The Symplectic Category.

Math 212b

March 19, 2001

Contents

1 Linear Lagrangian squares.

Let V and W be symplectic vector spaces with symplectic forms ω_V and ω_W. We put the direct sum symplectic form on $V \oplus W$ and denote it by $\omega_{V\oplus W}$. Let L be a Lagrangian subspace of V and set

$$H := L \oplus W.$$

Let Λ be a Lagrangian subspace of $V \oplus W$. Consider the exact square

$$\begin{array}{ccc} F & \longrightarrow & \Lambda \\ \big\downarrow & & \big\downarrow \iota_\Lambda \\ H & \underset{\iota_H}{\longrightarrow} & V \oplus W \end{array} \tag{1}$$

This means that we have the exact sequence

$$0 \to F \to H \oplus \Lambda \to V \oplus W \to \operatorname{coker}(\tau) \to 0 \tag{2}$$

where the middle map

$$\tau : H \oplus \Lambda \to V \oplus W$$

is given by

$$\tau(h, \lambda) = \iota_H(h) - \iota_\Lambda(\lambda).$$

Let $\mathrm{pr} : F \to \Lambda$ denote projection onto the second component and let $\rho : V \oplus W \to W$ denote the projection onto the second component (restricted to Λ) so

$$\alpha := \rho \circ \mathrm{pr} : F \to W. \tag{3}$$

Theorem 1 *The image of α is a Lagrangian subspace of W.*

Proof. If $w_1 = \alpha((v_1, w_1))$ and $w_2 = \alpha((v_2, w_2))$ then

$$\omega_W(w_1, w_2) = \omega_{V \oplus W}\left((v_1, w_1), (v_2, w_2)\right) - \omega_V(v_1, v_2) = 0.$$

The first term vanishes because Λ is Lagrangian, and the second term vanishes because L is Lagrangian. We have proved that α maps F onto an isotropic subspace of W. We want to prove that this subspace is Lagrangian. We do this be a dimension count:

We have the exact sequence

$$0 \to \ker(\alpha) \to F \to \operatorname{im}(\alpha) \to 0. \tag{4}$$

Write

$$\lambda = (v_1, w_1), \qquad h = (v_2, w_2)$$

so that $(h, \lambda) \in F$ when these two expressions are equal. To say that $\rho(\lambda) = 0$ means that $\lambda = (v, 0)$ so we may identify $\ker(\alpha)$ with the set of all $v \in L$ such that

$$(v, 0) \in \Lambda.$$

In this way we identify $\ker \alpha$ as a subspace of $V \oplus W$ consisting of all $(v, 0) \in \Lambda$ with $v \in L$.

On the other hand, $u \in \operatorname{im}(\tau)$ when

$$u = \iota_H(v_2, w_2) - \iota_\Lambda(v_1, w_1)$$

for

$$(v_1, w_1) \in \lambda, \quad (v_2, w_2) \in H.$$

So

$$\mathrm{im}(\tau) = H + \Lambda$$

and hence

$$\mathrm{im}(\tau)^{\perp} = H^{\perp} \cap \Lambda^{\perp}.$$

But $\Lambda^{\perp} = \Lambda$ since Λ is Lagrangian, and $H^{\perp} = L \oplus \{0\}$ since L is Lagrangian and $H = L \times W$. Hence

$$\ker(\alpha)^{\perp} = \mathrm{im}(\tau) \quad \text{in } V \oplus W.$$

In other words, the symplectic form on $V \oplus W$ induces a non-degenerate pairing between $\ker(\alpha)$ and $\mathrm{coker}(\tau)$. Thus we can write

$$\dim\ \mathrm{im}(\alpha) = \dim F - \dim \ker(\alpha) = \dim F - \dim\ \mathrm{coker}\,\tau.$$

From (2) we have

$$\dim F - \dim\ \mathrm{coker}(\tau) = \dim H + \dim \Lambda - \dim V - \dim W.$$

