Elliptic Engineering

Math 212b

January 18, 2001

Contents

I plan to study differential operators acting on vector bundles over manifolds. But it requires some effort to set things up, and I want to get to the key analytic ideas which are essentially repeated applications of integration by parts. So I will start with elliptic operators L acting on functions on the torus $\mathbf{T} = \mathbf{T}^n$, where there are no boundary terms when we integrate by parts. Then an immediate extension gives the result for elliptic operators on functions on manifolds, and also for boundary value problems such as the Dirichlet problem.

The key result is that the resolvent $R(L, z)$ is a compact operator if $z = -\lambda$ for λ sufficiently large, see Theorem 4 and that the corresponding eigenvectors are C^∞ functions, see Theorem 5.

The treatment here rather slavishly follows the treatment by Bers and Schechter in *Partial Differential Differential Equations* by Bers, John and Schecter AMS (1964).

1 The Sobolev spaces.

Let $\mathbf{P} = \mathbf{P}(\mathbf{T})$ denote the space of trigonometric polynomials. These are functions on the torus of the form

$$u(x) = \sum a_\ell e^{i\ell \cdot x}$$

where

$$\ell = (\ell_1, \ldots, \ell_n)$$

is an n-tuplet of integers and the sum is finite. For each integer t (positive, zero or negative) we introduce the scalar product

$$(u, v)_t := (2\pi)^n \sum_\ell (1 + \ell \cdot \ell)^t a_\ell \overline{b}_\ell. \tag{1}$$

For $t = 0$ this is the usual scalar product

$$(u, v)_0 = \int_{\mathbf{T}} u(x)\overline{v(x)}dx.$$

We will denote the norm corresponding to the scalar product $(\ ,\ )_s$ by $\|\ \|_s$. We then get the "generalized Schwartz inequality"

$$|(u, v)_s| \leq \|u\|_{s+t}\|v\|_{s-t} \tag{2}$$

for any t. This reduces to the usual Cauchy-Schwartz inequality when $t = 0$.

Clearly we have

$$\|u\|_s \leq \|u\|_t \quad \text{if } s \leq t.$$

If D^p denotes a partial derivative,

$$D = \frac{\partial^{|p|}}{\partial(x^1)^{p_1} \cdots \partial(x^n)^{p_m}}$$

then

$$D^p u = \sum (i\ell)^p a_\ell e^{i\ell \cdot x}$$

in the obvious notation, and hence

$$\|D^p u\|_t \leq \|u\|_{t+|p|} \tag{3}$$

and similarly

$$\|u\|_t \leq (\text{constant depending on } t) \sum_{|p| \leq t} \|D^p u\|_0 \quad \text{if } t \geq 0. \tag{4}$$

In particular,

Proposition 1 *The norms*

$$u \mapsto \|u\|_t$$

$t \geq 0$ *and*

$$u \mapsto \sum_{|p| \leq t} \|D^p\|_0$$

are equivalent.

If

$$\Delta = -\left(\frac{\partial^2}{\partial(x^1)^2} + \cdots + \frac{\partial^2}{\partial(x^n)^2}\right)$$

the operator $(1+\Delta)$ satisfies

$$(1+\Delta)u = \sum (1+\ell\cdot\ell)a_\ell e^{i\ell\cdot x}$$

and so

$$((1+\Delta)^t u, v)_s = (u, (1+\Delta)^t v)_s = (u,v)_{s+t}$$

and

$$\|(1+\Delta)^t u\|_s = \|u\|_{s+2t}. \tag{5}$$

We let $\mathbf{H}_t$ denote the completion of the space $\mathbf{P}$ with respect to the norm $\|\ \|_t$. Each $\mathbf{H}_t$ is a Hilbert space, and we have natural embeddings

$$\mathbf{H}_t \hookrightarrow \mathbf{H}_s \quad \text{if } s < t.$$

Equation (5) says that

$$(1+\Delta)^t : \mathbf{H}_{s+2t} \to \mathbf{H}_s$$

and is an isometry.

¿From the generalized Schwartz inequality we also have a natural pairing of $\mathbf{H}_t$ with $\mathbf{H}_{-t}$ given by the extension of $(\ ,\ )_0$, so

$$|(u,v)_0| \le \|u\|_t \|v\|_{-t}.$$

In fact, this pairing allows us to identify $\mathbf{H}_{-t}$ with the space of continuous linear functions on $\mathbf{H}_t$. Indeed, if ϕ is a continuous linear function on $\mathbf{H}_t$ the Riesz representation theorem tells us that there is a $w \in \mathbf{H}_t$ such that $\phi(u) = (u,w)_t$. Set

$$v := (1+\Delta)^t w.$$

Then

$$v \in \mathbf{H}_{-t}$$

and

$$(u,v)_0 = (u, (1+\Delta)^t w)_0 = (u,w)_t = \phi(u).$$

We record this fact as

$$\mathbf{H}_{-t} = (\mathbf{H}_t)^*. \tag{6}$$

We set

$$\mathbf{H}_\infty := \bigcap \mathbf{H}_t, \qquad \mathbf{H}_{-\infty} := \bigcup \mathbf{H}_t.$$

