The Maslov index and the metaplectic representation.

Math 212b

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Contents

In these lengthy notes I want to cover in excruciating detail some points relating to the Stone - van Neumann theorem, the Malsov index, and the metaplectic group. In this introduction I will give a sketchy survey of what this is all about, so you don't lose sight of the forest for the trees. The section we will really need for applications is the section on the Maslov index.

Let V be a symplectic vector space, **h** the associated Heisenberg algebra, and N the corresponding Heisenberg group. Let τ be any irreducible representation satisfying $\tau(\exp tE) = e^{2\pi it}I$. By the Stone - von Neumann theorem we know that τ is unique up to a unitary equivalence. Any $M \in Sp(V)$ acts as an automorphism of **h**, hence as an automporphism of N preserving the center, which we will continue to denote by M. Hence τ_M defined by

$$\tau_M(n) = \tau(Mn)$$

is another irreducible representation of N satisfying the same condition on the center. Hence there is a unitary map U_M detemined up to multiplication by a scalar of absolute value 1 such that

$$\tau_M(n) = U_M\tau(a)U_M^{-1}.$$

Then it follows that

$$U_{M_1}U_{M_2} = c(M_1, M_2)U_{M_1M_2}.$$

In other words the map $M \mapsto U_M$ is what is known as a projective representation of the group $Sp(V)$ with cocycle c. By general principles of group theory, this implies that this corresponds to an honest unitary representation of the universal cover of $Sp(V)$. In fact it is a representation of the double cover, and you have enough information to prove this fact. Indeed, if you take the τ to be Schrodinger realization ρ that we constructed in class, and use the operators U you developped in the homework, you can check that if U corresponds to the matrix M (remember that this was two to one) then

$$U\rho(a)U^{-1} = \rho(Ma).$$

Indeed, you only have to verify this for the operators U_d and V_P since they generate, and this is a direct verification.

We will spend a lot of time in these notes giving another verification of this fact and identifying the cocycle as being related to the square root of a certain determinant - the need to pass to the double cover coming from the two choices in the sign of the square root.

We will also give many other realizations of τ. Instead of constructing τ from a real Lagrangian subspace, we will use certain complex Lagrangian subspaces. They have the advantage that they possess a unique "vacuum state". What I mean is this. Suppose we look for an element in the representation space (say in the Schrodinger realization - remember that all are equivalent) which is annhilated by all the $\dot{\tau}(Q_j)$. If these are realized by

$$\frac{\partial}{\partial x_j},$$

then the (one dimensional space of) constants are the only guys annihilated by all the Q_j. They do not lie in the Hilbert space. Similarly, as the P_j are realized as multiplication by $2\pi x_j$ and only elements annihilated by all these are (multiples of) the delta function which again does not belong to the Hilbert space. But suppose we consider the element annihilated by the $Q_j - iP_j$. This will be (all multiples of) a Gaussian, which *does* belong to the Hilbert space. So by passing from real to certain complex Lagrangian subspaces, we get a canonical line lying in the Stone von Neumann representation.

We will begin by studying the space of "positive complex Lagrangian subspaces", and see that they form a natural generalization to $2n$ dimensions of the unit disk in the complex plane. We will associate to each such point in the "generalized unit disk" a tiny subspace of a certain huge Hilbert space, which will be the realization corresponding to this point of the Stone - von Neumann reprsentation. Each of these subspaces has a unique line of "vacuum vectors", and no two of these lines are orthogonal in the huge Hilbert space. Given any three non-orthogonal lines in a Hilbert space there is an associated invariant:

$$\arg\,(v_1, v_2)(v_2, v_3)(v_3, v_1), \quad 0 \neq v_i \in \ell_i,\ i = 1, 2, 3.$$

(The left hand side does not depend on the choice of the v_i.) In fact, although we will not prove it, this number is essentially e^{iA} where A is the area of the triangle spanned by the three points in the hyperbolic geometry of the unit disk.)

1 Complexifying a symplectic vector space.

Let V be a symplectic vector space over $\mathbf{R}$ with symplectic form $(\ ,\ )$. We let $V^{\mathbf{C}} := V \otimes_{\mathbf{R}} \mathbf{C}$ denote the complexification of V, so elements of $V^{\mathbf{C}}$ can be written as

$$z = x + iy, \quad x, y \in V$$

and then we defined $\overline{z}$ as

$$\overline{z} := x - iy.$$

A subspace U of $V^{\mathbf{C}}$ is called **totally complex** if

$$U \cap \overline{U} = \{0\}$$

and **totally real** if $U = \overline{U}$.

The symplectic form extends by complex bilinearity to give a complex symplectic form on $V^{\mathbf{C}}$:

$$(x + iy, u + iv) := (x, u) - (y, v) + i[(x, v) + (y, u)].$$

Then $H : V^{\mathbf{C}} \times V^{\mathbf{C}} \to \mathbf{C}$ defined by

$$H(z, w) := i(z, \overline{w})$$

is complex linear in z and complex antilinear in w and satisfies

$$H(w,z) = \overline{H(z,w)}.$$

It is a Hermitian form which is non-singular, but is not positive definite, since $H(x,x) = 0$ if $x \in V$ is real.

A (complex) subspace $L \subset V^{\mathbf{C}}$ is called Lagrangian if it is maximal isotropic. If L is Lagrangian so is $\overline{L}$. So if L is Lagrangian and totally complex, then L and $\overline{L}$ are non-singularly paired by $(\ ,\ )$, and hence the restriction of H to L is non-degenerate. We say that L is **positive definite** if the restriction of H to L is positive definite, and let D denote the set of positive definite Lagrangian subspaces of $V^{\mathbf{C}}$. We will find that each element of D gives rise to a complex structure and a positive definite Hermitian form on our original real symplectic space, V. We will also find that if we fix an element, $L_0 \in D$, then D "looks like" an n-dimensional generalization of the unit disk in that the symplectic group of V acts on D as "fractional linear transformations".

2 The space of totally complex Lagrangians.

Let L be a totally complex Lagrangian subspace of $V^{\mathbf{C}}$. In particular,

$$L \cap V = \{0\} \quad \text{and} \quad L \cap iV = \{0\}.$$

As a real vector space, $\dim L = 2n = \dim V$ and as a real vector space $V^{\mathbf{C}} = V \oplus iV$. Hence we can regard L as the graph of a bijective real linear map from V to iV, or what is the same thing, as the graph of a bijective real linear map from V to V. In other words, there exists a bijective real linear map $J : V \to V$ such that L consists of all elements of the form

$$x + iJx, \qquad x \in V.$$

The fact that L is Lagrangian says that

$$(x + iJx, y + iJy) = 0, \qquad \forall x, y \in V.$$

The real part of this equation says

$$(Jx, Jy) = (x, y) \quad \forall x, y \in V \tag{1}$$

i.e. $J \in Sp(V)$ is a symplectic transformation. The imaginary part of this equation says that J is skew adjoint relative to $(\ ,\ )$:

$$(Jx, y) + (x, Jy) = 0. \tag{2}$$

Another way of saying this is that $J \in sp(V)$, the Lie algebra of the symplectic group. Setting $y = Ju$ in this equation gives

$$(x, J^2u) = -(Jx, Ju) = -(x, u)$$

by (1). Hence

$$(J^2u+u,x)=0 \quad \forall x\in V$$

which implies that

$$J^2=-\mathrm{id}. \tag{3}$$

Thus J defines a complex structure on V.

Conversely, suppose that $J\in \mathrm{End}(V)$ satisfies (1) and (2) (and hence also (3)). In particular J is bijective so that $L:=\{x+iJx\}$ is totally complex and is Lagrangian. We have proved:

Proposition 1 *The space of totally complex Lagrangian subspaces of $V^{\mathbf{C}}$ is in one to one correspondence with the set of $J\in \mathrm{End}(V)$ which satisfy (1) and (2).*

Notice that any two of the three conditions (1), (2), and (3) imply the third.

