Lectures 1 and 2: The Stone -von Neumann theorem.

math212b

Jan 31 and Feb 5, 2002

Contents

1 Exponential of a matrix. **2**

2 The three dimensional Heisenberg algebra. **2**

3 The three dimensional Heisenberg group. **3**

4 Elementary symplectic facts. **4**
- 4.1 Symplectic vector spaces. 4
- 4.2 Special kinds of subspaces. 4
- 4.3 Normal forms. 5
- 4.4 Existence of Lagrangian subspaces. 5

5 The $(2d+1)$dimensional Heisenberg algebra and group. **5**

6 The $(2d+1)$dimensional Heisenberg group. **6**

7 The Weyl representation. **6**
- 7.1 The character. 6
- 7.2 Some commutation calculations. 7
- 7.3 The induction construction. 8

8 Recall about induced representations in general. **8**
- 8.1 The image of the center. 9
- 8.2 The Shr odinger realization. 9

9 Statement of the Stone - von Neumann theorem. **10**

10 The group algebra. **11**
- 10.1 Definition of the group algebra. 11

11 Representations and the group algebra. **11**
- 11.1 Unitary representations and the adjoint. 12

12 The Weyl transform. 12
12.1 Properties of the Weyl transform. 13

13 Hilbert-Schmidt Operators. 14

14 Proof of the irreducibility of $\rho(\ell)$. 15
14.1 The Weyl transform in the Schrödinger realization. 16
14.2 The partial Fourier transform. 16

15 Completion of the proof. 17

1 Exponential of a matrix.

$$\exp M = I + M + \frac{1}{2}M^2 + \frac{1}{3!}M^3 + \cdots$$

just as in the scalar case. If two matrices M and N commute, then it is true that $\exp(M+N) = (\exp M)(\exp N)$ but this is not true in general:

$$\exp(M+N) = I + M + N + \frac{1}{2}(M+N)^2 + \cdots$$

$$= I + M + N + \frac{1}{2}\left(M^2 + MN + NM + N^2\right) + \cdots$$

while

$$(\exp M)(\exp N) = I + M + N + \frac{1}{2}M^2 + MN + \frac{1}{2}N^2 + \cdots.$$

So although the constant and first order terms agree, the quadratic terms need not, and if fact we have

$$(\exp M)(\exp N) = \exp(M+N) + \frac{1}{2}[M,N] + \cdots$$

where

$$[M,N] = MN - NM$$

and the $\cdots$ represent terms cubic and higher.

2 The three dimensional Heisenberg algebra.

Consider the set of three by three matrices of the form

$$\begin{pmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix}.$$

If

$$M = \begin{pmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad M' = \begin{pmatrix} 0 & x' & z' \\ 0 & 0 & y' \\ 0 & 0 & 0 \end{pmatrix}$$

then their commutator $[M, M'] := MM' - M'M$ is given by

$$[M, M'] = \begin{pmatrix} 0 & 0 & xy - x'y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

So if we define the anti-symmetric form

$$\omega\left(\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x' \\ y' \end{pmatrix} \right) := xy' - x'y$$

we can write the preceding equation as

$$[M, M'] = \begin{pmatrix} 0 & 0 & \omega(X, X') \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

where

$$X = \begin{pmatrix} x \\ y \end{pmatrix}, \quad X' = \begin{pmatrix} x' \\ y' \end{pmatrix}.$$

3 The three dimensional Heisenberg group.

If

$$M = \begin{pmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix}$$

then

$$\exp M = \begin{pmatrix} 1 & x & z + \frac{1}{2}xy \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}.$$

Therefore

$$(\exp M)(\exp M') = \begin{pmatrix} 1 & x + x' & z + z' + \frac{1}{2}(xy + x'y') + xy' \\ 0 & 1 & y + y' \\ 0 & 0 & 1 \end{pmatrix}$$

while

$$\exp(M + M') = \begin{pmatrix} 1 & x + x' & z + z' + \frac{1}{2}(xy + xy' + x'y + xy') \\ 0 & 1 & y + y' \\ 0 & 0 & 1 \end{pmatrix}.$$

Now

$$\frac{1}{2}(xy + x'y') + xy' = \frac{1}{2}(xy + xy' + x'y + x'y') + \frac{1}{2}(xy' - yx')$$

and so if we describe the matrix M as being given by the pair (X, z) then the multiplication law is

$$(\exp(X, z))(\exp(X', z')) = \exp(X + X', z + z' + \frac{1}{2}\omega(X, X')).$$

The Heisenberg group we will study is the $2d + 1$ dimensional generalization of this three dimensional Heisenberg group.

4 Elementary symplectic facts.

4.1 Symplectic vector spaces.

Let V be a (usually finite dimensional) vector space over the real numbers. A symplectic structure on V consists of an antisymmetric bilinear form

$$\omega : V \times V \to \mathbf{R}$$

which is non-degenerate. So we can think of ω as an element of $\wedge^2 V^*$ when V is finite dimensional, as we shall assume until further notice. A vector space equipped with a symplectic structure is called a symplectic vector space.