The space $\mathbf{H}_0$ is just $L_2(\mathbf{T})$, and we can think of the space $\mathbf{H}_t$, $\ t > 0$ as consisting of those functions having generalized L_2 derivatives up to order t. Certainly a function of class C^t belongs to $\mathbf{H}_t$. With a loss of degree of differentiability the converse is true:

Lemma 1 [Sobolev.] *If* $u \in \mathbf{H}_t$ *and*

$$t \geq \left[\frac{n}{2}\right] + k + 1$$

then $u \in C^k(\mathbf{T})$ *and*

$$\sup_{x \in \mathbf{T}} |D^p u(x)| \leq \text{const.}\|u\|_t \quad \textit{for } |p| \leq k. \tag{7}$$

By applying the lemma to $D^p u$ it is enough to prove the lemma for $k = 0$. So we assume that $u \in \mathbf{H}_t$ with $t \geq [n/2] + 1$. Then

$$(\sum |a_\ell|)^2 \leq \sum (1 + \ell \cdot \ell)^t |a_\ell|^2 \sum (1 + \ell \cdot \ell)^{-t} < \infty.$$

So the Fourier series for u converges absolutely and uniformly. The right hand side of the above inequality gives the desired bound. QED

A **distribution** on $\mathbf{T}^n$ is a linear function T on $C^\infty(\mathbf{T}^n)$ with the continuity condition that

$$\langle T, \phi_k \rangle \to 0$$

whenever

$$D^p \phi_k \to 0$$

uniformly for each fixed p. If $u \in \mathbf{H}_{-t}$ we may define

$$\langle u, \phi \rangle := (\phi, \overline{u})_0$$

and since $C^\infty(\mathbf{T})$ is dense in $\mathbf{H}_t$ we may conclude

Lemma 2 $\mathbf{H}_{-t}$ *is the space of those distributions* T *which are continuous in the* $\| \; \|_t$ *norm, i.e. which satisfy*

$$\|\phi_k\|_t \to 0 \quad \Rightarrow \; \langle T, \phi_k \rangle \to 0.$$

We then obtain

Theorem 1 [Laurent Schwartz.] $\mathbf{H}_\infty$ *is the space of all distributions. In other words, any distribution belongs to* $\mathbf{H}_{-t}$ *for some* t.

Proof. Suppose that T is a distribution that does not belong to any $\mathbf{H}_{-t}$. This means that for any $k > 0$ we can find a C^∞ function ϕ_k with

$$\|\phi_k\|_k < \frac{1}{k}$$

and

$$|\langle T, \phi_k * \rangle| \geq 1.$$

But by Lemma 1 we know that $\|\phi_k\|_k < \frac{1}{k}$ implies that $D^p \phi_k \to 0$ uniformly for any fixed p contradicting the continuity property of T. QED

Suppose that ϕ is a C^∞ function on $\mathbf{T}$. Multiplication by ϕ is clearly a bounded operator on $\mathbf{H}_0 = L_2(\mathbf{T})$, and so it is also a bounded operator on $\mathbf{H}_t,\ \ t > 0$ since we can expand $D^p(\phi u)$ by applications of Leibniz's rule.

For $t = -s < 0$ we know by the Riesz representation theorem that

$$\|\phi u\|_t = \sup(v, \phi u)_0/\|v\|_s = \sup(u, \overline{\phi} v)/\|v\|_s \leq \|u\|_t \|\overline{\phi} v\|_s/\|v\|_s.$$

So in all cases we have

$$\|\phi u\|_t \leq (\text{const. depending on } \phi \text{ and } t)\|u\|_t. \tag{8}$$

Let

$$L = \sum_{|p| \leq m} \alpha_p(x) D^p$$

be a differential operator of degree m with C^∞ coefficients. Then it follows from the above that

$$\|Lu\|_{t-m} \leq \text{constant}\|u\|_t \tag{9}$$

where the constant depends on L and t.

Lemma 3 [Rellich's lemma.] *If $s < t$ the embedding $\mathbf{H_t} \hookrightarrow \mathbf{H}_s$ is compact.*

Proof. We must show that the image of the unit ball B of $\mathbf{H}_t$ in $\mathbf{H}_t$ can be covered by finitely many balls of radius ϵ. Choose N so large that

$$(1 + \ell \cdot \ell)^{(s-t)/2} < \frac{\epsilon}{2}$$

when $\ell \cdot \ell > N$. Let Z be the subspace of $\mathbf{H}_t$ consisting of all u such that $a_\ell = 0$ when $\ell \cdot \ell \leq N$. This is a space of finite codimension, and hence the unit ball of $\mathbf{H}_t/Z$ can be covered by finitely many balls of radius $\frac{\epsilon}{2}$. On the other hand, for $u \in B \cap Z$ we have

$$\|u\|_s^2 \leq (1 + N^2)^{s-t}\|u\|_t \leq \left(\frac{\epsilon}{2}\right)^2.$$

So the image of $B \cap Z$ is contained in a ball of radius $\frac{\epsilon}{2}$ and so the image of B is covered by finitely many balls of radius ϵ. QED