Suppose that we are given a totally complex Lagrangian subspace $L\subset V^{\mathbf{C}}$, or what amounts to the same thing its corresponding J. The map $\alpha=\alpha_L: V\to V^{\mathbf{C}}$ defined by

$$\alpha(v):=v-iJv$$

satisfies

$$\alpha(Jv)=Jv+iv=i(v-iJv)=i\alpha(v).$$

In other words $\alpha: V\to V^{\mathbf{C}}$ is complex linear relative to the complex structure on V given by J and the complex structure on $V^{\mathbf{C}}$ given by multiplication by i:

$$\alpha J=i\alpha.$$

Furthermore

$$\alpha(V)=\overline{L}.$$

Similarly, the map $\overline{\alpha}$ defined by

$$\overline{\alpha}(v):=\overline{\alpha(v)}=v+iJv$$

is complex antilinear and maps V onto L.

Since $\overline{\alpha}$ is antilinear, it is convenient to consider the pull back via $\overline{\alpha}$ of the complex conjugate of H to V. Actually, for matters of convenience, we throw in a factor of one half and define

$$\langle u,v\rangle_L:=-\frac{i}{2}(\alpha(u),\overline{\alpha}(v)). \tag{4}$$

From the definition of α we get

$$\begin{aligned}\langle u,v\rangle_L &= -\frac{i}{2}(u-iJu,v+iJv)\\ &= (u,Jv)-i(u,v)\end{aligned}$$

so an alternative expression for $\langle\ ,\rangle_L$ is

$$\langle u,v\rangle_L=(u,Jv)-i(u,v). \tag{5}$$

The first expression shows that $\langle u,v\rangle_L$ is linear in u and antilinear in v, relative to the complex structure given by J on V. Either expression shows that

$$\langle v,u\rangle_L = \overline{\langle u,v\rangle_L}$$

so $\langle\ ,\rangle_L$ defines a Hermitian form on V relative to this complex structure. The non-degeneracy of H when restricted to L implies that the Hermitian form $\langle\ ,\ \rangle_L$ is non-degenerate, and L will be positive definite if and only if $\langle\ ,\ \rangle_L$ is. So to parameterize the positive Lagrangian subspaces we must determine when $\langle\ ,\ \rangle_L$ is positive definite.

If we start with a complex structure, J, on V and a non-degenerate Hermitian form, $\langle\ ,\ \rangle$ relative to J, then the definition of a Hermitian form demands that $\langle Ju, Jv\rangle = \langle u,v\rangle$ for all $u,v \in V$. In particular, this must also hold for the imaginary part of $\langle\ ,\ \rangle$. So if $\mathrm{Im}\,\langle\ ,\ \rangle = -(\ ,\ )$ then J satisfies (1). We can thus state

Proposition 2 *The set of totally complex Lagrangian subspaces is in one to one correspondence with the set of complex structures and Hermitian forms on V which satisfy*

$$\mathrm{Im}\,\langle\ ,\ \rangle = -(\ ,\ ).$$

The corresponding Lagrangian subspace is totally positive if and only if the Hermitian form is positive definite.

Finally, let us construct a positive definite Hermitian form satisfying the conditions of the proposition. We know that we can find a basis

$$e_1,\dots,e_n,f_1,\dots,f_n$$

of V such that

$$(e_i,e_j) = (f_i,f_j) = 0\ \forall ij$$

and

$$(e_i,f_j) = \delta_{ij}.$$

Define J relative to this basis by

$$Je_i = f_i, \quad Jf_i = -e_i.$$

Clearly $J^2 = -\mathrm{id}$ and

$$(Je_i,Je_j) = (Jf_i,Jf_j) = 0, \ \ (Je_i,Jf_j) = -(f_i,e_j) = (e_j,f_i) = \delta_{ij}.$$

Hence J satisfies (3) and (1) and so defines a totally complex Lagrangian subspace. From the definition (5) we see that the e_i are an orthonormal basis for the corresponding Hermitian form which is therefore positive definite.

3 Parameterizing the positive Lagrangians.

Let D denote the set of positive Lagrangian subspaces of $V^{\mathbf{C}}$ and let $\overline{D}$ denote the set of positive semi-definite Lagrangian subspaces, i.e. those for which the restriction of the Hermitian form H is positive semi-definite. At the extreme, the real Lagrangian subspaces belong to $\overline{D}$ since $H(z,w) = i(z,\overline{w}) = i(z,w) = 0$ when z, w being to a real Lagrangian subspace, so the restriction of H to a real Lagrangian subspace is identically zero.

Let us fix one positive Lagrangian subspace, L_0 (for example the one constructed at the end of the preceding section). We denote all the corresponding objects with a subscript zero, for example $\alpha_0, J_0, \langle\ ,\ \rangle_0$. The space $\overline{L_0}$ is negative definite. Hence

$$L \cap \overline{L_0} = \{0\} \quad \forall L \in \overline{D}.$$

Both L_0 and $\overline{L_0}$ are complex subspaces of $V^{\mathbf{C}}$ and

$$V^{\mathbf{C}} = L_0 \oplus \overline{L_0}.$$

Hence we can think of any $L \in \overline{D}$ as being the graph of a complex linear map

$$S(L) : L_0 \to \overline{L_0}.$$

Then the composite

$$T(L) := \alpha_0^{-1} S(L) \overline{\alpha_0} : V \xrightarrow{\overline{\alpha_0}} L_0 \xrightarrow{S(L)} \overline{L_0} \xrightarrow{\alpha_0^{-1}} V$$

is an antilinear map of $V \to V$ relative to the complex structure induced by L_0, i.e.

$$TJ_0 = -J_0 T$$

for $T = T(L)$. By definition,

$$L = \{\overline{\alpha_0}x + \alpha_0 Tx,\} \qquad x \in V. \tag{6}$$

For any $x, y \in V$ we have

$$(\overline{\alpha_0}x + \alpha_0 Tx, \overline{\alpha_0}y + \alpha_0 Ty) = (\alpha_0 Tx, \overline{\alpha_0}y) + (\overline{\alpha_0}x, \alpha_0 Ty) = 2i\left[\langle Tx, y\rangle_0 - \langle Ty, x\rangle_0\right].$$

Thus L is Lagrangian if and only if T is symmetric with respect to the form $\langle\ ,\ \rangle_0$, i.e. satisfies

$$\langle Tx, y\rangle_0 = \langle Ty, x\rangle_0 \quad \forall x, y \in V.$$

If this holds, it certainly holds when we take real parts, i.e.

$$\mathrm{Re}\ \langle Tx, y\rangle_0 = \mathrm{Re}\ \langle Ty, x\rangle_0. \tag{7}$$

Conversely, if this holds, then, since T is antilinear, we have

$$\begin{aligned}
\mathrm{Im}\ \langle Tx, y\rangle_0 &= -\mathrm{Re}\ i\langle Tx, y\rangle_0 \\
&= -\mathrm{Re}\ \langle J_0 Tx, y\rangle_0 \\
&= \mathrm{Re}\ \langle TJ_0 x, y\rangle_0 \\
&= \mathrm{Re}\ \langle Ty, J_0 x\rangle_0 \\
&= -\mathrm{Re}\ \langle J_0 Ty, x\rangle_0 \\
&= \mathrm{Im}\ \langle Ty, x\rangle_0
\end{aligned}$$

so

$$\langle Tx, y\rangle_0 = \langle Ty, x\rangle_0.$$

We have thus proved

Proposition 3 *The set of all Lagrangian subspaces of* $V^{\mathbf{C}}$ *which satisfy* $L \cap \overline{L_0} = \{0\}$ *is in one to one correspondence with the set of all antilinear maps,* $T : V \to V$ *which are self-adjoint relative to the symmetric form* $\mathrm{Re}\langle\ ,\rangle$*, i.e. which satisfy (7).*

