

Strongly contractive semi-groups

Math 212b

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Throughout this lecture \mathbf{K} will denote a Hilbert space and $t \mapsto S(t), t \geq 0$ a strongly continuous semi-group of contractions defined on \mathbf{K} which tends strongly to 0 as $t \rightarrow \infty$ in the sense that

$$\lim_{t \rightarrow \infty} \|S(t)k\| = 0 \quad \text{for each } k \in \mathbf{K}.$$

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1 Examples.

1.1 Translation - truncation.

Let \mathbf{N} be some Hilbert space and consider the Hilbert space

$$L_2(\mathbf{R}, \mathbf{N}).$$

Let T_t denote the one parameter unitary group of right translations:

$$[T_t f](x) = f(x - t)$$

and let P denote the operator of multiplication by $\mathbf{1}_{(-\infty, 0]}$ so P is projection onto the subspace \mathbf{G} consisting of the f which are supported on $(-\infty, 0]$. We claim that

$$t \mapsto PT_t$$

is a semi-group acting on \mathbf{G} satisfying our conditions. The operator PT_t is a strongly continuous contraction since it is unitary operator on $L_2(\mathbf{R}, \mathbf{N})$ followed by a projection. Also

$$\|PT_t f\|^2 = \int_{-\infty}^{-t} |f(x)|^2 dx$$

tends strongly to zero. Clearly $PT_0 = \text{Id}$ on \mathbf{G} . We must check the semi-group property. We have

$$PT_s PT_t f = PT_s [T_t f + g] = PT_{s+t} f + PT_s g$$

where

$$g = PT_t f - T_t f$$

so

$$g \perp \mathbf{G}.$$

But

$$T_{-s} : \mathbf{G} \rightarrow \mathbf{G}$$

for $s \geq 0$. Hence $g \perp \mathbf{G} \Rightarrow T_s g = T_{-s}^* g \perp \mathbf{G}$.

1.2 Incoming representations.

The last argument is quite formal. We can axiomatize it as follows: Let \mathbf{H} a Hilbert space, and $t \mapsto U(t)$ a strongly continuous group of unitary operators on \mathbf{H} . A closed subspace $\mathbf{D} \subset \mathbf{H}$ is called **incoming** with respect to U if

$$U(t)\mathbf{D} \subset \mathbf{D} \quad \text{for } t \leq 0 \quad (1)$$

$$\bigcap_t U(t)\mathbf{D} = \{0\} \quad (2)$$

$$\overline{\bigcup_t \mathbf{D}} = \mathbf{H}. \quad (3)$$

Let $P_{\mathbf{D}} : \mathbf{H} \rightarrow \mathbf{D}$ denote orthogonal projection. The preceding argument goes over unchanged to show that S defined by

$$S(t) := P_{\mathbf{D}} U(t)$$

is a strongly continuous semi-group. We must prove that it converges strongly to zero. First observe that (1) implies that

$$U(-s)\mathbf{D} \supset U(-t)\mathbf{D} \quad \text{if } s < t$$

and since $U(-s)\mathbf{D}^\perp = [U(-s)\mathbf{D}]^\perp$ we get

$$U(-s)\mathbf{D}^\perp \subset U(-t)\mathbf{D}^\perp.$$

We claim that

$$\bigcup_{t>0} U(-t)\mathbf{D}^\perp$$

is dense in \mathbf{H} . If not, there is an $h \in \mathbf{H}$ such that $h \in [U(-t)\mathbf{D}^\perp]^\perp$ for all $t > 0$ which says that $U(t)h \perp \mathbf{D}^\perp$ for all $t > 0$ or $U(t)h \in \mathbf{D}$ for all $t > 0$ or

$$h \in U(-t)\mathbf{D} \quad \text{for all } t > 0$$

contradicting (2). Therefore, if $f \in \mathbf{D}$ and $\epsilon > 0$ we can find $g \perp \mathbf{D}$ and an $s > 0$ so that

$$\|f - U(-s)g\| < \epsilon$$

or

$$\|U(s)f - g\| < \epsilon.$$

Since $g \perp \mathbf{D}$ we have $P_{\mathbf{D}}[U(s)f - g] = P_{\mathbf{D}}U(s)f$ and hence

$$\|P_{\mathbf{D}}U(s)f\| < \epsilon,$$

proving that $P_{\mathbf{D}}U(s)$ tends strongly to zero.