2 Gårding's inequality.

Let $x,\ a$, and b be positive numbers. Then

$$x^a + x^{-b} \geq 1$$

because if $x \geq 1$ the first summand is ≥ 1 and if $x \leq 1$ the second summand is ≥ 1. Setting $x = \epsilon^{1/a} A$ gives

$$1 \leq \epsilon A^a + \epsilon^{-b/a} A^{-b}$$

if ϵ and A are positive. Suppose that $t_1 > s > t_2$ and we set $a = t_1 - s,\ b = s - t_2$ and $A = 1 + \ell \cdot \ell$. Then we get

$$(1 + \ell \cdot \ell)^s \leq \epsilon(1 + \ell \cdot \ell)^{t_1} + \epsilon^{-(s-t_2)/(t_1-s)}(1 + \ell \cdot \ell)^{t_2}$$

and therefore

$$\|u\|_s \leq \epsilon\|u\|_{t_1} + \epsilon^{-(s-t_2)/(t_1-s)}\|u\|_{t_2} \quad \text{if } t_1 > s > t_2, \quad \epsilon > 0 \tag{10}$$

for all $u \in \mathbf{H}_{t_1}$. This elementary inequality will be the key to several arguments in this section where we will combine (10) with integration by parts.

A differential operator $L = \sum_{|p| \leq m} \alpha_p(x) D^p$ with real coefficients and m even is called **elliptic** if there is a constant $c > 0$ such that

$$\sum_{|p|=m} a_p(x)\xi^p \geq c|\xi|^m. \tag{11}$$

We will assume until further notice that the operator L is elliptic.

Theorem 2 [Gårding's inequality.] *For every $u \in C^\infty(\mathbf{T})$ we have*

$$(u, Lu)_0 \geq c_1\|u\|^2_{m/2} - c_2\|u\|^2_0 \tag{12}$$

where c_1 and c_2 are constants depending on L.

Remark. If $u \in \mathbf{H}_{m/2}$, then both sides of the inequality make sense, and we can approximate u in the $\|\ \|_{m/2}$ norm by C^∞ functions. So once we prove the theorem, we conclude that it is also true for all elements of $\mathbf{H}_{m/2}$.

We will prove the theorem in stages:

1. When L is constant coefficient and homogeneous.
2. When L is homogeneous and approximately constant.
3. When the L can have lower order terms but the homogeneous part of L is approximately constant.
4. The general case.

Stage 1. $L = \sum_{|p|=m} \alpha_p D^p$ where the α_p are constants. Then

$$\begin{aligned}
(u, Lu)_0 &= \left(\sum a_\ell e^{i\ell\cdot x}, \sum_\ell \left(\sum_{|p|=m} \alpha_p(i\ell^p)\right) a_\ell e^{i\ell\cdot x}\right)_0 \\
&\geq c\sum_\ell (\ell\cdot\ell)^{m/2}|a_\ell|^2 \quad \text{by (11)} \\
&= c\sum[1 + (\ell\cdot\ell)^{m/2}]|a_\ell|^2 - c\|u\|^2_0 \\
&\geq cC\|u\|^2_{m/2} - c\|u\|_0
\end{aligned}$$

where

$$C = \sup_{r \geq 0} \frac{1 + r^{m/2}}{(1+r)^{m/2}}.$$

This takes care of stage 1.

Stage 2. $L = L_0 + L_1$ where L_0 is as in stage 1 and $L_1 = \sum_{|p|=m} \beta_p(x) D^p$ and

$$\max_{p,x} |\beta_p(x)| < \eta,$$

where η sufficiently small. (How small will be determined very soon in the course of the discussion.) We have

$$(u, L_0 u)_0 \geq c' \|u\|_{m/2}^2 - c\|u\|_0^2$$

from stage 1.

We integrate $(u, L_1 u)$ by parts $m/2$ times. There are no boundary terms since we are on the torus. In integrating by parts some of the derivatives will hit the coefficients. Let us collect all the these terms as I_2. The remain terms we collect as I_1, so

$$I_1 = \sum \int b_{p'+p''} D^{p'} u \overline{D^{p''} u} dx$$

where $|p'| = |p''| = m/2$. We can estimate this sum by

$$|I_1| \leq \eta \cdot \text{const.} \|u\|_{m/2}^2$$

and so will require that $\eta \cdot (\text{const.}) < c'$.

The remaining terms give a sum of the form

$$I_2 = \sum \int b_{p'q} D^{p'} u \overline{D^q u} dx$$

where $p' \leq m/2, q' < m/2$ so we have

$$|I_2| \leq \text{const.} \|u\|_{\frac{m}{2}} \|u\|_{\frac{m}{2}-1}.$$

Now let us take

$$s = \frac{m}{2} - 1,\ t_1 = \frac{m}{2},\ t_2 = 0$$

in (10) which yields, for any $\epsilon > 0$,

$$\|u\|_{\frac{m}{2}-1} \leq \epsilon \|u\|_{\frac{m}{2}} + \epsilon^{-m/2} \|u\|_0.$$