Let us now examine which of these is positive. The requirement is that

$$i(z, \overline{z}) = -i(\overline{z}, z)$$

be positive for all non-zero $z \in L$. Writing $z = \overline{\alpha_0}x + \alpha_0 Tx$ This becomes

$$-i(\alpha_0 x + \overline{\alpha_0}Tx, \overline{\alpha_0}x + \alpha_0 Tx) > 0.$$

Since L_0 and $\overline{L_0}$ are Lagrangian, we have

$$-\frac{i}{2}\langle\alpha_0 x + \overline{\alpha_0}Tx, \overline{\alpha_0}x + \alpha_0 Tx\rangle_0 = \langle x, x\rangle_0 - \langle Tx, Tx\rangle_0.$$

Thus

Proposition 4 *The space* D *is in one to one correspondence with the set of all antilinear maps* $T : V \to V$ *which satisfy (7) and*

$$\langle Tx, Tx\rangle_0 < \langle x, x\rangle_0, \quad \forall x \neq 0 \in V. \tag{8}$$

The set $\overline{D}$ *is in one to one correspondence with the set of all antilinear maps* $T : V \to V$ *which satisfy (7) and*

$$\langle Tx, Tx\rangle_0 \leq \langle x, x\rangle_0, \quad \forall x \neq 0 \in V. \tag{9}$$

The set of antilinear T satisfying (7) is clearly a real vector space. Furthermore, if T satisfies (7) and is antilinear with respect to J_0, then

$$\begin{aligned} \langle J_0 Tx, y\rangle_0 &= -\langle Tx, J_0 y\rangle_0 \\ &= -\langle TJ_0 y, x\rangle_0 \\ &= \langle J_0 Ty, x\rangle_0. \end{aligned}$$

Thus J_0T also satisfies (7) (and is anti-linear). In other words, the map

$$T \mapsto J_0 T$$

is a real linear transformation of the space of all such T unto itself, and the square of this map is the map $T \mapsto -T$. We have put a complex structure on this space. In particular, we have made D into a complex manifold.

Finally, let us determine the complex structure J_L and the Hermitian form $\langle\ ,\ \rangle_L$ associated to $L \in D$ in terms of $T = T(L)$. Let us substitute the definitions

$$\begin{aligned} \alpha_0(x) &= x - iJ_0x \\ \overline{\alpha_0}(x) &- x + iJ_0x \end{aligned}$$

into (6). We see that L consists of all

$$x + iJ_0x + Tx - iJ_0Tx = (I+T)x + i(I+T)J_0x$$

where I is the identity transformation of V. In view of (8) the map $I+T$ is invertible. So if we write $(I+T)x = y$, we see that L consists of all

$$y + i(I+T)J_0(I+T)^{-1}y, \qquad y \in V.$$

Thus

$$J_L = (I+T)J_0(I+T)^{-1} = J_0(I-T)(I+T)^{-1}. \tag{10}$$

Thus

$$\text{Re } \langle u, v\rangle_L = (u, J_Lv) = (u, J_0(I-T)(I+T)^{-1}).$$

Hence

$$\langle u, v\rangle_L = \langle u, (I-T)(I+T)^{-1}v\rangle_0. \tag{11}$$

4 The action of $Sp(V)$ on D.

The symplectic group of V (all real linear transformations which preserve the symplectic form) is denoted by $Sp(V)$. It is is clear that $Sp(V)$ acts transitively on D, and that the subgroup fixing L_0 is the unitary group $U(V) = U_0(V)$ consisting of all unitary transformations relative to the complex structure J_0 and the Hermitian form $\langle\ ,\ \rangle_0$. In this section we want to describe this action in terms of the T parameterization of D given in the preceding section, and in terms of a corresponding parameterization of $Sp(V)$. The end result will be a description of this action as "fractional linear transformations" of T, similar to the action of $Sl(2, \mathbf{R})$ on the unit disk.

First some preliminaries. For any linear transformation, g of V, let $g^\dagger$ denote its transpose with respect to the symmetric form Re $\langle\ ,\ \rangle_0$. Since the symplectic form is given in terms of this symmetric form by

$$(\ ,\ ) = \text{Re } \langle J_0\cdot, \cdot\rangle_0,$$

we see that g is symplectic if and only if

$$gJ_0g^\dagger = J_0.$$

For any invertible g, define

$$g^\sharp := (g^\dagger)^{-1}.$$

Thus the condition that an invertible transformation be symplectic is

$$gJ_0 = J_0 g^\sharp.$$

If g is symplectic, then $a = a(g)$ and $b = b(g)$ defined by

$$a(g) := \frac{1}{2}(g + g^\sharp), \quad b(g) := \frac{1}{2}(g - g^\sharp)$$

have the properties

$$a \text{ is } J_0 \text{ linear}, \ \ b \text{ is } J_0 \text{ anti-linear}, \tag{12}$$

$$ab^\dagger = ba^\dagger, \quad a^\dagger b = b^\dagger a \tag{13}$$

and

$$aa^\dagger - bb^\dagger = (a - b)(a^\dagger + b^\dagger) = aa^\dagger - bb^\dagger = I. \tag{14}$$

Clearly we can recover g from a and b as $g = a + b$ and $g^\sharp$ as $a - b$. Furthermore any real linear transformation is uniquely the sum of a complex linear and complex anti-linear transformation, so $g = a + b$ together with (12) determine a and b. Starting with (a, b) satisfying (12)-(14) and defining $g = a + b$, we see from (14) that $g^\dagger$ and hence g is invertible and then that it is symplectic. Thus we have established a one to one correspondence between the elements of the symplectic group and pairs (a, b) satisfying (12)-(14). If $g_1 = a_1 + b_1$ and $g_2 = a_2 + b_2$ then

$$g_1 g_2 = (a_1 a_2 + b_1 b_2) + (a_1 b_2 + b_1 a_2)$$

is the decomposition of $g_1 g_2$ into complex linear and antilinear parts. In short, we may identify $g \in Sp(V)$ with the "matrix"

$$\begin{pmatrix} a & b \\ b & a \end{pmatrix}$$

where (a, b) satisfy (12)-(14) and then multiplication becomes identified with matrix multiplication.