Since $\dim H = \frac{1}{2}\dim V + \dim W$ and $\dim \Lambda = \frac{1}{2}\dim V + \frac{1}{2}\dim W$ we obtain

$$\dim\ \alpha(F) = \frac{1}{2}\dim W$$

as desired. QED

From (2) it follows that $\mathrm{coker}(\tau) = \{0\}$ if and only if $H + \Lambda = V \oplus W$, in other words, the spaces H and Λ are transverse. We have thus proved

Proposition 1 *α is injective if and only if Λ and H are transverse.*

Whether or not α is injective, we may identify $\mathrm{im}(\alpha)$ with the set of all $w \in W$ such that there exists a $v \in V$ such that $(v, w) \in \Lambda$ with $v \in L$. In terms of the notation we shall introduce in the next section, it will be convenient to denote this Lagrangian subspace of W as $\Lambda \circ L$, and think of it as "the image of L under Λ".

2 The linear symplectic category.

Let X, Y, and Z be symplectic vector spaces with symplectic forms ω_X, ω_Y, and ω_Z. We will let X^- denote the vector space X equipped with the symplectic form $-\omega_X$. So $X^- \oplus Y$ denotes the vector space $X \oplus Y$ equipped with the symplectic form $-\omega_X \oplus \omega_Y$ and similarly $Y^- \oplus Z$ denotes the vector space $Y \oplus Z$ equipped with the symplectic form $-\omega_Y \oplus \omega_Z$. Let

$$\Lambda_1 \text{ be a Lagrangian subspace of } \quad X^- \oplus Y$$

and let

$$\Lambda_2 \text{ be a Lagrangian subspace of } \quad Y^- \oplus Z.$$

Consider the exact square

$$\begin{array}{ccc} F & \longrightarrow & \Lambda_2 \\ \big\downarrow & & \big\downarrow \\ \Lambda_1 & \longrightarrow & Y \end{array} . \tag{5}$$

This means that we have the exact sequence

$$0 \to F \to \Lambda_1 \oplus \Lambda_2 \to Y \to \operatorname{coker}(\tau) \to 0 \tag{6}$$

where

$$\tau : \Lambda_1 \oplus \Lambda_2 \to Y$$

is given by

$$\tau\left((u, y_1), (y_2, z)\right) = y_1 - y_2.$$

Thus the points of F consist of elements of the form

$$(x, y, y, z)$$

where $(x, y) \in \Lambda_1$ and $(y, z) \in \Lambda_2$. Let

$$\alpha : F \to X \oplus Z$$

be defined by

$$\alpha((x, v, v, z)) = (x, z).$$

The image of α is the set of all $(x, z) \in X \oplus Z$ such that there exists a $y \in Y$ with $(x, y) \in \Lambda_1$ and $(y, z) \in \Lambda_2$. If we think of Λ_1 and Λ_2 as *relations*, this set is what is usually called the composite of the two relations and is denoted by $\Lambda_2 \circ \Lambda_1$ Then we have

Theorem 2 *The composite $\Lambda_2 \circ \Lambda_1$ is a Lagrangian subspace of $X^- \oplus Z$.*

Proof. Take $V := Y \oplus Y$ with the symplectic form $\omega_{Y1} - \omega_{Y2}$ where ω_{Y2} denotes the pullback of ω_Y to $Y \oplus Y$ via projections onto the second factor, with ω_{Y1} the pullback via projection onto the first factor. Take $L = \Delta$ to be the diagonal in $Y \oplus Y$ so $\Delta := \{(y, y)\}$. Take $W = X \oplus Z$. Identify $V \oplus W$ with $X \oplus Y \oplus Y \oplus Z$ (by putting the two Y components in the middle), and let $\Lambda := \Lambda_1 \oplus \Lambda_2$. So in the terminology Theorem 1, $H = \{(x, y, y, z)\}$ and $\alpha(F) = \Lambda_2 \circ \Lambda_1$. Thus Theorem 2 is a consequence of Theorem 1. QED

Theorem 2 means that we get a category if we take as our objects the symplectic vector spaces, and, if X and Y are symplectic vectors spaces define the morphisms from X to Y to consist of the Lagrangian subspaces of $X^- \oplus Y$.