Substituting this into the above estimate for I_2 gives

$$|I_2| \leq \epsilon \cdot \text{const.} \|u\|_{m/2}^2 + \epsilon^{-m/2} \text{const.} \|u\|_{m/2} \|u\|_0.$$

Using $2ab \leq \epsilon a^2 + \epsilon^{-1} b^2$ we can replace the second term on the right by

$$\epsilon \cdot \text{const.} \|u\|_0^2$$

at the cost of slightly enlarging the constant in front of $\|u\|^2_{\frac{m}{2}}$. We have thus established that

$$|I_1| \leq \eta \cdot (\text{const.})_1 * \|u\|^2_{m/2}$$

where the constant depends only on m, and

$$I_2 \leq \epsilon(\text{const.})_2 * \|u\|^2_{m/2} + \epsilon^{-m/2}\text{const.}\|u\|^2_0$$

where the constants depend on L_1 but ϵ is at our disposal. So if $\eta(\text{const.})_1 < c'$ and we then choose ϵ so that $\epsilon(\text{const.})_2 < c' - \eta$ we obtain Gårding's inequality for this case.

Stage 3. $L = L_0 + L_1 + L_2$ where L_0 and L_1 are as in stage 2, and L_0 is a lower order operator. Here we integrate by parts and argue as in stage 2.

Stage 4, the general case. Choose an open covering of T such that the variation of each of the highest order coefficients in each open set is less than $\eta < c'$. (Recall that c' depended only on the c that entered into the definition of ellipticity.) Thus, if v is a smooth function supported in one of the sets of our cover, the action of L on v is the same as the action of an operator as in case 3) on v, and so we may apply Gårding's inequality. Choose a finite subcover and a partition of unity $\{\phi_i\}$ subordinate to this cover. Write $\phi_i = \psi_i^2$ (where we choose the ϕ so that the ψ are smooth). So $\sum \psi_i^2 \equiv 1$. Now

$$(\psi_i u, L(\psi_i u))_0 \geq c''\|\psi_i u\|^2_{m/2} - \text{const.}\|\psi_i u\|^2_0$$

where c'' is a positive constant depending only on c, η, and on the lower order terms in L. We have

$$(u, Lu)_0 = \int (\sum \psi_i^2 u)\overline{Lu}dx = \sum(\psi_i u, L\psi_i u)_0 + R$$

where R is an expression involving derivatives of the ψ_i and hence lower order derivatives of u. These can be estimated as in case 2) above, and so we get

$$(u, Lu)_0 \geq c''' \sum \|\psi_i u\|^2_{m/2} - \text{const.}\|u\|^2_0 \tag{13}$$

since $\|\psi_i u\|_0 \leq \|u\|_0$. Now $\|u\|_{m/2}$ is equivalent, as a norm, to $\sum_{p \leq m/2} \|D^p u\|_0$ as we verified in the preceding section. Also

$$\sum \|D^p(\psi_i u)\|_0 = \sum \|\psi_i D^p u\| + R'$$

where R' involves terms differentiating the ψ and so lower order derivatives of u. Hence

$$\sum \|\psi_i u\|^2_{m/2} \geq \text{pos. const.}\|u\|^2_{m/2} - \text{const.}\|u\|^2_0$$

by the integration by parts argument again. Hence by (13)

$$(u, Lu)_0 \geq c''' \sum \|\psi_i u\|^2_{m/2} - \text{const.}\|u\|^2_0$$

$$\geq \text{pos. const.}\|u\|^2_{m/2} - \text{const.}\|u\|^2_0$$

which is Gårding's inequality. QED

3 Consequences of Gårding's inequality.

Proposition 2 *For every integer t there is a constant $c(t) = c(t, L)$ and a positive number $\Lambda = \Lambda(t, L)$ such that*

$$\|u\|_t \leq c(t)\|Lu + \lambda u\|_{t-m} \tag{14}$$

when

$$\lambda > \Lambda$$

for all smooth u, and hence for all $u \in \mathbf{H}_t$.

Proof. Let s be some non-negative integer. We will first prove (14) for $t = s + \frac{m}{2}$. We have

$$\|u\|_t\|Lu+\lambda u\|_{s-\frac{m}{2}} = \|u\|_t\|(1+\Delta)^s Lu+\lambda(1+\Delta)^s u\|_{-s-\frac{m}{2}} \geq (u, (1+\Delta)^s Lu+\lambda(1+\Delta)^s u)_0$$

by the generalized Schwartz inequality (2).

The operator $(1 + \Delta)^s L$ is elliptic of order $m + 2s$ so Gårding's inequality gives

$$(u, (1 + \Delta)^s Lu + \lambda(1 + \Delta)^s u)_0 \geq c_1\|u\|^2_{s+\frac{m}{2}} - c_2\|u\|^2_0 + \lambda\|u\|^2_s.$$

Since $\|u\|_s \geq \|u\|_0$ we can combine the two previous inequalities to get

$$\|u\|_t\|Lu + \lambda u|^2_{t-m} \geq c_1\|u\|^2_t + (\lambda - c_2)\|u\|^2_0.$$

If $\lambda > c_2$ we can drop the second term and divide by $\|u\|_t$ to obtain (14).