Now let $L \in D$ so that, by (6)

$$\overline{\alpha_0} + \alpha_0 T(L) : V \to L.$$

Hence

$$g \circ [\overline{\alpha_0} + \alpha_0 T(L)] \ \text{ and } \overline{\alpha_0} + \alpha_0 T(gL)$$

are both bijective real linear maps of $V \to gL$. Thus the map

$$v \mapsto \left(g \circ [\overline{\alpha_0} + \alpha_0 T(L)]\right)^{-1} [\overline{\alpha_0} + \alpha_0 T(gL)] v := u$$

is an automorphism of V. We can write the relation between u and v as

$$g[\overline{\alpha_0} + \alpha_0 T(L)]u = [\overline{\alpha_0} + \alpha_0 T(gL)]v.$$

Since $\alpha_0 = I + iJ_0$ the real and imaginary parts of this equation become

$$\begin{aligned} g(I + T(L))u &= (I + T(gL))v \\ gJ_0(I - T(L))u &= J_0(I - T(gL))v. \end{aligned}$$

Using $gJ_0 = J_0 g^\sharp$ we can rewrite this pair of equations as

$$\begin{aligned} g[I + T(L)]u &= [I + T(gL)]v \\ g^\sharp[I - T(L)]u &= [I - T(gL)]v. \end{aligned}$$

Adding and subtracting these equations gives

$$\begin{aligned} v &= [bT(L) + a]u \\ T(gL)v &= [aT(L) + b]u \end{aligned}$$

where $a = a(g), b = b(g)$ as above. In particular the operator $aT(L) + b$ is invertible and

$$T(gL) = (aT(L) + b)(bT(L) + a)^{-1}. \tag{15}$$

Informally, we can think of this as saying that the transformation on D corresponding to

$$g = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$$

is the "fractional linear transformation"

$$T \mapsto \frac{aT + b}{bT + a}.$$

5 The cocycle.

Let $B(g, L)$ denote the "denominator" occurring in (15). That is, define

$$B(g, L) := bT + a. \tag{16}$$

Since a is J_0 linear, and b and T are both J_0 anti-linear, we see that B is J_0 linear. For $g_1, g_2 \in Sp(V)$ we have

$$\begin{aligned} B(g_1 g_2, L) &= (a_1 b_2 + b_1 a_2)T + a_1 a_2 + b_1 b_2 \\ &= b_1(a_2 T + b_2) + a_1(b_2 T + a_2) \\ &= b_1(a_2 T + b_2)(b_2 T + a_2)^{-1}(b_2 T + a_2) + a_1(b_2 T + a_2) \\ &= B(g_1, g_2 L)B(g_2, L). \end{aligned}$$

So B satisfies the cocycle condition

$$B(g_1 g_2, L) = B(g_1, g_2 L)B(g_2, L). \tag{17}$$

Since B is complex linear (relative to J_0) we may compute its determinant as a complex matrix (so as an $n \times n$ matrix if V is $2n$ dimensional). Define

$$\chi : Sp(V) \times D \to \mathbf{R}/2\pi\mathbf{Z}$$

by

$$e^{i\chi(g,L)} = \frac{\det_{\mathbf{C}} B(g,L)}{|\det_{\mathbf{C}} B(g,L)|}.$$

In other words,

$$\chi(g,L) := \arg\ \det_{\mathbf{C}} B(g,L). \tag{18}$$

Then we have

$$\chi(g_1g_2,L) = \chi(g_1,g_2L) + \chi(g_2,L). \tag{19}$$

6 Action on pairs of positive Lagrangians.

Let $L_1, L_2 \in D$ with the corresponding T_1, T_2 in terms of our parameterization. Since T_1 and T_2 are anti-linear,

$$A(L_1,L_2) := I - T_1T_2 \tag{20}$$

is linear. We claim that

$$A(gL_1,gL_2) = (B(g,L_1)^*)^{-1}A(L_1,L_2)B(g,L_2)^{-1}. \tag{21}$$

For this observe that $T = T^\dagger$ and similarly for gT so

$$\begin{aligned}
A(gL_1,gL_2) &= I - (aT_1+b)(bT_1+a)^{-1}(aT_2+b)(bT_2+a)^{-1} \\
&= I - [(aT_1+b)(bT_1+a)^{-1}]^\dagger(aT_2+b)(bT_2+a)^{-1} \\
&= I - (bT_1+a)^{-1\dagger}(aT_1+b)^\dagger(aT_2+b)(bT_2+a)^{-1} \\
&= (bT_1+a)^{-1\dagger}\left[(bT_1+a)^\dagger(bT_2+a) - (aT_1+b)^\dagger(aT_2+b)\right](bT_2+a)^{-1} \\
&= (bT_1+a)^{-1\dagger}\left[(T_1b^\dagger+a^\dagger)(bT_2+a) - (T_1a^\dagger+b^\dagger)(aT_2+b)\right](bT_2+a)^{-1} \\
&= B(g,L_1)^{-1\dagger}[I - T_1T_2]B(g,L_2)^{-1}.
\end{aligned}$$

Now for a complex linear map M, (such as B^{-1}) its adjoint, $M^\dagger$ with respect to the symmetric form $\mathrm{Re}\langle\ ,\ \rangle_0$ is the same as its adjoint, M^* with respect to $\langle\ ,\ \rangle_0$. This proves (21).

In particular, if we define

$$s(L_1,L_2) := \mathrm{argdet}_{\mathbf{C}} A(T_1,T_1) \tag{22}$$

we get

$$s(gL_1,gL_2) = s(L_1,L_2) + \chi(g,L_1) - \chi(g,L_2). \tag{23}$$

7 The Fock representations.

Let $N = N(V)$ be the Heisenberg group, and let $\mathcal{F} = \mathcal{F}(N(V))$ be the space of smooth functions on N. The group $N(V)$ acts on itself by right or left multiplication, and we get corresponding actions on $\mathcal{F}$ given by

$$[\mathcal{L}(a)f](n) := f(a^{-1}n)$$

and

$$[\mathcal{R}(a)f](n) := f(na).$$

We will use the corresponding lower case letter to denote the action of the Lie algebra, so for X in the Heisenberg algebra,

$$[\ell(X)f](n) := \frac{d}{dt}f((\exp -tX)n)_{|t=0}$$

and

$$[r(X)f](n) := \frac{d}{dt}f(n\exp tX)_{|t=0}.$$

Let H denote the Hilbert space completion of the space of all $f \in \mathcal{F}$ which satisfy

$$f(n(\exp tE)) = e^{-2\pi it}f(n) \tag{24}$$

and

$$\int_V |f|^2 dv < \infty. \tag{25}$$

This last equation is to be understood as follows: In view of (24), the function $|f|$ depends only on $N/\mathbf{R}E$ and hence, by the diffeomorphism $v \mapsto \exp v$ can be considered as a function on V. We demand that this function be square integrable relative to Lebesgue measure. We fix a choice of Lebesgue measure (which is determined up to positive scalar multiple) and this makes $\mathcal{H}$ into a Hilbert space. For each $L \in D$, let $\mathcal{H}_L$ denote the Hilbert space completion of those elements of $\mathcal{F}$ which satisfy (24), (25) and also

$$r(X)f = 0, \quad \forall X \in L. \tag{26}$$

We have thus attached a Hilbert space, $\mathcal{H}_L$ to each $L \in D$ and all of these Hilbert spaces are subspaces of the large Hilbert space, $\mathcal{H}$. Since right and left translations in any group commute, each of the spaces $\mathcal{H}_L$ is invariant under the action $\mathcal{L}$ of N and hence give rise to a representation of N. We shall soon see that these representations are irreducible.

If $L_1, L_2 \in D$, the scalar product in $\mathcal{H}$ induces a pairing

$$\mathcal{H}_{L_1} \times \mathcal{H}_{L_2} \to \mathbf{C}$$

between $\mathcal{H}_{L_1}$ and $\mathcal{H}_{L_2}$: sending a pair consisting of $f_1 \in \mathcal{H}_{L_1}$ and $f_2 \in \mathcal{H}_{L_2}$ into

$$(f_1, f_2)_{\mathcal{H}} = \int_V f_1\overline{f_2}dx.$$

This pairing induces a map

$$P_2 \circ I_1 : \ \mathcal{H}_{L_1} \to \mathcal{H}_{L_2}$$

where

$$I_L : \mathcal{H}_L \to \mathcal{H}$$

is injection as a subspace and

$$P_L : \mathcal{H} \to \mathcal{H}_L$$

is orthogonal projection onto the subspace and where we have written

$$I_1 = I_{L_1}, \quad P_2 = P_{L_2}.$$

We will find that for any $L_1, L_2 \in D$, the spaces $\mathcal{H}_{L_1}$ and $\mathcal{H}_{L_2}$ are not orthogonal, so $P_2 \circ I_1 \neq 0$. Once we have proved that the representations on the $\mathcal{H}_L$ are irreducible, this implies that we can find a positive scalar $c = c_{12}$ such that

$$I_{21} := cP_2 \circ I_1, \quad I_{21} : \mathcal{H}_{L_1} \to \mathcal{H}_{L_2}$$

is a canonical unitary intertwining operator between these two irreducible representations of N.