A basic example is $\mathbf{R}^2$ with

$$\omega_{\mathbf{R}^2}\left(\begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} c \\ d \end{pmatrix}\right) = \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc.$$

We will call this the standard symplectic structure on $\mathbf{R}^2$.

4.2 Special kinds of subspaces.

If W is a subspace of symplectic vector space V then $W^\perp$ denotes the symplectic orthocomplement of W:

$$W^\perp := \{v \in V |\ \omega(v, w) = 0,\ \forall w \in W\}.$$

A subspace is called

1. **symplectic** if $W \cap W^\perp = \{0\}$,
2. **isotropic** if $W \subset W^\perp$,
3. **coisotropic** if $W^\perp \subset W$, and
4. **Lagrangian** if $W = W^\perp$.

Since $(W^\perp)^\perp = W$ by the non-degeneracy of ω it follows that W is symplectic if and only if $W^\perp$ is. Also, the restriction of ω to any symplectic subspace W is non-degenerate, making W into a symplectic vector space. Conversely, to say that the restriction of ω to W is non-degenerate means precisely that $W \cap W^\perp = \{0\}$.

4.3 Normal forms.

For any non-zero $e \in V$ we can find an $f \in V$ such that $\omega(e, f) = 1$ and so the subspace W spanned by e and f is a two dimensional symplectic subspace. Furthermore the map

$$e \mapsto \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad f \mapsto \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

gives a symplectic isomorphism of W with $\mathbf{R}^2$ with its standard symplectic structure. We can apply this same construction to $W^\perp$ if $W^\perp \neq 0$. Hence by induction, we can decompose any symplectic vector space into a direct sum of two dimensional symplectic subspaces:

$$V = W_1 \oplus \cdots W_d$$

where dim $V = 2d$ (proving that every symplectic vector space is even dimensional) and where the W_i are pairwise (symplectically) orthogonal and where each W_i is spanned by e_i, f_i with $\omega(e_i, f_i) = 1$. In particular this shows that all $2d$ dimensional symplectic vector spaces are isomorphic, and isomorphic to a direct sum of d copies of $\mathbf{R}^2$ with its standard symplectic structure.

4.4 Existence of Lagrangian subspaces.

Let us collect the $e_1, \ldots, e_d$ in the above construction and let L be the subspace they span. It is clearly isotropic. Also, $e_1, \ldots, e_n, f_1, \ldots, f_d$ form a basis of V. If $v \in V$ has the expansion

$$v = a_1 e_1 + \cdots a_d e_d + b_1 f_1 + \cdots + b_d f_d$$

in terms of this basis, then $\omega(e_i, v) = b_i$. So $v \in L^\perp \Rightarrow v \in L$. Thus L is Lagrangian. So is the subspace M spanned by the f's.

Conversely, if L is a Lagrangian subspace of V and M is a complementary Lagrangian subspace. Then ω induces a non-degenerate linear pairing of L with M and hence any basis $e_1, \cdots e_d$ picks out a dual basis $f_1, \cdots .f_d$ of M giving a basis of the above form.

5 The $(2d+1)$dimensional Heisenberg algebra and group.

Let V be a symplectic vector space. So V comes equipped with a skew symmetric non-degenerate bilinear form ω. By the choice of a pair of transverse Lagrangian subspaces, and then dual bases in these subspaces, we obtain a basis

$$P_1, \ldots, P_n, Q_1, \ldots, Q_n$$

of V with

$$\begin{aligned} \omega(P_i, P_j) &= 0 \\ \omega(Q_i, Q_j) &= 0 \\ \omega(P_i, Q_j) &= \delta_{ij}. \end{aligned} \tag{1}$$

We make

$$\mathbf{h} := V \oplus R$$

into a Lie algebra by defining

$$[X,Y] := \omega(X,Y)E$$

where $E = 1 \in \mathbf{R}$ and

$$[E,E] = 0 = [E,X] \quad \forall X \in V.$$

The Lie algebra $\mathbf{h}$ is called the **Heisenberg algebra**. It is a generalization to $2d+1$ dimensions of the three dimensional Heisenberg algebra introduced above. It is a nilpotent Lie algebra. In fact, the Lie bracket of any three elements is zero. If we write out the brackets in terms of the basis above we get

$$\begin{aligned} [P_i, Q_j] &= \delta_{ij}E \\ [P_i, P_j] &= 0 \\ [Q_i, Q_j] &= 0 \end{aligned}$$

which, together with

$$[E, P_j] = 0 = [E, Q_j]$$

are the famous Heisenberg "canonical commutation relations" up to inessential (or essential) factors such as $\hbar$ and i.