We now prove the proposition for the case $t = \frac{m}{2} - s$ by the same sort of argument: We have

$$\|u\|_t\|Lu + \lambda u\|_{-s-\frac{m}{2}} = \|(1 + \Delta)^{-s}u\|_{s+\frac{m}{2}}\|Lu + \lambda u\|_{-s-\frac{m}{2}}$$

$$\geq ((1 + \Delta)^{-s}u, L(1 + \Delta)^s u + \lambda u)_0.$$

Now use the fact that $L(1+\Delta)^s$ is elliptic and Gårding's inequality to continue the above inequalities as

$$\geq c_1\|(1 + \Delta)^{-s}u\|^2_{s+\frac{m}{2}} - c_2\|(1 + \Delta)^{-s}u\|^2_0 + \lambda\|u\|^2_{-s}$$

$$= c_1\|u\|^2_t - c_2\|u\|_{-2s} + \lambda\|u\|^2_s \geq c_1\|u\|^2_t$$

if $\lambda > c_2$. Again we may then divide by $\|u\|_t$ to get the result. QED

The operator $L + \lambda I$ is a bounded operator from $\mathbf{H}_t$ to $\mathbf{H}_{t-m}$ (for any t). Suppose we fix t and choose λ so large that (14) holds. Then (14) says that $(L+\lambda I)$ is invertible on its image, and bounded there with a bound independent of $\lambda > \Lambda$, and this image is a closed subspace of $\mathbf{H}_{t-m}$.

Let us show that this image is all of $\mathbf{H}_{t-m}$ for λ large enough. Suppose not, which means that there is some $w \in \mathbf{H}_{t-m}$ with

$$(w, Lu + \lambda u) = 0$$

for all $u \in \mathbf{H}_t$. Integration by parts gives the adjoint differential operator L^* characterized by

$$(\phi, L\psi)_0 = (L^*\phi, \psi)_0$$

for all smooth functions ϕ and ψ, and by passing to the limit this holds for all element of $\mathbf{H}_r$ for $r \geq m$. The operator L^* has the same leading term as L and hence is elliptic. So let us choose λ sufficiently large that (14) holds for L^* as well as for L. Now

$$0 = \left((1+\Delta)^{t-m}w, Lu + \lambda u\right)_0 = \left(L^*(1+\Delta)^{t-m}w + \lambda(1+\Delta)^{t-m}w, u\right)_0$$

for all $u \in \mathbf{H}_t$ which is dense in $\mathbf{H}_0$ so

$$L^*(1+\Delta)^{t-m}w + \lambda(1+\Delta)^{t-m}w = 0$$

and hence (by (14)) $(1+\Delta)^{t-m}w = 0$ so $w = 0$. We have proved

Proposition 3 *For every t and for λ large enough (depending on t) the operator $L+\lambda I$ maps $\mathbf{H}_t$ bijectively onto $\mathbf{H}_{t-m}$ and $(L = \lambda I)^{-1}$ is bounded independently of λ.*

As an immediate application we get the important

Theorem 3 *If u is a distribution and $Lu \in \mathbf{H}_s$ then $u \in \mathbf{H}_{s+m}$.*

Proof. Write $f = Lu$. By Schwartz' theorem, we know that $u \in \mathbf{H}_k$ for some k. So $f + \lambda u \in \mathbf{H}_{\min(k,s)}$ for any λ. Choosing λ large enough, we conclude that $u = (L+\lambda I)^{-1}(f+\lambda u) \in \mathbf{H}_{\min(k+m,s+m)}$. If $k+m < s+m$ we can repeat the argument to conclude that $u \in \mathbf{H}_{\min(k+2m,s+m)}$. we can keep going until we conclude that $u \in \mathbf{H}_{s+m}$. QED

Notice as an immediate corollary that any solution of the homogeneous equation $Lu = 0$ is C^∞.

We now obtain a second important consequence of Proposition 3. Choose λ so large that the operators

$$(L+\lambda I)^{-1} \quad \text{and} \quad (L^*+\lambda I)^{-1}$$

exist as operators from $\mathbf{H}_0 \to \mathbf{H}_m$. Follow these operators with the injection $\iota_m : \mathbf{H}_m \to \mathbf{H}_0$ and set

$$M := \iota_m \circ (L+\lambda I)^{-1}, \qquad M^* := \iota_m \circ (L^*+\lambda I)^{-1}.$$

Since ι_m is compact (Rellich's lemma) and the composite of a compact operator with a bounded operator is compact, we conclude

Theorem 4 *The operators M and M^* are compact.*

For example, if L, and hence M is self-adjoint, we can apply the theory of compact self-adjoint transformations to M, and hence conclude all the information we would need about the spectrum of L. More generally, we could apply the theory of Fredholm operators (operators of the form $I-K$ where K is compact).