We now turn to the proof that the representation of N on each $\mathcal{H}_L$ is irreducible. By the Stone-von Neumann theorem, we know that each such representation is a multiple of the unique irreducible representation for which $\rho(\exp tE) = e^{2\pi it}$ id. We must prove that this multiple is one. For this, it suffices to prove that there is (up to scalar multiple), a unique "vacuum state", by which we mean the following: Fix some $L_0 \in D$ which we then use to parameterize D as in the preceding sections. For any representation, σ, satisfying the Stone -von Neumann condition (and hence a multiple of the unique irreducible satisfying this condition) let us define a "vacuum vector" to be one which satisfies

$$\dot{\sigma}(X)f = 0 \quad \forall X \in \overline{L_0}. \tag{27}$$

It will be enough for us to prove that the set of vacuum vectors in $\mathcal{H}_L$ is a one dimensional subspace: for this implies that the unique irreducible, H_0, satisfying the Stone - von Neumann condition contains non-zero vacuum vectors, and hence that the dimension of the space of vacuum vectors in any multiple, $H_0 \otimes H_1$ is at least equal to dim H_1.

Let $N^{\mathbf{C}}$ denote the complex simply connected Lie group obtained by exponentiating the complexification of the Heisenberg algebra $h(V)$. Thus N embeds as a subgroup of $N^{\mathbf{C}}$.

In the group $N^{\mathbf{C}}$ it makes sense to talk of $\exp X, X \in L$ or $X \in \overline{L_0}$.

We have the direct sum decomposition

$$V^{\mathbf{C}} = L \oplus \overline{L_0}.$$

Let p_L and p_0^* denote the corresponding projections according to this direct sum decomposition, both being complex linear maps. Let e denote the identity element of $N^{\mathbf{C}}$ (and of N).

Lemma 1 *There exists a unique function defined on $N^{\mathbf{C}}$ such that*

1. $\phi(1) = 1$,

2. $\phi((\exp X)u) = \phi(u) \quad \forall X \in \overline{L_0},\ u \in N^{\mathbf{C}}$,

3. $\phi(u \exp Y) = \phi(u) \quad \forall Y \in L,\ u \in N^{\mathbf{C}}$,

4. $\phi(u(\exp tE)) = e^{-2\pi it}\phi(u) \quad \forall\ t \in \mathbf{R},\ u \in N^{\mathbf{C}}$.

The function ϕ is holomorphic on $N^{\mathbf{C}}$ and is given by

$$\phi\left((\exp v) \exp zE\right) = e^{\pi i(p_0^* v, p_L v) - 2\pi iz}.$$

Proof . Writing $v = p_L v + p_0^* v$ we obtain

$$\exp v = (\exp p_0^* v)(\exp p_L v)(\exp -\frac{1}{2}(p_0^* v, p_L v)E)$$

from the multiplication law in $N^{\mathbf{C}}$. The formula for ϕ then follows from our requirements. By inspection, this formula shows that ϕ is holomorphic. QED

We now come to the main theorem of this section:

Theorem 1 *For any $L \in D$ there is a unique $f = f_L \in \mathcal{F}$ which satisfies (24), (26), such that $f(e) = 1$ and*

$$\ell(X)f = 0 \quad \forall X \in \overline{L_0}. \tag{28}$$

In fact, this function f_L, which by (24) is determined by its value on V is given by

$$f_L(\exp v) = e^{-\frac{\pi}{2}\langle I - Tv, v\rangle_0} \tag{29}$$

where $T = T(L)$. In particular, this function belongs to $\mathcal{H}$ and hence to $\mathcal{H}_L$ and is, up to scalar multiple, the unique vacuum vector in $\mathcal{H}_L$ so that $\mathcal{H}_L$ is irreducible.

Proof. Let us first observe that the restriction of the function ϕ to the real subgroup $N \subset N^{\mathbf{C}}$ satisfies all the conditions of the theorem. Indeed, the map

$$X \mapsto \phi(u(\exp X)), \quad X \in V^{\mathbf{C}}$$

is holomorphic and hence

$$\frac{d}{dt}\phi(u(\exp itX))_{|t=0} = i\frac{d}{dt}\phi(u(\exp tX))_{|t=0}.$$

Hence, for $u \in N, Y \in L$

$$r(Y)\phi(n) = \frac{d}{dt}\phi(n(\exp tY))_{|t=0} = 0$$

by property 3) of ϕ, verifying (26). A similar argument applied to left multiplication (and using property 2)) verifies (28). Condition (24) is property 4) and $\phi(e) = 1$ is property 1) of ϕ.

To verify that the restriction of ϕ to N is given by (29) we note that

$$L = (\overline{\alpha_0} + \alpha_0 T)V, \quad \overline{L_0} = \alpha_0 V$$

and $\alpha_0 v = v - iJ_0 v$ so every $v \in V$ can be written as

$$v = \frac{1}{2}(\alpha_0 + \overline{\alpha_0})v = \frac{1}{2}(\overline{\alpha_0} + \alpha_0 T)v + \frac{1}{2}\alpha_0(I - T)v$$

so the restrictions of p_L and p_0^* to $V \subset V^{\mathbf{C}}$ are given by

$$p_L(v) = \frac{1}{2}(\overline{\alpha_0} + \alpha_0 T)v, \quad p_0^* v = \frac{1}{2}\alpha(I - Tv).$$

Hence

$$-\frac{i}{2}(p_L v, p_0^* v) = -\frac{i}{8}(\overline{\alpha_0} v, \alpha_0(I - T)v) = -\frac{1}{4}\langle v, (I - T)v\rangle_0.$$

Now to the uniqueness: Suppose that g were some other function satisfying the conditions. Then $h := g/f$ is well defined since f is everywhere positive, and h satisfies $h(n_1 n_2) = h(n_2 n_1), \forall n_1, n_2 \in N$ since $h(n \exp tE) = h(n)$. Thus h is really a function on the commutative group obtained from N by quotienting out its center, and on a commutative group right and left multiplication coincide. So the operators $\ell(X)$ and $r(X)$ coincide. Since L and $\overline{L_0}$ span all of V^C, it follows from (26) and (28) that $\ell(X)h \equiv 0$ for all $X \in V$, hence that h is a constant, equal to one by our normalization. The rest of the theorem now follows. QED