6 The $(2d+1)$dimensional Heisenberg group.

We will let N denote the simply connected Lie group with this Lie algebra. We may identify the $2n+1$ dimensional vector space $V + \mathbf{R}$ with N via the exponential map, and with this identification the multiplication law on N reads

$$\exp(v + tE)\exp(v' + t'E) = \exp\left(v + v' + (t + t' + \frac{1}{2}\omega(v,v'))E\right). \quad (2)$$

Let dv be the Euclidean (Lebesgue) measure on V. Then the measure $dvdt$ is invariant under left and right multiplication. So the group N is unimodular. For those of you who are unfamiliar with the notion of the exponential map for Lie algebras and Lie groups, just start with (2) as a definition of multiplication, where exp is just a weird symbol, which generalizes the construction in the three dimensional as given above, where exp meant matrix exponentiation.

7 The Weyl representation.

7.1 The character.

If ℓ is a Lagrangian subspace of V, then $\ell \oplus \mathbf{R}$ is an Abelian subalgebra of $\mathbf{h}$, and in fact is maximal abelian. Similarly

$$L := \exp(\ell \oplus \mathbf{R})$$

is a maximal Abelian subgroup of N.

Define the function

$$f : N \to T^1$$

$$f(\exp(v + tE)) := e^{2\pi i t}.$$

We have

$$f\left((\exp(v + tE))(\exp(v' + t'E))\right) = e^{2\pi i (t + t' + \frac{1}{2}\omega(v,v'))}. \tag{3}$$

Therefore

$$f(h_1 h_2) = f(h_1) f(h_2)$$

for

$$h_1, \quad h_2 \in L.$$

We say that the restriction of f to L is a **character** of L.

I want to consider the quotient space

$$N/L$$

which has a natural action of N (via left multiplication). In other words N/L is a homogeneous space for the Heisenberg group N. Let ℓ' be a Lagrangian subspace transverse to ℓ. Every element of N has a unique expression as

$$(\exp y)(\exp(x + sE)) \quad \text{where } y \in \ell' \quad x \in \ell.$$

This allows us to make the identification

$$N/L \sim \ell'$$

and the Euclidean measure dv' on ℓ' then becomes identified with the (unique up to scalar multiple) measure on N/L invariant under N.

7.2 Some commutation calculations.

For later use we record the following "commutation calculation" at the group level: Let $y \in \ell'$ and $x \in \ell$. Then

$$\exp(-x)(\exp y) = \exp(y - x - \frac{1}{2}\omega(x, y)E)$$

while

$$\exp(y)\exp(-x) = \exp(y - x - \frac{1}{2}\omega(y, x)E)$$

so, since ω is antisymmetric, we get

$$(\exp(-x))(\exp y) = (\exp y)(\exp(-x))\exp(-\omega(x, y)E). \tag{4}$$

7.3 The induction construction.

We continue with the above notation. In particular, we have chosen a Lagrangian subspace ℓ, have the corresponding subgroup L and the quotient space N/L. We are going to construct a unitary representation of N which is known in group theory language as the representation of N **induced** from the character f of L.

Its definition is as follows: Consider the space of continuous functions ϕ on N which satisfy

$$\phi(nh) = f(h)^{-1}\phi(n) \quad \forall\; n \in N \quad h \in L \tag{5}$$

and which in addition have the property that the function on N/L

$$n \mapsto |\phi(n)|$$

(which is well defined on N/L on account of (5)) is square integrable on N/L. We let $H(\ell)$ denote the Hilbert space which is the completion of this space of continuous functions relative to this L_2 norm. So $\phi \in H(\ell)$ is a "function" on N satisfying (5) with norm

$$||\phi||^2 = \int_{N/L} |\phi|^2 d\dot{n}$$

where $d\dot{n}$ is left invariant measure on N/L.

The representation ρ_ℓ of N on $H(\ell)$ is given by left translation:

$$(\rho_\ell(m)\phi)(n) := \phi\left(m^{-1}n\right). \tag{6}$$

For the rest of this section we will keep ℓ fixed, and so may write H for $H(\ell)$ and ρ for ρ_ℓ. The dependence on ℓ will become important for us later.

8 Recall about induced representations in general.

If K is a subgroup of a group G and we have a representation γ of K on a vector space W, we can consider the set

$$E := (G \times W)/K$$

where K acts to the right on G and via γ on W and diagonally on $G \times W$ so

$$k(g, w) := (gk^{-1}, \gamma(k)w).$$

Projection onto the first factor descends to a projection

$$E \to X$$

where $X = G/K$ making E into a vector bundle over X. A section of E corresponds to a function

$$f : G \to W$$

satisfying

$$f(gk^{-1}) = \gamma(k)f(g).$$

If γ is unitary and if $X = G/K$ has a measure invariant under the action of G we can speak of the space H of L^2 sections and then we get a unitary representation of G on H. Here

$$(\tau(a)f)(g) = f(a^{-1}g).$$

In terms of sections, the section s associated to f assigns the value $[(g, f(g))]$ to the point $x = gK$. The section corresponding to $\tau(a)f$ assigns the value

$$as(a^{-1}x) = a[(a^{-1}g, f(a^{-1}g))] = [(g, f(a^{-1}g))]$$

to the point x.