In any event, we can conclude from Theorem 3 that

Theorem 5 *The eigenvectors of $(L+\lambda I)^{-1}$ (and hence of L) are C^∞.*

We want to extend the results obtained above for the torus in two directions. One is to consider functions defined in a **domain** = bounded open set $\mathcal{G}$ of $\mathbf{R}^n$ and the other is to consider functions defined on a compact manifold. In both cases a few elementary tricks allow us to reduce to the torus case. Here is an illustration:

4 Interior regularity.

Suppose that u is a locally square integrable function satisfying $Lu = f$ and f is locally square integrable. Then we claim that u has L^2 derivatives up to order m. More generally, if f has L_2 derivatives locally up to order t then u has L_2 derivatives locally up to order $t+m$. If f is C^∞ so is u.

All of these statements are local. So let us choose a coordinate neighborhood $\mathcal{O}_3$ in the manifold case, or an open proper subset $\mathcal{O}_3 \subset \mathcal{G}$ in the domain case. (Proper means that the closure of $\mathcal{O}_3$ is a compact subset of $\mathcal{G}$.) Choose $\mathcal{O}_1$ and $\mathcal{O}_2$ with

$$\overline{\mathcal{O}}_1 \subset \mathcal{O}_2, \quad \overline{\mathcal{O}}_2 \subset \mathcal{O}_3$$

It is enough to prove the desired smoothness properties in $\mathcal{O}_1$.

Let ϕ be a function with support contained in $\mathcal{O}_3$ and such that $\phi \equiv 1$ on $\mathcal{O}_2$. Map $\mathcal{G}_3$ diffeomorphically onto an open subset of a torus $\mathbf{T}$. To say that

$$Lu = f$$

means that

$$\int f\zeta dx = \int uL^*\zeta dx$$

for any smooth function of compact support. Take $\zeta = \phi\psi$ where ψ has compact support. The preceding equation becomes

$$\int \phi f\psi = \int uL^*(\phi\psi)dx = \int [\phi uL^*\psi + uK^*\psi]dx$$

for all ψ of compact support, where K^* is an operator of degree less than m with smooth coefficients having support in $\mathcal{O}_3$. Since ϕ and the coefficients of K^* are supported in $\mathcal{O}_3$, we can rewrite this as

$$\int_{\mathbf{T}} \phi f w dx = \int_{\mathbf{T}} [\phi uL^*w + uK^*w]dx$$

for any C^∞ function w on $\mathbf{T}$. Now extend f and the coefficients of L to all of $\mathbf{T}$, so that f is locally square integrable and the coefficients of L are still smooth. The above equation now reads

$$\phi f = L(\phi u) + Ku.$$

Now $\phi f - Ku \in \mathbf{H}_{1-m}$ so we know that $\phi u \in \mathbf{H_1}$. Repeating the process with ϕu instead of u (and replacing $\mathcal{O}_2$ by an open set containing $\overline{\mathcal{O}_1}$ and whose closure is contained in $\mathcal{O}_2$ we conclude that $\phi u \in \mathbf{H}_{2-m}$ this time with a different ϕ. Repeating the process we conclude that u has locally $t+m$ derivatives.

5 Extension of the basic lemmas to manifolds.

Let $E \to M$ be a vector bundle over a manifold. We assume that M is equipped with a density which we shall denote by $|dx|$ and that E is equipped with a positive definite (smoothly varying) scalar product, so that we can define the L_2 norm of a smooth section s of E of compact support:

$$\|s\|_0^2 := \int_M |s|^2(x)d|x|.$$

Suppose for the rest of this section that M is compact. Let $\{U_i\}$ be a finite cover of M by coordinate neighborhoods over which E has a given trivialization, and ρ_i a partition of unity subordinate to this cover. Let ϕ_i be a diffeomorphism or U_i with an open subset of $\mathbf{T}^n$ where n is the dimension of M. Then if s is a smooth section of E, we can think of $(\rho_i s)\circ\phi_i^{-1}$ as an $\mathbf{R}^m$ or $\mathbf{C}^m$ valued function on $\mathbf{T}^n$, and consider the sum of the $\|\cdot\|_k$ norms applied to each component. We shall continue to denote this sum by $\|\rho_i f \circ \phi_i^{-1}\|$ and then define

$$\|f\|_k * := \sum_i \|\rho_i f \circ \phi_i^{-1}\|_k$$

where the norms on the right are in the norms on the torus. These norms depend on the trivializations and on the partitions of unity. But any two norms are equivalent, and the $\| \ \|_0$ is equivalent to the "intrinsic" L_2 norm defined above. We define the Sobolev spaces $\mathbf{W}_k$ to be the completion of the space of smooth sections of E relative to the norm $\| \ \|_k$ for $k \geq 0$, and these spaces are well defined as topological vector spaces independently of the choices. Since Sobolev's lemma holds locally, it goes through unchanged. Similarly Rellich's lemma: if s_n is a sequence of elements of $\mathbf{W}_\ell$ which is bounded in the $\| \ \|_\ell$ norm for $\ell > k$, then each of the elements $\rho_i s_n \circ \phi_i^{-1}$ belong to $\mathbf{H}_\ell$ on the torus, and are bounded in the $\| \ \|_\ell$ norm, hence we can select subsequence of $\rho_1 s_n \circ \phi_1^{-1}$ which converges in $\mathbf{H}_k$, then a subsubsequnce such that $\rho_i s_n \circ \phi_i^{-1}$ for $i = 1, 2$ converge etc. arriving at a subsequence of s_n which converges in $\mathbf{W}_k$.

