the set of all Hilbert-Schmidt operators on $L_2(\mathbf{R}^n)$.

Now if a bounded operator C commutes with all Hilbert-Schmidt operators on a Hilbert space, then $CE_{ij} = E_{ij}C$ implies that $c_{ij} = c\delta_{ij}$, i.e. $C = c\mathrm{Id}$. So we have proved that every bounded operator that commutes with all the $\rho_\ell(n)$ must be a constant. Thus $\rho(\ell)$ is irreducible.

6 Completion of the proof.

We fix ℓ, ℓ' as above, and have the representation ρ realized as σ on $L_2(\ell')$ which is identified with $L_2(\mathbf{R}^n)$ all as above. We want to prove that any representation τ satisfying (7) is isomorphic to a multiple of σ.

We consider the "twisted convolution" (10) on the space of Schwartz functions $\mathcal{S}(\mathcal{V})$. If $\psi \in \mathcal{S}(V)$ then its Weyl kernel $K_\psi(\xi, y)$ is a rapidly decreasing function of (ξ, y) and we get all operators with rapidly decreasing kernels as such images of the Weyl transform W_σ sending ψ into the kernel giving $\sigma(\phi)$.

Consider some function $u \in \mathcal{S}(\ell')$ with

$$\|u\|_{L_2(\ell')} = 1.$$

Let P_1 be the projection onto the line through u, so P_1 is given by the kernel

$$p_1(x, y) = \overline{u(y)} u(x).$$

We know that it is given as

$$p_1 = W_\sigma(\psi) \quad \text{for some} \ \ \psi \in \mathcal{S}(V).$$

Since $P_1^2 = P_1, P_1^* = P_1$ and

$$P_1\sigma(n)P_1 = \alpha(n)P_1 \quad \text{with} \quad \alpha(n) = (\sigma(n)u, u).$$

Recall that $\phi \mapsto \sigma(\phi)$ takes convolution into multiplication, and that K_{psi} is the kernel sending ψ into $\sigma(\phi)$ where $\phi \in \Phi$ corresponds to $\psi \in \mathcal{S}(V)$. Then in terms of our twisted convolution $\star$ given by (10) we have

$$\psi \star \psi = \psi, \quad \psi^* = \psi, \quad \psi \star n \star \psi = \alpha(n)\psi, \tag{15}$$

Let τ be any unitary representation of N on a Hilbert space H satisfying (7). We can form $W_\tau(\psi)$.

Lemma 6.1 *The set of linear combinations of the elements*

$$\tau(n)W_\tau(\psi)x, \quad x \in H, \quad n \in N$$

is dense in H.

Proof. Suppose that $y \in H$ is orthogonal to all such elements and set $n = \exp w$. Then

$$0 = (y, \tau(n)W_{tau}(\psi)\tau(n)^{-1}x) = \int_V (y, \tau(\exp w)\tau(\exp(v)\tau(\exp(-w)\psi(v)dv$$

$$= \int_V (y, \tau(\exp(v{+}B(w,v)E)x)\psi(v)dv = \int_V (y, \tau(\exp v)x)e^{-2\pi i\omega(w,v)}dv = \mathcal{F}[(y, \tau(\exp v)x)\psi].$$

The function is square brackets whose Fourier transform is being taken is continuous and rapidly vanishing. Since its Fourier transform vanishes, it must vanish. Since ψ does not vanish everywhere, there is some value v_0 with $\psi(v_0) \neq 0$, and hence

$$(y, \tau(\exp v_0)x) = 0 \quad \forall x \in H.$$

Writing $x = \tau(\exp v_0)^{-1}z$ we see that y is orthogonal to all of H and hence $y = 0$. QED

Now from equation (15) we see that $W_\tau(\psi)$ is an orthogonal projection onto a subspace, call it H_1 of H. We are going to show that H is isomorphic to $H(\ell) \otimes H_1$ as a HIlbert space and as a representation of N.

We wish to define

$$I : H(\ell) \otimes H_1 \to H, \quad \rho(n)u \otimes b \mapsto \tau(n)b$$

where $b \in H_1$.

We first check that if

$$b_1 = W_\tau(\psi)x_1 \quad \text{and} \quad b_2 = W_\tau(\psi)x_2$$

then for any $n_1, n_2 \in N$ we have

$$(\tau(n_1)W_\tau(\psi)x_1, \tau(n_2)W_\tau(\psi)x_2)_H = (\rho(n_1)u, \rho(n_2)u)_{H(\ell)}(b_1, b_2)_{H_1}. \qquad (16)$$

Proof. The left hand side equals

$$(\tau(n_1)W_\tau(\psi)x_1, \tau(n_2)W_\tau(\psi)x_2) = (W_\tau(\psi)\tau(n_2^{-1}n_1)W_\tau(\psi)x_1, x_2)$$
$$= \alpha(n_2^{-1}n_1)(W_\tau(\psi)x_1, x_2).$$

Since $W_\tau(\psi)$ is a projection and from the definition of α we have

$$\alpha(n_2^{-1}n_1) = (\rho(n_2^{-1}n_1)u, u) = (\rho(n_1u, \rho(n_2)u)$$

and

$$(W_\tau(\psi)x_1, x_2) = (W_\tau(\psi)x_1, W_\tau(\psi)x_2)_H == (w_1, w_2)_{H_1},$$

the last expression above equals the right hand side of (16). Now define

$$I: \quad \sum_{i=1}^{N} \rho(n_i)u \otimes b_i \mapsto \sum \tau(n_i)b_i.$$

This map is well defined, for if

$$\sum_{i=1}^{N} \rho(n_i)u \otimes b_i = 0$$

then

$$\|\sum_{i=1}^{N} \rho(n_i)u \otimes b_i\|_{H(\ell)\otimes H_1} = 0$$

and (16) then implies that

$$\|\sum_{i=1}^{N} \rho(n_i)u \otimes b_i\|_{H(\ell)\otimes H_1} = \|\sum_{i=1}^{N} \tau(n_i)b_i\| = 0.$$

Equation (16) also implies that the map I is an isometry where defined. Since ρ is irreducible, the elements $\sum_{i=1}^{N} \rho(n_i)u$ are dense in $H(\ell)$, and so I extends to an isometry from $H(\ell) \otimes H_1$ to H. By Lemma 6.1 this map is surjective. Hence I stands to a unitary isomorphism (which clearly is also a morphism of N modules) between $H(\ell) \otimes H_1$ and H. This completes the proof of the Stone - von Neumann Theorem.