Our construction of the Heisenberg representation amounts to the choice of $G = N, K = L$ and γ the one dimensional representation corresponding to the character f on L.

8.1 The image of the center.

Back to the special case of the Heisenberg group that we are considering. Since $\exp tE$ is in the center of N, we have

$$\rho(\exp tE)\phi(n) = \phi\left((\exp -tE)n\right) = \phi\left(n(\exp -tE)\right) = e^{2\pi it}\phi(n).$$

In other words

$$\rho\left(\exp tE\right) = e^{2\pi it}\mathrm{Id}_H. \tag{7}$$

The Stone - von Neumann theorem (Theorem 9.1 below) characterizes all unitary representations of N which satisfy this condition.

8.2 The Shr odinger realization.

Suppose we choose a complementary Lagrangian subspace ℓ' and then identify N/L with ℓ' as in the preceding section. Condition (5) becomes

$$\phi\left((\exp y)(\exp(x))(\exp tE)\right) = \phi(\exp y)e^{-2\pi it}.$$

So $\phi \in H$ is completely determined by its restriction to $\exp \ell'$. In other words the map

$$\phi \mapsto \psi, \quad \psi(y) := \phi(\exp y)$$

defines a unitary isomorphism

$$R : H \to L_2(\ell')$$

and if we set

$$\sigma := R\rho R^{-1}$$

then

$$\begin{array}{lcll} [\sigma(\exp x)\psi](y) & = & e^{2\pi i\omega(x,y)}\psi(y) & x \in \ell,\ y \in \ell' \\ [\sigma(\exp u)\psi](y) & = & \psi(y-u) & y, u \in \ell' \\ \sigma(\exp(tE)) & = & e^{2\pi it}\mathrm{Id}_{L_2(\ell')}. & \end{array} \tag{8}$$

The first of these equations follows from (4) and the definition (6) and the last two follow immediately from (6).

We define the infinitesimal version of the representation ρ by

$$\dot{\rho}(X) := \frac{d}{dt}\rho(\exp(tX))_{|t=0}$$

for $X \in \mathbf{h}$ with a similar notion and notation for σ. Under the P, Q basis (with $P_i \in \ell$ chosen above), we may identify $L_2(\ell')$ with $L_2(\mathbf{R}^n)$. Then it follows from (8) that

$$\begin{array}{lcl} \dot{\sigma}(P_j) & = & 2\pi i x_j \\ \dot{\sigma}(Q_j) & = & -\frac{\partial}{\partial x_j} \\ \dot{\sigma}(E) & = & 2\pi i \mathrm{Id} \end{array} \tag{9}$$

This is the Schrodinger version of the Heisenberg commutation relations.

9 Statement of the Stone - von Neumann theorem.

The Stone-von Neumann theorem asserts that the representation σ, and hence the representation ρ is irreducible and is the unique irreducible representation (up to isomorphism) satisfying (8). In fact, to be more precise, the theorem asserts that any unitary representation of N such that

$$\exp(tE) \mapsto e^{2\pi it}\mathrm{Id}$$

must be isomorphic to a **multiple** of ρ in the following sense:

Let H_1 and H_2 be Hilbert spaces. We can form their tensor product as vector spaces, and this tensor product inherits a scalar product determined by

$$(u \otimes v, x \otimes y) = (u, x)(v, y).$$

The completion of this (algebraic) tensor product with respect to this scalar product will be denoted by $H_1 \otimes H_2$ and will be called the (Hilbert space) tensor product of H_1 and H_2. If we have a representation τ of a group G on H_1 we get a representation

$$g \mapsto \tau(g) \otimes \mathrm{Id}_{H_2}$$

on $H_1 \otimes H_2$ which we call a multiple of the representation τ. We can now state:

Theorem 9.1 **[The Stone-von-Neumann theorem.]** *The representation* $\rho(\ell)$ *of* N *is irreducible, and any representation such that* $\exp(tE) \mapsto e^{2\pi i t}\mathrm{Id}$ *is isomorphic to a multiple of* $\rho(\ell)$.

10 The group algebra.

10.1 Definition of the group algebra.

If G is a locally compact Hausdorff topological group with a given choice of Haar measure, we defined the convolution of two continuous functions of compact support on G by

$$(\phi_1 \star \phi_2)(g) := \int_G \phi_1(u)\phi_2(u^{-1}g)du.$$

If ψ is another continuous function on G we have

$$\int_G (\phi_1 \star \phi_2)(g)\psi(g)dg = \int_{G\times G} \phi_1(u)\phi_2(h)\psi(uh)dudh.$$

This right hand side makes sense if ϕ_1 and ϕ_2 are distributions of compact support and ψ is smooth. Also the left hand side makes sense if ϕ_1 and ϕ_2 belong to $L_1(G)$ and ψ is bounded, etc.