This category is a vast generalization of the symplectic group because of the following observation: Suppose that the Lagrangian subspace $\Lambda \subset X^- \oplus Y$ projects bijectively onto X under the projection of $X \oplus Y$ onto the first factor. This means that Λ is the graph of a linear transformation T from X to Y:

$$\Lambda = \{(x, Tx)\}.$$

T must be injective since if $Tx = 0$ the fact that Λ is isotropic implies that $x \perp X$ so $x = 0$. Also T is surjective since if $y \perp \mathrm{im}(T)$, then $(0, y) \perp \Lambda$, implying that $(0, y) \in \Lambda$ since Λ is maximal isotropic, and by the bijectivity of the projection this imp;lies that $y = 0$. In other words T is a bijection. The fact that Λ is isotropic then says that

$$\omega_Y(Tx_1, Tx_2) = \omega(x_1, x_2),$$

i.e. T is a symplectic isomorphism. If $\Lambda_1 = \mathrm{graph}\, T$ and $\Lambda_2 = \mathrm{graph}\, S$ then

$$\Lambda_2 \circ \Lambda_1 = \mathrm{graph}\, S \circ T$$

so composition of Lagrangian relations reduces to composition of symplectic isomorphisms in the case of graphs. In particular, If we take $Y = X$ we see that $\mathrm{Symp}(X)$ is a subgroup of Morph (X, X) in our category.

3 Clean squares.

We recall Bott's definition: If X, Y, and Z are smooth manifolds, with $f : X \to Z$ and $g : Y \to Z$ smooth maps, we say that the square

$$\begin{array}{ccc} W & \longrightarrow & Y \\ \downarrow & & \downarrow g \\ X & \xrightarrow[f]{} & Z \end{array}$$

is **clean** if $W := \{(x, y) | f(x) = g(y)\}$ is a submanifold of $X \times Y$ and if at every $w \in W$ the square

$$\begin{array}{ccc} TW_w & \longrightarrow & TY_y \\ \downarrow & & \downarrow dg_y \\ TX_x & \xrightarrow[df_x]{} & TZ_z \end{array}$$

is exact, where $w = (x, y)$ and $z = f(x) = g(y)$. If the maps f and g are transversal, then the corresponding square is clean. But there may be some situations (for example the presence of a group action) where we only have clean intersections.

We can immediately translate Theorems 1 and 2 into assertions about symplectic manifolds and their Lagrangian submanifolds:

Theorem 3 *i) Let X and Y be symplectic manifolds, L a Lagrangian submanifold of X and Λ a Lagrangian submanifold of $X \times Y$. Let $H = L \times Y$. If*

$$\begin{array}{ccc} F & \longrightarrow & \Lambda \\ \downarrow & & \downarrow \iota_\Lambda \\ H & \xrightarrow[\iota_H]{} & X \times Y \end{array} \tag{7}$$

is a clean square, then the image of F under the composite map

$$\alpha = \rho \circ \mathrm{pr}$$

is an immersed Lagrangian submanifold of Y. Here $\mathrm{pr} : F \to \Lambda$ *denotes projection onto the second factor Λ restricted to F and ρ denotes the map Λ to Y.*

ii) Let Λ_1 be a Lagrangian submanifold of $X^- \times Y$ and Λ_2 be a Lagrangian submanifold of $Y^- \times Z$. If (5) is a clean square, then $\Lambda_2 \circ \Lambda_1$ is an immersed Lagrangian submanifold of $X^- \times Z$.

4 Pullback of Lagrangian submanifolds of the cotangent bundle.

Let M and N be differentiable manifolds, and

$$f : M \to N$$

be a differentiable map. If B is a subset of T^*N, we define its pullback df^*B under f as the subset of T^*M given by

$$df^*(B) := \{(x, \xi) | \exists\ b = (y, \eta) \in B \text{ with } f(x) = y,\ df_x^*\eta = \xi\}. \tag{8}$$

Suppose that $B = L$ is a Lagrangian submanifold of T^*N. We wish to give sufficient conditions on f which guarantee that f^*L is a Lagrangian submanifold of T^*M. For this let us write $\operatorname{graph} f \subset N \times M$:

$$\operatorname{graph} f = \{(f(m), m)\}.$$

Let us set

$$X = T^*N, \quad Y = T^*M.$$

The normal bundle $\mathcal{N}(W)$ to a submanifold W of any manifold Z (in the cotangent bundle) is always a Lagrangian submanifold. Take

$$\Lambda := \mathcal{N}(\operatorname{graph} f) \subset T^*(N \times M) = X \times Y.$$