A differential operator L mapping sections of E into sections of E is an operator whose local expression (in terms of a trivialization and a coordinate chart) has the form

$$Ls = \sum_{|p| \le m} \alpha_p(x) D^p s$$

Here the a_p are linear maps (or matrices if our trivializations are in terms of $\mathbf{R}^m$).

Under changes of coordinates and trivializations the change in the coefficients are rather complicated, but the **symbol** of the differential operator

$$\sigma(L)(\xi) := \sum_{|p|=m} a_p(x)\xi^p \qquad \xi \in T^*M_x$$

is well defined.

If we put a Riemann metric on the manifold, we can talk about the length $|\xi|$ of any cotangent vector.

If L is a differential operator from E to itself (i.e. $F{=}E$) we shall call L **even elliptic** if m is even and there exists some constant C such that

$$\langle v, \sigma(L)(\xi)v\rangle \ge C|\xi|^m |v|^2 *$$

for all $x \in M,\ v \in E_x,\ \xi \in T^*M_x$ and $\langle , \rangle$ denote the scalar product on E_x. Rellich's lemma holds. Indeed, locally, this is just a restatement of the (vector valued version) of Rellich's lemma that we have already proved for the torus. But Stage 4 in the proof extends unchanged (other than the replacement of scalar valued functions by vector valued functions) to the more general case.

6 Example: Hodge theory.

We assume knowledge of the basic facts about differentiable manifolds, in particular the existence of an operator $d : \Omega^k \to \Omega^{k+1}$ with its usual properties, where Ω^k denotes the space of exterior k-forms. Also, if M is orientable and carries a Riemann metric then the Riemann metric induces a scalar product on the exterior powers of T^*M and also picks out a volume form. So there is an induced scalar product $(\ , \ ) = (\ , \ )_k$ on Ω^k and a formal adjoint δ of d

$$\delta : \ \Omega^k \to \Omega^{k-1}$$

and satisfies

$$(d\psi, \phi) = (\phi, \delta\phi)$$

where ϕ is a $(k+1)$-form and ψ is a k-form. Then

$$\Delta := d\delta + \delta d$$

is a second order differential operator on Ω^k and satisfies

$$(\Delta\phi, \phi) = \|d\phi\|^2 + \|\delta\phi\|^2$$

where $\|\phi\|^2 = (\phi, \phi)$ is the intrinsic L_2 norm (so $\| \ \| = \| \ \|_0$ in terms of the notation of the preceding section). Furthermore, if

$$\phi = \sum_I \phi_I dx^I$$

is a local expression for the differential form ϕ, where

$$dx^I = dx_{i_1} \wedge \cdots \wedge dx_{i_k} \quad I = (i_1, \ldots, i_k)$$

then a local expression for Δ is

$$\Delta\phi = -\sum g^{ij} \frac{\partial\phi_I}{\partial x^i \partial x^j} + \cdots$$

where

$$g^{ij} = \langle dx^i, dx^j \rangle$$

and the $\cdots$ are lower order derivatives. In particular Δ is elliptic.

Let $\phi \in \Omega^k$ and suppose that

$$d\phi = 0.$$

Let $\mathcal{C}(\phi)$, the **cohomology class** of ϕ be the set of all $\psi \in \Omega^k$ which satisfy

$$\phi - \psi = d\alpha, \quad \alpha \in \Omega^{k-1}$$

and let

$$\overline{\mathcal{C}(\phi)}$$

denote the closure of $\mathcal{C}$ in the L_2 norm. It is a closed subspace of the Hilbert space obtained by completing Ω^k relative to its L_2 norm. Let us denote this space by L_2^k, so $\overline{\mathcal{C}(\phi)}$ is a closed subspace of L_2^k.

Proposition 4 *If $\phi \in \Omega^k$ and $d\phi = 0$, there exists a unique $\tau \in \overline{\mathcal{C}(\phi)}$ such that*

$$\|\tau\| \leq \|\psi\| \quad \forall \ \psi \in \mathcal{C}(\phi).$$

Furthermore, τ is smooth,and

$$d\tau = 0 \quad \textit{and} \quad \delta\tau = 0.$$

If choose a minimizing sequence for $\|\psi\|$ in $\mathcal{C}(\phi)$.

If we choose a minimizing sequence for $\|\psi\|$ in $\mathcal{C}(\phi)$ we know it is Cauchy, cf. the proof of the existence of orthogonal projections in a Hilbert space. So we know that τ exists and is unique. For any $\alpha \in \Omega^{k+1}$ we have

$$(\tau, \delta\alpha) = \lim(\psi, \delta\alpha) = \lim(d\psi, \alpha) = 0$$

as ψ ranges over a minimizing sequence. The equation $(\tau, \delta\alpha) = 0$ for all $\alpha \in \Omega^{k+1}$ says that τ is a weak solution of the equation $d\tau = 0$.