1.3 Scattering residue.

Let $t \mapsto U(t)$ be a strongly continuous one parameter unitary group on a Hilbert space \mathbf{H} , let \mathbf{D}_- be an incoming subspace for U and let \mathbf{D}_+ be an outgoing subspace (i.e. incoming for $t \mapsto U(-t)$). Suppose that

$$\mathbf{D}_- \perp \mathbf{D}_+$$

and let

$$\mathbf{K} := [\mathbf{D}_- \oplus \mathbf{D}_+]^\perp.$$

Let

$$P_\pm := \text{orthogonal projection onto } \mathbf{D}_\pm^\perp.$$

Let

$$Z(t) := P_+U(t)P_-, \quad t \geq 0.$$

Claim:

$$Z(t) : \mathbf{K} \rightarrow \mathbf{K}.$$

Proof. Since P_+ occurs as the leftmost factor in the definition of Z , the image of $Z(t)$ is contained in \mathbf{D}_+^\perp . We must show that

$$x \in \mathbf{D}_-^\perp \rightarrow P_+U(t)x \in \mathbf{D}_-^\perp$$

since $Z(t)x = P_+U(t)x$ as $P_-x = x$ if $x \in \mathbf{D}_-^\perp$. Now $U(-t) : \mathbf{D}_- \rightarrow \mathbf{D}_-$ is one of the conditions for incoming, and so

$$U(t) : \mathbf{D}_-^\perp \rightarrow \mathbf{D}_-^\perp.$$

So

$$U(t)x \in \mathbf{D}_-^\perp.$$

QED

Now since $\mathbf{D}_- \subset \mathbf{D}_+^\perp$ the projection P_+ is the identity on \mathbf{D}_- , in particular

$$P_+ : \mathbf{D}_- \rightarrow \mathbf{D}_-$$

and hence, since P_+ is self-adjoint,

$$P_+ : \mathbf{D}_-^\perp \mapsto \mathbf{D}_+^\perp.$$

We will now use $Z(t)$ to denote the restriction of $Z(t)$ to \mathbf{K} . We claim that $t \mapsto Z(t)$ is a semi-group. Indeed, we have

$$P_+U(t)P_+x = P_+U(t)x + P_+U(t)[P_+x - x] = P_+U(t)x$$

since $[P_+x - x] \in \mathbf{D}_+$ and $U(t) : \mathbf{D}_+ \rightarrow \mathbf{D}_+$. Also $Z(t) = P_+U(t)$ on \mathbf{K} since P_- is the identity on \mathbf{K} . Therefore we may drop the P_- on the right when restricting to \mathbf{K} and we have

$$Z(s)Z(t) = P_+U(s)P_+U(t) = P_+U(s)U(t) = P_+U(s+t) = Z(s+t)$$

proving that Z is a semigroup.

We now show that Z is strongly contracting. For any $x \in \mathbf{H}$ and any $\epsilon > 0$ we can find a $T > 0$ and a $y \in \mathbf{D}_+$ such that

$$\|x - U(-T)y\| < \epsilon$$

since $\bigcup_{t < 0} U(t)\mathbf{D}_+$ is dense. For $x \in \mathbf{K}$ we get

$$\|Z(t)x - P_+U(t)U(-T)y\| = \|P_+U(t)[x - U(-T)y]\| < \epsilon.$$

But for $t > T$

$$U(t)U(-T)y = U(t-T)y \in \mathbf{D}_+ \quad \text{so } P_+U(t)U(-T)y = 0$$

and hence

$$\|Z(t)x\| < \epsilon.$$

We have proved that Z is a strongly contractive semi-group on \mathbf{K} which tends strongly to zero. This example will be of primary importance to us in computations.

2 Breit-Wigner.

Suppose that \mathbf{K} is one dimensional, and that

$$Z(t)d = e^{-\mu t}d$$

for $d \in \mathbf{K}$ where

$$\operatorname{Re} \mu > 0.$$

This is obviously a strongly contractive semi-group in our sense. Consider the space $L_2(\mathbf{R}, \mathbf{N})$ where \mathbf{N} is a copy of \mathbf{K} but with the scalar product whose norm is

$$\|d\|_{\mathbf{N}}^2 = 2 \operatorname{Re} \mu \|d\|^2.$$

Let

$$f_d(t) = \begin{cases} e^{\mu t} d & t \leq 0 \\ 0 & t > 0. \end{cases}.$$

Then

$$\|f_d\|^2 = \int_{-\infty}^0 e^{2 \operatorname{Re} \mu t} (2 \operatorname{Re} \mu) \|d\|^2 dt = \|d\|^2$$

so the map

$$R: d \mapsto f_d$$

is an isometry. Also

$$P(T_t f_d)(s) = P f_d(s - t) = e^{-\mu t} f_d(s)$$

so

$$(PT_t) \circ R = R \circ Z(t).$$

This is an example of the representation theorem in the next section.