Thus a point of Λ is of the form

$$(f(x), \eta, x, -df^*\eta).$$

A point of $H = L \times Y$ is of the form

$$(y, \eta, x, \xi), \quad (y, \eta) \in L.$$

So if we form the square (7), a point of F is of the form

$$(f(x), \eta, x, -df_x^*\eta)$$

and hence

$$\alpha(F) = -df^*(L).$$

Now multiplication by -1 carries Lagrangian submanifolds of T^*M into Lagrangian submanifolds. We conclude (interchange some orders in products):

Theorem 4 *Let $f : M \to N$ be a smooth map and L a Lagrangian submanifold of T^*N. If $T^*M \times L$ and $\mathcal{N}(\text{graph}\, f)$ intersect cleanly in $T^*(M \times N)$ then df^*L is a Lagrangian submanifold of T^*M.*

A special case of clean intersection is transverse intersection. Let us examine when this happens: The points of H were described as all $(y, \eta, x, \xi), \quad (y, \eta) \in L$. Here (x, ξ) can vary freely. The points of Λ were described as all $(f(x), \eta, x, -df_x^*\eta)$ where η can vary freely in $T^*N_{f(x)}$. So the issue of transversality comes down to the behavior of the first component, i.e. H and Λ are transverse if and only if the maps

$$f : M \to N \quad \text{and } \pi : L \to N$$

are transverse. So we have

Proposition 2 *Let $f : M \to N$ be a smooth map and L a Lagrangian submanifold of T^*N. If the maps f, and the restriction of the projection $\pi : T^*N \to N$ to L are transverse, then df^*L is a Lagrangian submanifold of T^*M.*

Here are two examples of the proposition:

- Suppose that $\pi : T^*N \to N$ is a diffeomorphism when restricted to the Lagrangian submanifold L. This means that L is given by a section: $\beta : N \to T^*N$, where β assigns a covector to each point of N, i.e. β is a one form on N. The restriction of the canonical one form α to L then pulls back to a one form on N which is just β. The fact that L is Lagrangian asserts that β is closed, so locally β is the differential of a function ψ on N. Then $df^*(L)$ consists of all covectors on M of the form $(x, df_x^*\beta(f(x)))$, i.e. is given by the section of T^*M given by $f^*\beta$ which is locally of the form $d(f^*\psi)$. In this case the pull-back of the Lagrangian manifold L corresponds (locally) to the pull back of a function.

- Suppose that $L = \mathcal{N}(S)$ is the normal bundle to a submanifold S of N. The transversality condition becomes the condition that the map f is transversal to S. Then $f^{-1}(S)$ is a submanifold of M. If $x \in f^{-1}(S)$ and $\xi = df_x^*\eta$ with $(f(x), \eta) \in \mathcal{N}(S)$ then ξ vanishes when restricted to $T(f^{-1}(S))$, i.e. $(x, \xi) \in \mathcal{N}(f^{-1}(S))$. More precisely, the transversality asserts that at each $x \in f^{-1}(S)$ we have $df_x(TM_x) + TS_{f(x)} = TN_x$ so

  $$TM_x / T(f^{-1}(S))_x \cong TN_{f(x))} / TS_{f(x)}$$

  and so we have an isomorphism of the dual spaces

  $$\mathcal{N}_x(f^{-1}(S)) \cong \mathcal{N}_{f(x)}(S).$$

  In short, the pullback of $\mathcal{N}(S)$ is $\mathcal{N}(f^{-1}(S))$.