We claim that

$$(\tau, d\beta) = 0 \quad \forall\ \beta \in \Omega^{k-1}$$

which says that τ is a weak solution of $\delta\tau = 0$. Indeed, for any $t \in \mathbf{R}$,

$$\|\tau\|^2 \le \|\tau + td\beta\|^2 = \|\tau\|^2 + t^2\|d\beta\|^2 + 2t(\tau, d\beta)$$

so

$$-2t(\tau, d\beta) \le t^2\|d\beta\|^2.$$

If $(\tau, d\beta) \ne 0$, we can choose

$$t = -\epsilon\frac{(\tau, d\beta)}{|(\tau, d\beta)|}, \quad \epsilon > 0$$

so

$$|(\tau, d\beta)| \le \epsilon|d\beta|^2.$$

As ϵ is arbitrary, this implies that $(\tau, d\beta) = 0$.

So $(\tau, \Delta\psi) = (\tau, [d\delta + \delta d]\psi) = 0$ for any $\psi \in \Omega^k$. Hence τ is a weak solution of $\Delta\tau = 0$ and so is smooth. The space $\mathcal{H}^k$ of weak, and hence smooth solutions of $\Delta\tau = 0$ is finite dimensional by the general theory. It is called the space of Harmonic forms. We have seen that there is a unique harmonic form in the cohomology class of any closed form, s the cohomology groups are finite dimensional. In fact, the general theory tells us that

$$L_2^k \bigoplus_\lambda E_\lambda^k$$

(Hilbert space direct sum) where E_λ^k is the eigenspace with eigenvalue λ of Δ. Each E_λ is finite dimensional and consists of smooth forms, and the $\lambda \to \infty$. The eigenspace E_0^k is just $\mathcal{H}^k$, the space of harmonic forms. Also, since

$$(\Delta\phi, \phi) = \|d\phi\|^2 + \|\delta\phi\|^2$$

we know that all the eigenvalues λ are non-negative.

Since $d\Delta = d(d\delta + \delta d) = d\delta d = \Delta d$, we see that

$$d : E_\lambda^k \to E_\lambda^{k+1}$$

and similarly

$$\delta : E_\lambda^k \to E_\lambda^{k-1}.$$

For $\lambda \ne 0$, if $\phi \in E_\lambda^k$ and $d\phi = 0$, then $\lambda\phi = \Delta\phi = d\delta\phi$ so $\phi = d(1/\lambda)\delta\phi$ so d restricted to the E_λ is exact, and similarly so is δ. Furthermore, on $\bigoplus_k E_\lambda^k$ we have

$$\lambda I = \Delta = (d + \delta)^2$$

so we have

$$E_\lambda^k = dE_\lambda^{k-1} \oplus \delta E_\lambda^{k+1}$$

and this decomposition is orthogonal since $(d\alpha,\delta\beta)=(d^2\alpha,\beta)=0$.

As a first consequence we see that

$$L_2^k=\mathcal{H}^k\oplus\overline{d\Omega^{k-1}}\oplus\overline{\delta\Omega^{k-1}}$$

(the Hodge decomposition). If H denotes projection onto the first component, then Δ is invertible on the image of $I-H$ with a an inverse there which is compact. So if we let N denote this inverse on im $I-H$ and set $N=0$ on $\mathcal{H}^k$ we get

$$\begin{aligned}\Delta N&=I-H\\ Nd&=dN\\ \delta N&=N\delta\\ \Delta N&=N\Delta\\ NH&=0\end{aligned}$$

which are the fundamental assertions of Hodge theory, together with the assertion proved above that $H\phi$ is the unique minimizing element in its cohomology class.

We have seen that

$$d+\delta:\ \bigoplus_k E_\lambda^{2k}\to\bigoplus_k E_\lambda^{2k+1}\ \text{ is an isomorphism for }\lambda\neq 0 \qquad (15)$$

which of course implies that

$$\sum_k(-1)^k\dim E_\lambda^k=0$$

This shows that the index of the operator $d+\delta$ acting on $\bigoplus L_2^k$ is the Euler characteristic of the manifold. (The index of any operator is the difference between the dimensions of the kernel and cokernel).

Let $P_{k,\lambda}$ denote the projection of L_2^k onto E_λ^k. So

$$e^{-t\Delta}=\sum e^{-\lambda t}P_{k,\lambda}$$

is the solution of the heat equation on L_2^k. As $t\to\infty$ this approaches the operator H projecting L_2^k onto $\mathcal{H}_k$. Letting Δ_k denote the operator Δ on L_2^k we see that

$$\operatorname{tr}e^{-t\Delta_k}=\sum e^{-\lambda_k}$$

where the sum is over all eigenvalues λ_k of Δ_k counted with multiplicity. It follows from (15) that the alternating sum over k of the corresponding sum over non-zero eigenvalues vanishes. Hence

$$\sum(-1)^k\operatorname{tr}e^{-t\Delta_k}=\chi(M)$$

is independent of t. The index theorem computes this trace for small values of t in terms of local geometric invariants.

The operator $d + \delta$ is an example of a Dirac operator whose general definition we will not give here. The corresponding assertion and local evaluation is the content of the celebrated Atiyah-Singer index theorem, one of the most important theorems discovered in the twentieth century.