8 Creation and annihilation operators.

Suppose we take $L = L_0$ so $T = 0$. Let us write f_0 instead of f_{L_0} so

$$f_0(\exp v) = e^{-\frac{\pi}{2}\langle v,v\rangle_0}.$$

Similarly, let us write $\mathcal{H}_0$ instead of $\mathcal{H}_{L_0}$. We write the most general element $f \in \mathcal{H}_0$ as

$$f(\exp v) = g(v) f_0(\exp v).$$

Since g is expressed as the quotient of two functions satisfying (24) it is a function on the Abelian group, V as we pointed out in the preceding section. The condition that

$$r(X)f = 0 \ \ \forall X \in L$$

translates into the condition that g be holomorphic as a function of v when we use the complex structure determined by J_0 on V. We now compute the action of $\ell(X)$ for $X \in L_0$ and for $X \in \overline{L_0}$. For $X = X_1 + iX_2 \in L_0,\ X_1, X_2 \in V$ we have

$$[\ell(X)f](\exp v)$$

$$\begin{aligned}
&= \frac{d}{dt}[e^{\pi i(tX_1,v)} f((\exp v)(\exp(-tX_1) + ie^{\pi it(X_2,v)} f((\exp v)(\exp -tX_2))]_{|t=0} \\
&= \pi i(X,v) f(\exp v) + r(X) f(\exp v) \\
&= \pi i(X,v) f(\exp v)
\end{aligned}$$

so $\ell(X)$ acts on g by

$$\ell(X): \quad g \mapsto \pi i(X, \cdot) g \quad X \in L_0. \tag{30}$$

On the other hand, since $\ell(X) f_0 = 0, \ \ \forall X \in \overline{L_0}$, we get

$$\ell(X): \quad g \mapsto -D_X g \quad X \in \overline{L_o}. \tag{31}$$

In short, we have identified $\mathcal{H}_0$ with the space of holomorphic functions on V which have finite norm relative to the scalar product

$$(g_1, g_2) := \int_V g_1 \overline{g_2} e^{-2\pi\langle v,v\rangle_0} dv.$$

The action of $X \in L_0$ is given by the "creation operator" (30) and the action of $X \in \overline{L_0}$ is given by the annihilation operator (31).

9 The pairing.

Let $L_1, L_2 \in D$. We will show that $P_2 \circ I_1 \neq 0$ by showing that the corresponding vacuum vectors $f_1 = f_{L_1}$ and $f_2 = f_{L_2}$ have non-zero scalar product. Indeed

$$\psi(T_1, T_2) := \int_V f_1 \overline{f_2} dv = \int_V e^{-\pi\langle v,v\rangle_0 + \frac{\pi}{2}[\langle T_1 v, v\rangle_0 + \langle v, T_2 v\rangle_0]} dv.$$

This expression is holomorphic as a function of T_1 and anti-holomorphic as a function of T_2. We claim that

$$\psi(T_1, T_2)^2 = K \det_{\mathbf{C}}(I - T_1 T_2)^{-1} \tag{32}$$

where $K > 0$ is a positive constant (independent of T_1 and T_2). Notice that the function

$$(T_1, T_2) \mapsto \det_{\mathbf{C}}(I - T_1 T_2)$$

is also holomorphic as a function of T_1 and anti-holomorphic as a function of T_2 since we must move the J_0 past the T_1 if we multiply T_2 on the left by J_0 before computing the determinant. If we have two functions, ψ_1 and ψ_2 which are both holomorphic as functions of T_1 and anti-holomorphic as functions of T_2, then $\psi_1(T, T) = \psi_2(T, T)$ for all T implies that $\psi_1(T_1, T_2) = \psi_2(T_1, T_2)$ for all T_1 and T_2. Applied to both sides of (32), we need only prove (32) when $T_1 = T_2 = T$, and the exponent in the exponential in the integral defining ϕ becomes

$$-\pi \mathrm{Re}\ \langle (I - Tv, v\rangle_0.$$

The operator $I - T$ is a symmetric operator with respect to the orthogonal form Re $\langle\ ,\ \rangle_0$ with strictly positive eigenvalues. Hence it has a unique positive symmetric square root, and we have

$$\phi(T, T) = \int_V e^{\pi \mathrm{Re}\ \langle ((I-T)^{\frac{1}{2}} v, (I-T)^{\frac{1}{2}} v\rangle_0} dv.$$

Setting $v = (I - T)^{\frac{1}{2}} w$ we obtain, by change of variables,

$$\phi(T, T) = \det(I - T)^{-\frac{1}{2}} \int_V e^{-\pi \langle w, w \rangle_0} dw$$

and the integral occurring on the right of this equation is strictly positive, and is independent of T_1 and T_2. We take it as our K. The determinant that occurs on the right is a real determinant, of a $2n \times 2n$ matrix. So to complete the proof of (32) it is enough to show that

$$\det_{\mathbf{R}}(I - T) = \det_{\mathbf{C}}(I - T^2). \tag{33}$$

Since T is symmetric relative to our orthogonal form it is diagonizable. If e is an eigenvector of T with eigenvalue, μ, so $Te = \mu e$, then $f = J_0 e$ satisfies $Tf = TJ_0 e = -J_0 Te = -\mu f$. The space spanned by e and f is a complex one dimensional subspace on which is an eigenspace for the complex linear transformation T^2 has eigenvalue μ^2. So decomposing V into a direct sum of the two dimensional real subspaces (which one dimensional complex subspaces) we see that

$$\det_{\mathbf{C}}(I - T^2) = \prod_{\mu > 0}(1 - \mu^2) = \prod_{\mu > 0}(1 - \mu)(1 + \mu) = \det_{\mathbf{R}}(I - T)$$

which is what was to be proved.

In particular, we see that $\psi(T_1, T_2) \neq 0$, and that

$$2\arg\, \psi(T_1, T_2) = -\arg\det_{\mathbf{C}}(I - T_1 T_2) = -s(L_1, L_2) \tag{34}$$

in the terminology of (20) and (22).

10 Intertwinings.

Since the spaces $\mathcal{H}_{L_1}$ and $\mathcal{H}_{L_2}$ are not orthogonal for any $L_1, L_2 \in D$, there exists a unique positive constant c_{21} such that

$$I_{21} := c_{21} P_2 \circ I_1 : \ \mathcal{H}_{L_1} \to \mathcal{H}_{L_2}$$

is a unitary intertwining operator. We have seen that $I_{21} f_1 = \beta(21) f_2$ where $\beta(21)$ is a complex number whose argument is the same as the argument of $\psi(T_1, T_2)$ and so satisfies (34). Suppose that we have three elements, $L_1, L_2, L_3 \in D$. Then

$$I_{13} I_{32} I_{21} : \mathcal{H}_{L_1} \to \mathcal{H}_{L_1}$$

is a unitary intertwining operator, and hence, by Schur's lemma, is some multiple of the identity:

$$I_{13} I_{32} I_{21} = \mu(1, 2, 3) I$$

where μ is a complex number of absolute value one. We then have

$$-2\arg\, \mu(1, 2, 3) = s(1, 2) + s(2, 3) + s(3, 1). \tag{35}$$

(What we have done, can be expressed in terms of the lines of vacuum states of each representation states as follows: Given three non-perpendicular lines, ℓ_1, ℓ_2, ℓ_3 in a Hilbert space $\mathcal{H}$, we get an invariant

$$\arg\,(v_1, v_2)(v_2, v_3)(v_3, v_1), \quad 0 \neq v_i \in \ell_i,\ i = 1, 2, 3.$$

That is, the answer does not depend on the choice of the vectors v_i.)