5 Pushforward of Lagrangian submanifolds of the cotangent bundle.

Let $f : M \to N$ be a smooth map and A a subset of T^*M. The pushforward $df_* A \subset T^*N$ is defined by

$$df_* A = \{(y, \eta) \in T^*N | \exists (x, \xi) \in A \text{ with } y = f(x) \text{ and } (x, df_x^* \eta) \in A\}.$$

We wish to have conditions that say that if $A = L$ is a Lagrangian submanifold of T^*M then $df_* L$ is a Lagrangian submanifold of T^*N. This time think of graph(f) in the usual way;

$$\text{graph}(f) = \{(x, f(x))\}$$

so that

$$\mathcal{N}(\text{graph}(f)) = \{(x, f(x), -df_x^* \eta, \eta)\}$$

where $\eta \in T^*N_{f(x)}$, and think of this as a subset of $X \times Y$ where $X = T^*M$ and $Y = T^*N$ (by permuting the second and third variables above). Then the H in Theorem 3 is given by

$$H = \{(x, y, \xi, \eta) | \ (x, \xi) \in L\}$$

so that $\alpha(F) = -df_*(L)$. We thus have

Theorem 5 *If $L \times T^*N$ intersects $\mathcal{N}(\text{graph}(f))$ cleanly, then $df_*(L)$ is a Lagrangian submanifold of T^*N.*

If f has constant rank, then the dimension of $df_x^* T^*N_{f(x)}$ does not vary, so that $df^*(T^*N)$ is a sub-bundle of T^*M. In H, the variables x, y and η vary freely, so if the intersection of L with $df^*(T^*N)$ is transverse, the conditions of the theorem are fulfilled. We have

Theorem 6 *Suppose that $f : M \to N$ has constant rank. If L is a Lagrangian submanifold of T^*M which intersects df^*T^*N transversaly then $df_* L$ is a Lagrangian submanifold of T^*N.*

For example, if f is an immersion, then $df^*T^*N = T^*M$ so all Lagrangian submanifolds are transverse to df^*T^*N.

Corollary 1 *If f is an immersion, then $df_* L$ is a Lagrangian submanifold of T^*N.*

At the other extreme, suppose that $f : M \to N$ is a fibration. Then $\mathcal{H} := df^*T^*N$ consists of the "horizontal sub-bundle", i.e those covectors which vanish when restricted to the tangent space to the fiber. So

Corollary 2 *Let $f : M \to N$ be a fibration, and let $\mathcal{H}$ be the bundle of the horizontal covectors in T^*M. If L is a Lagrangian submanifold of T^*M which intersects $\mathcal{H}$ transversaly, then $df_* L$ is a Lagrangian submanifold of T^*N.*

An important special case of this corollary for us will be when $L = \text{graph } d\phi$. Then $L \cap \mathcal{H}$ consists of those points where the "vertical derivative", i.e. the derivative in the fiber direction vanishes. At such points $d\phi$ descends to give a covector at $n = f(m)$. If the intersection is transverse, the set of such covectors is then a Lagrangian submanifold of T^*N.

One way of describing this construction is as follows: Consider the Lagrangian submanifold of $T^*(\mathbf{R})$ given by graph dt where t is the standard coordinate on $\mathbf{R}$. Then $L = \text{graph}\, d\phi = d\phi^*(\ \Lambda_1)$ and hence we can write the image of L as $df_* d\phi^* \Lambda_1$ assuming the necessary transversality. We will see that every Lagrangian submanifold locally has such a parametrization.

If $N = N \times S$ and we write a typical element of M as (x, θ) then the above construction yields $df_* L$ as the set of covectors of the form $d\phi_{x,\theta} = d\phi_x \phi$ at points where $d_\theta \phi = 0$.

6 Envelopes.

Suppose that instead of Λ_1 as above we take

$$\Lambda_0 := \mathcal{N}(0) \subset T^*(\mathbf{R})$$

and suppose that $\phi : M \to \mathbf{R}$ is transversal to $\{0\}$, i.e. that 0 is a regular value of ϕ. Then

$$d\phi^* \Lambda_0 = \mathcal{N}(\phi^{-1}(0)).$$

If this intersects $\mathcal{H}$ transversaly, then

$$df_* \left(\mathcal{N}(\phi^{-1}(0)) \right)$$

is a Lagrangian submanifold of T^*N. This construction generalizes the classical notion of an *envelope* which has more or less disappeared from the standard curriculum:

Suppose that $M = N \times S$ where S is some auxiliary "parameter space". We have assumed that $\phi : N \times S \to \mathbf{R}$ has 0 as a regular value, so that

$$Z := \phi^{-1}(0)$$

is a hypersurface of codimension one in $N \times S$. Let $\phi_s : N \to \mathbf{R}$ be the map obtained by holding s fixed:

$$\phi_s(x) := \phi(x, s).$$

Suppose we make the stronger assumption that each ϕ_s has 0 as a regular value, so that

$$Z_s := \phi_s^{-1}(0) = Z \cap (N \times \{s\})$$

is a submanifold and

$$Z = \bigcup_s Z_s$$

as a set. The Lagrangian submanifold $d\phi^*\Lambda_0 = \mathcal{N}(Z)$ consists of all points of the form

$$(x, s, td\phi_N(x, s), td_S\phi(x, s)) \quad \text{such that } \phi(x, s) = 0.$$

Here t is an arbitrary real number. The sub-bundle $\mathcal{H}$ consists of all points of the form

$$(x, s, \xi, 0).$$

So the transversality condition asserts that the map

$$z \mapsto d\left(\frac{\partial \phi}{\partial s}\right)$$

have rank equal to dim S on Z. The image Lagrangian submanifold $df_*(\mathcal{N}(Z))$ then consists of all covectors $td_N\phi$ where

$$\phi(x, s) = 0 \quad \text{and} \quad \frac{\partial \phi}{\partial s}(x, s) = 0,$$

a system of $p+1$ equations in $n+p$ variables, where $p = \dim S$. Our transversality assumptions say that these equations define a submanifold of $N \times S$. If we make the stronger hypothesis that the last p equations can be solve for s as a function of x, then the first equation becomes

$$\phi(x, s(x)) = 0$$

which defines a hypersurface $\mathcal{E}$ called the envelope of the surfaces Z_s. Furthermore, by the chain rule,

$$d\phi(\cdot, s(\cdot)) = d_N\phi(\cdot, s(\cdot)) + d_S\phi(\cdot, s(\cdot))d_N s(\cdot) = d_N\phi(\cdot, s(\cdot))$$

since $d_S\phi = 0$ at the points being considered. So if we set

$$\psi := \phi(\cdot, s(\cdot))$$

we see that under these restrictive hypotheses $df_*(\mathcal{N}(Z))$ consists of all multiples of $d\psi$, i.e.

$$df_*(\mathcal{N}(Z)) = \mathcal{N}(\mathcal{E})$$

is the normal bundle to the envelope. In the classical theory, the envelope "develops singularities". But form our point of view it is natural to consider the Lagrangian submanifold $df_*(Z)$. This will not be globally a normal bundle to a hypersurface because its projection on N (from T^*N) may have singularities. But as a submanifold of T^*N it is fine:
Examples:

- Suppose that S is an oriented curve in the plane, and at each point $s \in S$ we draw the normal ray to S at s. We might think of this line as a light ray propagating down the normal. The initial curve is called an "initial wave front" and the curve along which the the light tends to focus is called the "caustic". Focusing takes place where "nearby normals intersect" i.e. at the envelope of the family of rays. These are the points which are the loci of the centers of curvature of the curve, and the corresponding curve is called the evolute.

- We can let S be a hypersurface in n- dimensions, say a surface in three dimensions. We can consider a family of lines emanating from a point source (possible at infinity), and reflected by by S. The corresponding envelope is called the "caustic by reflection". In Descartes' famous theory of the rainbow he considered a family of parallel lines (light rays from the sun) which were refracted on entering a spherical raindrop, internally reflected by the opposite side and refracted again when exiting the raindrop. The corresponding "caustic" is the Descartes cone of 42 degrees.
- If S is a submanifold of $\mathbf{R}^n$ we can consider the set of spheres of radius r centered at points of S. The corresponding envelope consist of "all points at distance r from S". But this develops singularities past the radii of curvature. Again, from the Lagrangian point of view there is no problem.

7 Local parameterizations of Lagrangian submanifolds.

See *Geometric Asymptotics* pp. 154-160

8 Homogeneous Lagrangian submanifolds.

See *Geometric Asymptotics* pp. 160-165 and 349-352.