11 Representations and the group algebra.

If τ is a continuous representation of a locally compact Hausdorff group with Haar measure dg then for any function ϕ (say continuous and of compact support) define

$$\tau(\phi) := \int_G \phi(g)\tau(g)dg.$$

Then by Fubini, the left invariance of dg and Fubini again we have

$$\begin{aligned}
\tau(\phi_1 \star \phi_2) &= \tau\left(\int_G \phi_1(u)\phi_2(u^{-1}g)dg\right) \\
&= \int\int \phi_1(u)\phi_2(u^{-1}g)\tau(g)dudg \\
&= \int\int \phi_1(u)\phi_2(u^{-1}g)\tau(g)dgdu \\
&= \int\int \phi_1(u)\phi_2(h)\tau(g)dhdu \\
&= \tau(\phi_1)\tau(\phi_2).
\end{aligned}$$

11.1 Unitary representations and the adjoint.

If τ is unitary representation of G on a Hilbert space, so that

$$\tau(g^{-1}) = \tau(g)^*$$

and if dg is unimodular, then recalling the definition of ϕ^* as

$$\phi^*(g) = \overline{\phi(g^{-1})}$$

we have

$$\tau(\phi^*) = \int \overline{\phi(g^{-1})}\tau(g)dg = \int \overline{\phi(g)}\tau(g^{-1}dg = \int (\phi(g)\tau(g))^* \, dg = \tau(\phi)^*.$$

In other words, τ takes the adjoint of ϕ in the group algebra into the adjoint of $\tau(\phi)$.

12 The Weyl transform.

Let τ be a representation of N satisfying our condition

$$\tau(tE) = e^{2\pi i t}\mathrm{Id}.$$

Then τ descends to a representation of

$$B := N/\exp(\mathbf{Z}E)$$

since $\tau(\exp(kE)) = \mathrm{Id}$ for $k \in \mathbf{Z}$.

Let Φ denote the collection of continuous functions on N which satisfy

$$\phi(n \exp tE) = e^{-2\pi i t}\phi(n).$$

Every $\phi \in \Phi$ can be considered as a function on B, and every $n \in B$ has a unique expression as $n = (\exp v)(\exp tE)$ with $v \in V$ and $t \in \mathbf{R}/\mathbf{Z}$. We take as our left invariant measure on B the measure $dvdt$ where dv is Lebesgue measure on V and dt is the invariant measure on the circle with total measure one. The set of elements of Φ are then determined by their restriction to $\exp(V)$. Then for $\phi_1, \phi_2 \in \Phi$ of compact support (as functions on B) we have (with $\star$ denoting convolution on B)

$$(\phi_1 \star \phi_2)(\exp v)$$

$$\begin{aligned}
&= \int_V \int_T \phi_1((\exp u)(\exp tE))\phi_2((-\exp u)(\exp(-tE))(\exp v))dudt \\
&= \int_V \phi_1(\exp u)\phi_2((\exp -u)(\exp v))du \\
&= \int_V \phi_1(\exp u)\phi_2(\exp(v-u)\exp(-\frac{1}{2}\omega(u,v)E))du \\
&= \int_V \phi_1(\exp u)\phi_2(\exp(v-u))e^{\pi i \omega(u,v)}du.
\end{aligned}$$

So if we use the notation

$$\psi(u) = \phi(\exp u)$$

and $\psi_1 \star \psi_2$ for the ψ corresponding to $\phi_1 \star \phi_2$ we have

$$(\psi_1 \star \psi_2)(v) = \int_V \psi_1(u)\psi_2(v-u)e^{\pi i\omega(u,v)}du. \tag{10}$$

We thus get a "twisted" convolution on V.

If $\phi \in \Phi$ and if we define ϕ^* as above, then $\phi^* \in \Phi$ and the corresponding transformation on the ψ's is

$$\phi^*(\exp v) = \overline{\psi(-v)}.$$

We now define

$$W_\tau(\psi) = \tau(\phi) = \int_B \phi(b)\tau(b)db = \int\int \phi((\exp u)(\exp tE)\tau((\exp u)(\exp tE))dtdu = \int_V \psi(v)\tau(\exp v)dv.$$

The last equation holds because of the opposite transformation properties of τ and $\phi \in \Phi$.

The Weyl transform carries functions on V into operators on H. As such it can be considered as a version of "quantization".