11 The metaplectic representation.

The group $Sp(V)$ acts as automorphisms of N, preserving the center. Hence it acts on the space $\mathcal{H}$. We denote this action by C. So

$$[C(g)f](n) := f(g^{-1}n).$$

Since

$$g^{-1}(n_1(gn_2)) = (g^{-1}n_1)(g^{-1}gn_2) = (g^{-1}n_1)n_2,$$

if $r(X)f = 0$, then $r(gX)[C(g)f] = 0$. In other words,

$$C(g) : \mathcal{H}_L :\to \mathcal{H}_{gL}.$$

This map intertwines the representation

$$a \mapsto \mathcal{L}_L(a)$$

of N on $\mathcal{H}_L$ and the representation

$$a \mapsto \mathcal{L}(ga)$$

of N on

$$\mathcal{H}_{gL}.$$

Indeed, for $g \in Sp(V)$ and $a, n \in N$ we have

$$\begin{aligned} [C(g)\mathcal{L}(a)C(g^{-1})f](n) &= [C(g)\mathcal{L}(a)f](gn) \\ &= [C(g)f](g(a^{-1}n)) \\ &= [C(g)f](ga^{-1} \cdot gn) \\ &= f((ga)^{-1}n) \\ &= [\mathcal{L}(ga)f](n). \end{aligned}$$

We also have the intertwining operator

$$I_{L,gL} : \mathcal{H}_{gL} \to \mathcal{H}_L.$$

We can compose them, and so define

$$\rho_L(g) := I_{L,gL} \circ C(g) :\ \mathcal{H}_L \to \mathcal{H}_L.$$

The map

$$\rho_L(g)$$

intertwines the irreducible representations of the Heisenberg group:

$$\rho_L(g)\mathcal{L}_L(a)\rho_L(g^{-1}) = \mathcal{L}_L(ga). \tag{36}$$

Since both representations are irreducible, this equation determines R up to multiplication by a scalar factor of absolute value one. We thus get a projective representation of $Sp(G)$. To determine the multiplier, observe that $Sp(V)$ invariance implies that

$$C(g)I_{L_1L_2} = I_{gL_1gL_2}C(g)$$

so

$$\begin{aligned}\rho_L(g_1)\rho_L(g_2) &= I_{L,g_1L}C(g_1)I_{L,g_2L}C(g_2)\\ &= I_{L,g_1L}I_{g_1L,g_1g_2L}C(g_1g_2)\\ &= \mu(L,g_1g_2L,g_1L)I_{L,g_1g_2L}C(g_1g_2)\end{aligned}$$

so

$$\rho_L(g_1g_2) = c(g_1,g_2)\rho_L(g_1)\rho_L(g_2) \quad \text{where } c(g_1,g_2) = \mu(L,g_1L,g_1g_2L). \tag{37}$$

We now apply (35), (23) and (19) to obtain

$$\begin{aligned}-2\arg\,\mu(L,g_1L,g_2L) &= s(L,g_1L)+s(g_1L,g_1g_2L)+s(g_1g_2L,L)\\ &= s(L,g_1L)+\chi(g_1,L)-\chi(g_1,g_2L)+s(L,g_2L)+s(g_1g_2L,L)\\ &= [s(L,g_1L)+\chi(g_1,L)]+[s(L,g_2L)+\chi(g_2,L)]\\ &\qquad -[s(L,g_1g_2L)+\chi(g_1g_2,L)].\end{aligned}$$

So let us define

$$s_L(g) := e^{i[s(L,gL)+\chi(g,L)]}.$$

Then

$$c(g_1,g_2)^2 = \frac{s_L(g_1)s_L(g_2)}{s_L(g_1g_2)}.$$

Define the group G_L to consist of all pairs (g,u) where $g \in Sp(V)$ and u is a complex number with $|u| = 1$ satisfying

$$u^2 = s_L(g)^{-1}.$$

Define the multiplication law

$$(g_1,u)(g_2,u) := (g_1g_2, u_1u_2c(g_1,g_2)^{-1})$$

which is well defined. So G_L is a double cover of $Sp(V)$ due to the ambiguity in the choice of square root defining u. Define

$$\rho_L(g,u) := u\rho_L(g).$$

Then ρ_L is an ordinary representation of G_L.

12 Extensions to the boundary.

We now want to describe how to extend the preceding results so as to include positive semidefinite Lagrangian subspaces, i.e. elements of $\overline{D}$. So let $L \in \overline{D}$ and define

$$\lambda := L \cap V$$

so that

$$\lambda^{\mathbf{C}} = L \cap \overline{L}.$$

The extreme case, of most interest to us, will be where L is real, i.e. $L = \overline{L}$ and λ is a real Lagrangian subspace of V. The other extreme, where $\lambda = 0$, corresponds to points of D. In all cases λ is a real isotropic subspace of V. Let

$$m = \lambda \oplus \mathbf{R}E,$$

so that m is an abelian subalgebra of the Heisenberg algebra, and we let M denote the corresponding subgroup of N. The corresponding quotient, N/M is diffeomorphic to V/λ and the left invariant measures on N/M correspond to Lebesgue measures on V/λ. We pick one of them. We now consider the set of (say smooth) functions on N which satisfy

$$f(n \cdot (\exp(v + tE)) = e^{-2\pi i t} f(n) \quad \forall v \in \lambda, t \in \mathbf{R} \tag{38}$$

and

$$\int_{N/M} |f^2| < \infty. \tag{39}$$

We let $\mathcal{H}(\lambda)$ denote the Hilbert space completion of this space relative to the L^2 norm.

For $L \in D$, where $\lambda = \{0\}$ independently of L we have

$$\mathcal{H}(\{0\}) = \mathcal{H},$$

the large Hilbert space that we were working with before. But on the boundary, the "large" Hilbert space will depend on L. At the extreme of real L, the spaces $\mathcal{H}(\lambda)$ are all different. We still have the action

$$[L(a)f](n) = f(a^{-1}n)$$

giving a representation of N on $\mathcal{H}(\lambda)$ which satisfies the Stone-von Neumann condition, and hence is some multiple of the basic irreducible representation. We have the corresponding infinitesimal action

$$[\ell(X)f](n) := \frac{d}{dt} f((\exp -t)n)_{|t=0}$$

on the smooth elements of $\mathcal{H}(\lambda)$, for X in the Lie algebra of the Heisenberg group, and similarly the infinitesimal right action $r(X)$.

Finally, as before, we define

$$\mathcal{H}_L \subset \mathcal{H}(\lambda)$$

to be the subspace which is the closure of the space of C^∞ elements in $\mathcal{H}(\lambda)$ which satisfy (26). For real Lagrangian L this is no supplementary condition at all, so $\mathcal{H}_L = \mathcal{H}(\lambda)$.

Inside of $\mathcal{H}(\lambda)$ and $\mathcal{H}_L$ we define the subspace of C^∞ vectors, denoted by $\mathcal{H}^\infty(\lambda)$ and $\mathcal{H}_L^\infty$ to be the completion relative to the Frechet norms

$$f \mapsto \int_{V/L\cap V} |\ell^{j_1}(X_1)\cdots \ell^{j_{2n+1}}(X_{2n+2})f|^2 dx$$

where $X_1, \ldots, X_{2n+1}$ ranges over a basis of the Heisenberg algebra and $J = (j_1, \ldots, j_{2n+1})$ are non-negative integers. We denote these semi-norms by by $\| \,\|_J$ and set

$$\| \,\|_m = \sum_{|J|\leq m} \| \,\|_J, \quad |J| := j_1 + \cdots + j_{2n+1}.$$

For example, in the case of a totally real Lagrangian subspace, the space of C^∞ vectors of $\mathcal{H}(\lambda) = \mathcal{H}_L$ can be identified with the Schwartz space in the Schrodinger realization, while for the case of a totally positive L, the the space $\mathcal{H}_L^\infty$ can be identified with holomorphic functions such that $z^p\partial^q f/\partial z^q$ are square integrable with respect to a Gaussian measure, as we have seen.