9 The wave front set according to Sato and Hormander.

9.1 In $\mathbf{R}^n$.

Let X be an open set in $\mathbf{R}^n$. Let u be distribution defined on X. A point $(x_0, \xi_0) \in X \times (\mathbf{R}^n \setminus 0)$ is called a **smooth codirection** for u if, for some neighborhood U of x_0 and V of ξ_0 we have for each smooth ϕ of compact support in U (we write $\phi \in C_0^\infty(U)$) we have

$$\mathcal{F}(\phi u)(\tau\xi) = O(\tau^{-N}) \text{ as } \tau \to \infty \;\text{ uniformly in } \xi \in V. \tag{9}$$

The set of smooth codirections is obviously an open conical set in $X \times (\mathbf{R}^n \setminus 0)$. Its complement is called the **wave front set** of u and is denoted by $WF(u)$.

Proposition 3 *(x_0, ξ_0) is a smooth codirection for u if and only if for any real valued function $\psi(x, a)$ of $(x, a) \in \mathbf{R}^n \times \mathbf{R}^p$ with*

$$d_x\psi(x_0, a_0) = \xi_0$$

there is an open neighborhood U_0 of x_0 and A_0 of a_0 such that for any $\phi \in C_0^\infty(U_0)$ we have

$$\langle u, e^{-i\tau\psi(\cdot,a)}, \phi\rangle = O(\tau^{-N})$$

uniformly in $a \in A_0$.

Proof. Clearly if this condition holds, we can take $p = n$ and $\psi(x, a) = x \cdot a$ so that (9) holds. For the converse we use the method of stationary phase: Suppose that (x_0, ξ_0) is a smooth codirection for u. Choose a $\phi' \in C_0^\infty(U_0)$ which is identically one on a neighborhood of $\mathrm{Supp}(\phi)$. By the Plancherel formula,

$$\begin{aligned} \langle u, e^{-i\tau\psi(\cdot,a)}\phi\rangle &= \langle \mathcal{F}(\phi u), \mathcal{F}^{-1}(e^{-i\tau\psi(\cdot,a)})\rangle \\ &= (2\pi)^{-n} \int\int e^{i\xi\cdot x} e^{-i\tau\psi(x,a)} \phi'(x)\mathcal{F}(\phi u)(\xi) dx d\xi. \end{aligned}$$

In this last integral make the change of variables $\xi \mapsto \tau\xi$. The integral becomes

$$(2\pi)^{-n} \int\int e^{i\tau[\xi\cdot x - \psi(x,a)]} \phi'(x)\mathcal{F}(\phi u)(\tau\xi) dx d\xi.$$

Consider the integral with respect to x, i.e. set

$$I(\tau, \xi, a) := \int e^{i\tau[\xi\cdot x - \psi(x,a)]} \phi'(x) dx.$$

For U_0, V_0 small enough, we have

$$|d_x\psi(x, a) - \xi| > \epsilon > 0$$

if

$$(x, a) \in U_0 \times A_0, \quad \xi \notin V$$

where V is the neighborhood of ξ_0 in the definition of being a smooth codirection. So if we let Q denote the vector field

$$Q := |\xi - d_x\psi|^{-2}(\xi - d_x\psi) \cdot \frac{\partial}{\partial x}$$

then repeated integration by parts gives

$$|I(\tau, \xi, a)| \le C_k\tau^{-k}(1 + |\xi|)^{-k}, \quad a \in A_0,\ \xi \notin V.\ \ \tau > 1.$$

Since ϕu is a distribution of compact support, its Fourier transform grows at most polynomially in all directions, and so is overcome by the above in directions $\xi \notin V$. But for $\xi \in V$ we are assuming that $|\mathcal{F}(\phi u)| \le C_k'\tau^{-k}$. This proves the proposition.

9.2 On manifolds.

If u is a generalized section of a vector bundle over a manifold, we may take the condition in Prop. 3 as the *definition* of the smooth codirections in the cotangent bundle (where now ϕ has to be a section of the appropriate dual bundle). They form an open cone in $T^*X \setminus 0$ whose complement is the wave front set of u.