12.1 Properties of the Weyl transform.

If $\phi \in \Phi$ then $\delta_m \star \phi$ is given by

$$(\delta_m \star \phi)(n) = \phi(m^{-1}n)$$

which belongs to Φ if ϕ does and if

$$m = \exp(w)$$

then

$$(\delta_m \star \phi)(\exp u) = e^{\pi i\omega(w,u)}\psi(u-w).$$

Similarly,

$$(\phi \star \delta_m)(\exp u) = e^{-\pi i\omega(w,u)}\psi(u-w).$$

Let us write $w \star \psi$ for the function on V corresponding to $\delta_m \star \phi$ under our correspondence between elements of Φ and functions on V.

Then the facts that we have proved such as

$$\tau(\phi_1 \star \phi_2) = \tau(\phi_1)\tau(\phi_2)$$

translate into

$$\begin{aligned} W_\tau(\psi_1 \star \psi_2) &= W_\tau(\psi_1)W_\tau(\psi_2) &(11)\\ W_\tau(\psi^*) &= W_\tau(\psi)^* &(12)\\ W_\tau(w \star \psi) &= \tau(\exp w)W_\tau(\psi) &(13)\\ W_\tau(\psi \star w) &= W_\tau(\psi)\tau(\exp w). &(14)\end{aligned}$$

Equation (11) says that the Weyl transform carries twisted co convolution into multiplication, and equations (13) and (14) are versions of (11) extended to delta functions. Equation (12) says that the Weyl transform carries adjoints on functions into adjoints on operators.

13 Hilbert-Schmidt Operators.

Let H be a separable Hilbert space. An operator A on H is called **Hilbert-Schmidt** if in terms of some orthonormal basis $\{e_i\}$ we have

$$\sum \|Ae_i\|^2 < \infty.$$

Since

$$Ae_i = \sum (Ae_i, e_j)e_j$$

this is the same as the condition

$$\sum_{ij} |(Ae_i, e_j)|^2 < \infty$$

or

$$\sum |a_{ij}|^2 < \infty$$

where

$$a_{ij} := (Ae_i, e_j)$$

is the matrix of A relative to the orthonormal basis. This condition and sum does not depend on the orthonormal basis and is denoted by

$$\|A\|^2_{HS}.$$

This norm comes from the scalar product

$$(A, B)_{HS} = \operatorname{tr} B^*A = \sum (B^*Ae_i, e_i) = \sum (Ae_i, Be_i).$$

Indeed,

$$\begin{aligned}
(A^*Ae_i, e_i) &= (Ae_i, Ae_i) \\
&= \left(\sum_j (Ae_i, e_j)e_j, Ae_i\right) \\
&= \sum_j (Ae_i, e_j)(e_j, Ae_i) \\
&= \sum_j a_{ij}\overline{a_{ij}} \\
&= \sum_j |a_{ij}|^2,
\end{aligned}$$

and summing over i gives $\|A\|_{HS}^2$.

The rank one elements

$$E_{ij}, \quad E_{ij}(x) := (x, e_j)e_i$$

form an orthonormal basis of the space of Hilbert-Schmidt operators. We can identify the space of Hilbert-Schmidt operators with the tensor product $H \otimes \overline{H}$ where $\overline{H}$ is the space H with scalar multiplication and product given by the complex conjugate, e.g multiplication by $c \in \mathbf{C}$ is given by multiplication by $\overline{c}$ in H.

If $H = L_2(V, dy)$ (where V can be any measure space with measure dy, but we will be interested in our case) we can describe the space of Hilbert-Schmidt operators as follows: Let $\{e_i\}$ be an orthonormal basis of $H = L_2(V)$ and consider the rank one operators E_{ij} introduced above. Then

$$\begin{aligned}(E_{ij}\psi)(x) = (\psi, e_j)e_i(x) &= \int_V \psi(y)\overline{e_j(y)}e_i(x)dy \\ &= \int_Y K_{ij}(x,y)\psi(y)dy\end{aligned}$$

where

$$K_{ij}(x,y) = e_i(x)\overline{e_j(y)}.$$

This has norm one in $L_2(V \times V)$ and hence the most general Hilbert-Schmidt operator A is given by the $L_2(V \times V)$ kernel

$$K = \sum a_{ij}K_{ij}$$

with a_{ij} the matrix of A as above.

14 Proof of the irreducibility of $\rho(\ell)$.

Let us consider the case where $\tau = \rho = \rho(\ell)$. I claim that the map W_ρ defined on the elements of Φ of compact support extends to an isomorphism from $L_2(V)$ to the space of Hilbert-Schmidt operators on $H(\ell)$. Indeed, write

$$W_\rho(\psi) = \int_V \psi(v)\rho(\exp v)dV$$

and decompose

$$V = \ell \oplus \ell'$$

$$v = y + x, \quad s \in \ell, \ y \in \ell'$$

so

$$\exp(y + x) = \exp(y)\exp(x)\exp(-\frac{1}{2}\omega(y,x))$$

so

$$\rho(\exp(y + x)) = \rho(y)\rho(x)e^{-i\pi\omega(y,x)}$$

and hence

$$W_\rho(\psi) = \int\int \psi(y + x)\rho(\exp y)\rho(\exp x)e^{-\pi i\omega(y,x)}dxdy.$$

14.1 The Weyl transform in the Schrödinger realization.

So far the above would be true for any τ, not necessarily ρ. Now let us use the explicit realization of ρ as σ on $L_2(\mathbf{R}^n)$ in the form given in (8).