If we take the Fourier transform of f_d we obtain the function

$$\sigma \mapsto \frac{1}{\sqrt{2\pi}} \frac{1}{\mu - i\sigma} d$$

whose norm as a function of σ is proportional to the Breit-Wigner function

$$\frac{1}{\mu^2 + \sigma^2}.$$

It is this “bump” appearing in graph of a scattering experiment which signifies a “resonance”, i.e. an “unstable particle” whose lifetime is inversely proportional to the width of the bump.

3 The representation theorem for strongly contractive semi-groups.

Let $t \mapsto S(t)$ be a strongly contractive semi-group on a Hilbert space \mathbf{K} as in the preceding section. We want to prove that the pair \mathbf{K}, S is isomorphic to a restriction of Example 1.

Theorem 1 [Lax-Phillips.] *There exists a Hilbert space \mathbf{N} and an isometric map R of \mathbf{K} onto a subspace of $PL_2(\mathbf{R}, \mathbf{N})$ such that*

$$S(t) = R^{-1} P T_t R$$

for all $t \geq 0$.

Proof. Let B be the infinitesimal generator of S , and let $D(B)$ denote the domain of B . The sesquilinear form $f, g \mapsto$

$$-(Bf, g) - (f, Bg)$$

is non-negative definite since B satisfies

$$\operatorname{Re} (Bf, f) \leq 0.$$

Dividing out by the null vectors and completing gives us a Hilbert space \mathbf{N} whose scalar product we will denote by $(\cdot, \cdot)_N$. If $k \in D(B)$ so is $S(t)k$ for every $t \geq 0$. Let us define

$$f_k(-t) = S(t)k$$

so we can think of f as a map from $(-\infty, 0]$ to \mathbf{N} . We have

$$\|f(-t)\|_N^2 = \|S(t)k\|_N^2 = -2\operatorname{Re} (BS(t)k, k)_N = -\frac{d}{dt}\|S(t)k\|^2.$$

Integrating this from 0 to r gives

$$\int_{-r}^0 \|f(s)\|_N^2 = \|k\|^2 - \|S(r)k\|^2.$$

By hypothesis, the second term on the right tends to zero as $r \rightarrow \infty$. This shows that the map

$$R : k \mapsto f_k$$

is an isometry of $D(B)$ into $L_2((-\infty, 0], \mathbf{N})$, and since $D(B)$ is dense in \mathbf{K} , we conclude that it extends to an isometry of \mathbf{D} with a subspace of $PL_2(\mathbf{R}, \mathbf{N})$ (by extension by zero, say). Also

$$RS(t)k = f_{S(t)k}$$

is given by

$$f_{S(t)k}(s) = S(-s)S(t)k = S(t-s)k = S(-(s-t)k) = f_k(s-t)$$

for $s < 0$, and $t > 0$ so

$$RS(t)k = PT_t Rk.$$

Thus $R\mathbf{K}$ is an invariant subspace of $PL_2(\mathbf{R}, N)$ and the intertwining equation of the theorem holds. QED

We can strengthen the conclusion of the theorem for elements of $D(B)$:

Proposition 1 *If $k \in D(B)$ then f_k is continuous in the \mathbf{N} norm for $t \leq 0$.*

Proof. For $s, t, > 0$ we have

$$\begin{aligned} \|f_k(-s) - f_k(-t)\|_N^2 &= -2\operatorname{Re} (B[S(t) - S(s)]k, [S(t) - S(s)]k) \\ &\leq 2\|[S(t) - S(s)]Bk\| \|[S(t) - S(s)]k\| \end{aligned}$$

by the Cauchy-Schwartz inequality. Since S is strongly continuous the result follows. QED

Let us apply this construction to the semi-group associated to an incoming space \mathbf{D} for a unitary group U on a Hilbert space \mathbf{H} . Let $d \in \mathbf{D}$ and $f_d = Rd$ as above. We know that $U(-r)d \in \mathbf{D}$ for $r > 0$ by (1). Notice also that

$$S(r)U(-r)d = PU(r)U(-r)d = Pd = d$$

for $d \in \mathbf{D}$. Then for $t \leq -r$ we have, by definition,

$$f_{U(-r)d}(t) = S(-t)U(-r)d = PU(-t)U(-r)d = SU(-t-r)d = f_d(t+r)$$

and so

$$\begin{aligned} \|U(-r)d\|^2 &= \int_{-\infty}^0 \|f_{U(-r)d}\|_{\mathbf{N}}^2 dx \geq \int_{-\infty}^{-r} \|f_{U(-r)d}(x)\|_{\mathbf{N}}^2 dx \\ &= \int_{-\infty}^0 |f_d(x)|_{\mathbf{N}}^2 dx = \|d\|^2. \end{aligned}$$