9.3 Distributions defined by oscillatory integrals.

Let X be an open set in $\mathbf{R}^n$ and θ range over $\mathbf{R}^p \setminus 0$. Let $\phi(x,\theta)$ be positively homogeneous of degree one in θ meaning that

$$\phi(x, t\theta) = t\phi(x,\theta) \quad \text{for} \quad t > 0.$$

We will also assume that ϕ has no critical points as a function of (x,θ).

In what follows we will let L (with a possible subscript) denote a vector field in (x,θ) space which is homogeneous of degree -1 in θ. (For example $\partial/\partial\theta_k$.) Let us call a function $a = a(x,\theta)$ a **symbol** of degree μ if

$$[L_1 \ldots L_k a]\,(a, \tau\theta) = O(\tau^{\mu-k})$$

for any k vector fields of degree -1. For example, if a were homogeneous of degree μ it would satisfy this condition.

Suppose that $w \in C_0^\infty(X)$ and consider the integral

$$I_\phi(aw) := \int\int e^{i\phi(x,\theta)} a(x,\theta) w(x) dx d\theta. \tag{10}$$

A priori, this integral is only defined for symbols a which vanish when θ is large. But we can extend so that it is defined as a linear function of w, i.e. as a distribution for any symbol. We do so as follows: Choose a vector field L homogeneous of degree -1 such that

$$L\phi \equiv 1.$$

We can find such a vector field locally on the sphere bundle $|\theta| = 1$ by the hypothesis that $d\phi \neq 0$. We can then piece it together on the entire sphere bundle by a partition of unity, and then extend it off the sphere so as to be homogeneous of degree -1. since $L\phi$ is homogeneous of degree zero, $L\phi = 1$ everywhere on $X \times (\mathbf{R}^p \setminus 0)$.

Let $\chi \in C^\infty(X \times \mathbf{R}^p)$ be such that $\chi = 0$ for $|\theta| \leq \frac{1}{2}$ and $\chi \equiv 1$ for $|\theta| \geq 1$. Let

$$M = \frac{1}{i}\chi L + (1 - \chi).$$

Then M is a smooth first order differential operator defined on all of $X \times \mathbf{R}^p$ and

$$Me^{i\phi} = e^{i\phi}.$$

Let K be the transpose of M. Then K maps a symbol of degree μ into a symbol of degree $\mu - 1$, and we have

$$I(aw) = \int\int e^{i\phi} K^r(aw) dx d\theta \tag{11}$$

for any integer r. If we choose r large enough so that $\mu - r < -p$ the above integral converges for all symbols of degree μ. So we have given a meaning to (10) valid for any symbol by means of (11), and this defines a generalized function, call it u. In the above construction, if ϕ depends continuously on some other parameter, t, so does the distribution u.

Claim

$$WF(u) \subset d\pi_* d\phi^* \Lambda_1 \tag{12}$$

Recall that this is the set of all $d_x\phi(x,\theta)$ such that $d_\theta\phi(x,\theta) = 0$. In view of the homogeneity of ϕ, this is a homogeneous Lagrangian submanifold of T^*X, i.e. stable under multiplication by $T > 0$. We have to estimate

$$\begin{aligned} \langle u, e^{-i\tau\psi} w\rangle &= \int\int e^{i[\phi(x,\theta-\tau\psi(x,\sigma)]} b(x,\theta) w'(x) dx d\theta \\ &= \tau^p \int\int e^{i\tau[\phi(x,\theta)-\psi(x,sigma)} b(x,\tau\theta) w'(x) dx d\theta. \end{aligned}$$

Here we have replaced a and w by b and w' after enough integration by parts as in (11).

We need to show that this is rapidly decreasing in τ uniformly in σ in a neighborhood of σ_0 if $\operatorname{Supp} w$ is contained in a neighborhood U of x_0 and

$$d_x\psi(x_0,\sigma_0) \neq d_x\phi(x_0,\theta) \text{ with } \ d_\theta\phi(x_0,\theta) = 0.$$

If we write

$$\rho = \rho(x,\theta,\sigma) = \phi(x,\theta) - \psi(x,\sigma)$$

then

$$d_{(x,\theta)}\rho \neq 0$$

and we may apply the method of stationary phase to conclude that this integral vanishes faster than any inverse power of τ. QED