We obtain

$$[W_\sigma(\psi)(f)](\xi) = \int\int e^{-\pi i\omega(y,x)}\psi(y+x)e^{2\pi i\omega(x,\xi-y)}f(\xi-y)dxdy.$$

Collecting the exponential terms, the result in the exponent is $\pi i\times$

$$-\omega(y,x) + 2\omega(x,\xi-y) = 2\omega(x,\xi) - \omega(x,y)$$

which under the substitution $y \mapsto \xi - y$ becomes

$$\omega(x,\xi+y).$$

So making the change of variables $y \mapsto \xi - y$ in the previous expression for $[W_\sigma(\psi)(f)](\xi)$ we get

$$[W_\sigma(\psi)(f)](\xi) = \int\int e^{-\pi i\omega(\xi-y,x)}e^{2\pi i\omega(x,y)}\psi(\xi-y+x)f(y)dxdy.$$

If we define

$$K_\psi(\xi,y) := \int e^{\pi i\omega(x,y+\xi)}\psi(\xi-y+x)dx$$

we have

$$[W_\sigma(\psi)f)](\xi) = \int K_\psi(\xi,y)f(y)dy.$$

14.2 The partial Fourier transform.

Here we have identified ℓ' with $\mathbf{R}^n$ and $V = \ell' + \ell$ where ℓ is the dual space of ℓ' under ω. So if we consider the partial Fourier transform

$$\mathcal{F}_x : L_2(\ell' \oplus \ell) \to L_2(\ell' \oplus \ell')$$

$$(\mathcal{F}_x\psi)(y,\xi) = \int e^{-2\pi i\omega(x,\xi)}\psi(y+x)dx$$

(which is an isomorphism) we have

$$K_\psi(\xi,y) = (\mathcal{F}_x\psi)(\xi-y,-\frac{1}{2}(y+\xi)).$$

We thus see that the set of all K_ψ is the set of all Hilbert-Schmidt operators on $L_2(\mathbf{R}^n)$.

Now if a bounded operator C commutes with all Hilbert-Schmidt operators on a Hilbert space, then $CE_{ij} = E_{ij}C$ implies that $c_{ij} = c\delta_{ij}$, i.e. $C = c\mathrm{Id}$. So we have proved that every bounded operator that commutes with all the $\rho_\ell(n)$ must be a constant. Thus $\rho(\ell)$ is irreducible.

15 Completion of the proof.

We fix ℓ, ℓ' as above, and have the representation ρ realized as σ on $L_2(\ell')$ which is identified with $L_2(\mathbf{R}^n)$ all as above. We want to prove that any representation τ satisfying (7) is isomorphic to a multiple of σ.

We consider the "twisted convolution" (10) on the space of Schwartz functions $\mathcal{S}(\mathcal{V})$. If $\psi \in \mathcal{S}(V)$ then its Weyl kernel $K_\psi(\xi, y)$ is a rapidly decreasing function of (ξ, y) and we get all operators with rapidly decreasing kernels as such images of the Weyl transform W_σ sending ψ into the kernel giving $\sigma(\phi)$.

Consider some function $u \in \mathcal{S}(\ell')$ with

$$\|u\|_{L_2(\ell')} = 1.$$

Let P_1 be the projection onto the line through u, so P_1 is given by the kernel

$$p_1(x, y) = \overline{u(y)}u(x).$$

We know that it is given as

$$p_1 = W_\sigma(\psi) \quad \text{for some} \quad \psi \in \mathcal{S}(V).$$

We have $P_1^2 = P_1, P_1^* = P_1$ and

$$P_1\sigma(n)P_1 = \alpha(n)P_1 \quad \text{with} \quad \alpha(n) = (\sigma(n)u, u).$$

Recall that $\phi \mapsto \sigma(\phi)$ takes convolution into multiplication, and that K_ψ is the kernel giving the operator $W_\sigma(\psi) = \sigma(\phi)$ where $\phi \in \Phi$ corresponds to $\psi \in \mathcal{S}(V)$. Then in terms of our twisted convolution $\star$ given by (10) the above three equations involving P_1 get translated into

$$\psi \star \psi = \psi, \quad \psi^* = \psi, \quad \psi \star n \star \psi = \alpha(n)\psi, \tag{15}$$

Now let τ be any unitary representation of N on a Hilbert space H satisfying (7). We can form $W_\tau(\psi)$.