Since $U(-r)$ is unitary, we have equality throughout which implies that

$$f_{U(-r)d}(t) = 0 \quad \text{if } t > -r.$$

We have thus proved that if

$$r > 0$$

then

$$f_{U(-r)d}(t) = \begin{cases} f_d(t+r) & \text{if } t \leq -r \\ 0 & \text{if } -r < t \leq 0 \end{cases}. \quad (4)$$

4 The Sinai representation theorem.

This says that

Theorem 2 *If \mathbf{D} is an incoming subspace for a unitary one parameter group, $t \mapsto U(t)$ acting on a Hilbert space H then there is a Hilbert space \mathbf{N} , a unitary isomorphism*

$$R : \mathbf{H} \rightarrow L_2(\mathbf{R}, \mathbf{N})$$

such that

$$RU(t)R^{-1} = T_t$$

and

$$R\mathbf{D} = PL_2(\mathbf{R}, \mathbf{N}),$$

the subspace consisting of functions which vanish on $(0, \infty]$ a.e.

Proof. We apply the results of the last section. For each $d \in \mathbf{D}$ we have obtained a function $f_d \in L_2((-\infty, 0], \mathbf{N})$ and we extend f_d to all of \mathbf{R} by setting $f_d(s) = 0$ for $s > 0$. We thus defined an isometric map R from \mathbf{D} onto a subspace of $L_2(\mathbf{R}, \mathbf{N})$. Now extend R by setting

$$R(U(r)d)(t) = f_d(t - r).$$

Equation (4) assures us that this definition is consistent in that if d is such that $U(r)d \in \mathbf{D}$ then this new definition agrees with the old one. We have thus extended the map R to $\bigcap U(t)\mathbf{D}$ as an isometry satisfying

$$RU(t) = T_t R.$$

Since $\bigcap U(t)\mathbf{D}$ is dense in \mathbf{H} the map R extends to all of \mathbf{H} . Also by construction $RP_{\mathbf{D}} = PR$.

We must still show that R is surjective. For this it is enough to show that we can approximate any simple function with values in \mathbf{N} by an element of the image of R . Recall that the elements of the domain of B , the infinitesimal generator of $P_{\mathbf{D}}U(t)$, are dense in \mathbf{N} , and for $d \in D(B)$ the function f_d is continuous, satisfies $f(t) = 0$ for $t > 0$, and $f(0) = n$ where n is the image of d in \mathbf{N} . Hence

$$(I - P)U(\delta)d$$

is mapped by R into a function which is approximately equal to n on $[0, \delta]$ and zero elsewhere. Since the image of R is translation invariant, we see that we can approximate any simple function by an element of the image of R , and since R is an isometry, the image of R must be all of $L_2(\mathbf{R}, \mathbf{N})$.

5 The Stone - von Neumann theorem.

Let us show that the Sinai representation theorem implies a version (for $n = 1$) of the Stone - von Neumann theorem:

Theorem 3 *Let $\{U(t)\}$ be a one parameter group of unitary operators, and let B be a self-adjoint operator on a Hilbert space \mathbf{H} . Suppose that*

$$U(t)BU(-t) = B - tI. \tag{5}$$

Then we can find a unitary isomorphism R of \mathbf{H} with $L_2(\mathbf{R}, \mathbf{N})$ such that

$$RU(t)R^{-1} = T_t$$

and

$$RB R^{-1} = m_x,$$

where m_x is multiplication by the independent variable x .

Remark. If A denotes the infinitesimal generator of U , then differentiating (5) with respect to t and setting $t = 0$ gives

$$[A, B] = -I$$

which is a version of the Heisenberg commutation relations. So (5) is a “partially integrated” version of these commutation relations, and the theorem asserts that (5) determines the form of U and B up to the possible “multiplicity” given by the dimension of \mathbf{N} .

Proof. By the spectral theorem, write

$$B = \int \lambda dE_\lambda$$

where $\{E_\lambda\}$ is the spectral resolution of B , and so we obtain the spectral resolutions

$$U(t)BU(-t) = \int \lambda d[U(t)E_\lambda U(-t)]$$

and

$$B - tI = \int (\lambda - t) dE_\lambda = \int dE_{\lambda+t}$$

by a change of variables.

We thus obtain

$$U(t)E_\lambda U(-t) = E_{\lambda+t}.$$

Remember that E_λ is orthogonal projection onto the subspace associated to $(-\infty, \lambda]$ by the spectral measure associated to B . Let D denote the image of E_0 . Then the preceding equation says that $U(t)D$ is the image of the projection E_t . The standard properties of the spectral measure - that the image of E_t increase with t , tend to the whole space as $t \rightarrow \infty$ and tend to $\{0\}$ as $t \rightarrow -\infty$ are exactly the conditions that D be incoming for $U(t)$. Hence the Sinai representation theorem is equivalent to the Stone - von -Neumann theorem in the above form. QED