Lemma 15.1 *The set of linear combinations of the elements*

$$\tau(n)W_\tau(\psi)x, \quad x \in H, \quad n \in N$$

is dense in H.

Proof. Suppose that $y \in H$ is orthogonal to all such elements and set $n = \exp w$. Then for any $x \in H$

$$0 = (y, \tau(n)W_\tau(\psi)\tau(n)^{-1}x) = \int_V (y, \tau(\exp w)\tau(\exp(v)\tau(\exp(-w)\psi(v)dv$$

$$= \int_V (y, \tau(\exp(v + \omega(w, v)E)x)\psi(v)dv = \int_V (y, \tau(\exp v)x)e^{-2\pi i\omega(w,v)}\psi(v)dv$$

$$= \mathcal{F}[(y, \tau(\exp v)x)\psi].$$

The function is square brackets whose Fourier transform is being taken is continuous and rapidly vanishing. Indeed, x and y are fixed elements of H and τ is unitary, so the expression $(y, \tau(\exp v)x)$ is bounded by $\|y\|\|x\|$ and is continuous, and ψ is a rapidly decreasing functions of v. Since the Fourier transform of the function

$$v \mapsto (y, \tau(\exp(v))x)\psi(v)$$

vanishes, the function itself must vanish. Since ψ does not vanish everywhere, there is some value v_0 with $\psi(v_0) \neq 0$, and hence

$$(y, \tau(\exp v_0)x) = 0 \quad \forall x \in H.$$

Writing $x = \tau(\exp v_0)^{-1}z$ we see that y is orthogonal to all of H and hence $y = 0$. QED

Now from the first two equations in (15) we see that $W_\tau(\psi)$ is an orthogonal projection onto a subspace, call it H_1 of H. We are going to show that H is isomorphic to $H(\ell) \otimes H_1$ as a Hilbert space and as a representation of N.

We wish to define

$$I : H(\ell) \otimes H_1 \to H, \quad \rho(n)u \otimes b \mapsto \tau(n)b$$

where $b \in H_1$.

We first check that if

$$b_1 = W_\tau(\psi)x_1 \quad \text{and} \quad b_2 = W_\tau(\psi)x_2$$

then for any $n_1, n_2 \in N$ we have

$$(\tau(n_1)W_\tau(\psi)x_1, \tau(n_2)W_\tau(\psi)x_2)_H = (\rho(n_1)u, \rho(n_2)u)_{H(\ell)} \cdot (b_1, b_2)_{H_1}. \tag{16}$$

Proof. Since $\tau(n)$ is unitary and $W_\tau(\psi)$ is self-adjoint, we can write the left hand side of (16) as

$$(\tau(n_1)W_\tau(\psi)x_1, \tau(n_2)W_\tau(\psi)x_2)_H = (W_\tau(\psi)\tau(n_2^{-1}n_1)W_\tau(\psi)x_1, x_2)_H$$

and by the last equation in (15) this equals

$$= \alpha(n_2^{-1}n_1)(W_\tau(\psi)x_1, x_2)_H.$$

From the definition of α we have

$$\alpha(n_2^{-1}n_1) = (\rho(n_2^{-1}n_1)u, u)_{H(\ell)} = (\rho(n_1)u, \rho(n_2)u)_\ell$$

since $\rho(n_2)$ is unitary. This is the first factor on the right hand side of (16).

Since $W_\tau(\psi)$ is a projection we have

$$(W_\tau(\psi)x_1, x_2)_H = (W_\tau(\psi)x_1, W_\tau(\psi)x_2)_H = (b_1, b_2)_{H_1},$$

which is the second factor on the right hand side of (16). We have thus proved (16).

Now define

$$I: \quad \sum_{i=1}^{N} \rho(n_i)u \otimes b_i \mapsto \sum \tau(n_i)b_i.$$

This map is well defined, for if

$$\sum_{i=1}^{N} \rho(n_i)u \otimes b_i = 0$$

then

$$\|\sum_{i=1}^{N} \rho(n_i)u \otimes b_i\|_{H(\ell)\otimes H_1} = 0$$

and (16) then implies that

$$\|\sum_{i=1}^{N} \rho(n_i)u \otimes b_i\|_{H(\ell)\otimes H_1} = \|\sum_{i=1}^{N} \tau(n_i)b_i\|_H = 0.$$

Equation (16) also implies that the map I is an isometry where defined. Since ρ is irreducible, the elements $\sum_{i=1}^{N} \rho(n_i)u$ are dense in $H(\ell)$, and so I extends to an isometry from $H(\ell) \otimes H_1$ to H. By Lemma 15.1 this map is surjective. Hence I extends to a unitary isomorphism (which clearly is also a morphism of N modules) between $H(\ell) \otimes H_1$ and H. This completes the proof of the Stone - von Neumann Theorem.