Let $L_1, L_2 \in \overline{D}$, and write $\mathcal{H}_i$ for $\mathcal{H}_{L_i}$ etc. as usual. Let

$$\lambda_{12} := L_1 \cap L_2 \cap V$$

and M_{12} the subgroup of N whose Lie algebra is $\lambda_{12} + \mathbf{R}E$. Thus N/M_{12} is diffeomorphic to V/λ_{12} and invariant measures on N/M_{12} correspond to Lebesgue measures on V/λ_{12}. We choose one of them. We now try to define the sesquilinear pairing

$$\langle g_1, g_2\rangle_{12} := \int_{V/\lambda_{12}} g_1\overline{g_2}dv. \tag{40}$$

We will show that that this is well defined for $g_i \in \mathcal{H}_i^\infty$. We use a series of lemmas:

Lemma 2 *Let $L \in \overline{D}$ and suppose that $f \in C^\infty(N)$ satisfies*

$$r(X)f = 0, \quad \forall X \in L$$

and

$$f(n(\exp tE)) = e^{-2\pi it}f(n) \quad \forall n \in N.$$

Then

$$[\ell(Y)f](\exp v) = i(Y, v)f(\exp v) \quad \forall v \in V, Y \in L. \tag{41}$$

Indeed, write $Y = w_1 + iw_2$, $w_i \in V$. Then

$$[\ell(Y)f](\exp v) = \left[\frac{d}{dt}[f\left((\exp v)(\exp tw_1)e^{it(w_1,v)}\right)] + i\frac{d}{dt}[f\left((\exp v)(\exp tw_2)e^{it(w_2,v)}\right)] + i\frac{d}{dt}\right]_{|t=0}.$$

In carrying out the differentiation, use $r(Y)f = 0$ so that only the exponential term contributes. QED

Recall that for any L, the kernel of the restriction of the Hermitian form $H(z,w) = i(z,\overline{w})$ to L is $\lambda^{\mathbf{C}}$. Using this fact, we will prove

Lemma 3 *Let $L_1, L_2 \in \overline{D}$ with*

$$\dim \lambda_{12} = j, \quad \dim \lambda_1 = p.$$

Then there exists Darboux basis $P_1, \ldots, P_n, Q_1, \ldots, Q_n$ of V and an integer k with $1 \leq j \leq k \leq s$ such that

1. $\lambda_{12} = \mathbf{R}P_1 \oplus \cdots \oplus \mathbf{R}P_j$
2. $\mathbf{R}P_{j+1} \oplus \cdots \oplus \mathbf{R}P_k \oplus \mathbf{R}P_{k+1} \oplus \cdots \mathbf{R}P_s$ *is a complement to λ_{12} in λ_1,*
3. $\mathbf{C}Q_{j+1} \oplus \cdots \oplus Q_k \oplus \mathbf{C}(P_{k+1} + iQ_{k+1}) \oplus \cdots \oplus \mathbf{C}(P_s + iQ_s)$ *is a subspace of λ_2.*

Proof. In case $s = j$ properties 2) and 3) disappear and the assertion is merely that $\lambda_1 = \lambda_{12}$ is an isotropic subspace of V for which 1) is obvious. So we proceed by induction on $s - j$. If $s > j$, choose P_s to be any element of λ_1 which does not belong to λ_{12}. Let p denote the projection of $V^{\mathbf{C}} = V \oplus iV$ on to V along iV.Then $p(L_2) = \lambda_2^{\perp}$. We consider two cases:

If $p^{-1}(P_s) \cap L_2 = \{0\}$, then P_s is not perpendicular to λ_2, so we can find a $Q_s \in \lambda_2$ such that $(P_s, Q_s) = 1$.

Otherwise, we can find Q_s such that $P_s + iQ_s \in L_2$, and the positivity of L_2 implies that $0 < i(P_s + iQ_s, P_s - iQ_s) = 2(P_s, Q_s)$.

In either case, the plane spanned by P_s, Q_s is symplectic, and we can pass from V to V', the orthogonal complement of this plane, and consider $L_i' := L_i \cap V'^{\mathbf{C}}$. For the corresponding λ's we have $s' - j' = s - j - 1$. QED

Lemma 4 *There exists a positive continuous function ϕ_1 defined on V such that the map*

$$g_1 \mapsto \frac{g_1}{\phi_1}$$

is a continuous map of

$$L^2(V/\lambda_1) \to L^2(V/\lambda_{12})$$

while the map

$$g_2 \mapsto \phi_1 g_2$$

is a continuous map of

$$\mathcal{H}_2^{\infty} \to L^2(V/\lambda_2).$$

Proof. Choose a basis as in the preceding lemma and write the most general element of V as $v = \sum x_i P_i + \sum y_i Q_i$. Also set $z_j = x_j + iy_j$. If $g_1 \in L^2(V/\lambda_1)$ then g_1 does not depend on on $x_1, \ldots, x_s$. The map

$$g_1 \mapsto \frac{g_1}{(1 + |x_{j+1}|) \cdots (1 + |x_s|)}$$

introduces the necessary damping factor to guarantee that the image has a convergent square integral, i.e.lies in $L^2(V/\lambda_{12})$. So we set

$$\phi_1 := \prod_{j+1}^{s} (1 + |x_r|).$$

Since $\ell(Q_r)g_2 = -ix_r g_2, \quad j < r \le k$ and $\ell(P_r + iQ_r)g_2 = z_r g_2$ for $k < r \le s$, we see that the product of $g_2 \in \mathcal{H}_2^\infty$ by any monomial in the variable $x_j, \ldots, x_s$ will still be in $L^2(V/\lambda_2)$. QED

Now for $g_1 \in \mathcal{H}_1^\infty, \ g_2 \in \mathcal{H}_g^\infty$ the map

$$g_1 \mapsto \frac{\phi_2 |g_1|}{\phi_1}$$

is a continuous map of $\mathcal{H}_1^\infty \to L^2(V\lambda_{12})$ and similarly with 1 and 2 interchanged. Since

$$|g_1 \overline{g_2}| = \frac{\phi_2 |g_1|}{\phi_1} \cdot \frac{\phi_1 |g_2|}{\phi_2}$$

we see that the sesquilinear form (40) is bicontinuous on $\mathcal{H}_1^\infty \times \mathcal{H}_2^\infty$.

It is non-zero, and hence, since both $\mathcal{H}_1$ and $\mathcal{H}_2$ are copies of the irreducible Stone von-Neumann representation, we may multiply by a positive constant so as to get a canonical unitary intertwining operator

$$I_{21} : \mathcal{H}_1 \to \mathcal{H}_2$$

such that for all $g_1 \in \mathcal{H}_1^\infty, \ g_2 \in \mathcal{H}_2^\infty$ we have

$$(I_{21} g_1, g_2)_2 = \langle g_1, g_2 \rangle_{12}.$$

In the case of totally real Lagrangian subspaces I_{21} is given by a partial Fourier transform.

If L_1, L_2, L_3 are three elements of $\overline{D}$ we once again get a scalar of absolute value one defined by

$$I_{31} \circ I_{23} \circ I_{21} = \mu(L_1, L_2, L_3) I.$$

Our purpose in the next section will be to compute $\mu(L_1, L_2, L_3)$ for the case of totally real Lagrangian subspaces in terms of the Maslov index.The main result will be (50).

13 The Maslov index.

Let ℓ_1, ℓ_2, ℓ_3 be three Lagrangian subspaces of a symplectic vector space, V. On the $3n$ dimensional space

$$\ell_1 \oplus \ell_2 \oplus \ell_3$$

define the quadratic form

$$Q(x) = Q_{123}(x) := (x_1, x_2) + (x_2, x_3) + (x_3, x_1), \quad \text{for } x = x_1 \oplus x_2 \oplus x_3. \tag{42}$$

Define the **Maslov index**

$$\tau(\ell_1, \ell_2, \ell_3) := \text{sign } Q_{123}. \tag{43}$$

Interchanging x_1 and x_2 in this definitions changes the sign of the first term and interchanges the last two terms with a change of sign. Similarly when we interchange x_2 and x_3. Hence

$$\tau(\ell_2, \ell_1, \ell_3) = -\tau(\ell_1, \ell_2, \ell_3), \quad \tau(\ell_1, \ell_3, \ell_2) = -\tau(\ell_1, \ell_2, \ell_3) \tag{44}$$

and hence

$$\tau(\ell_2, \ell_3, \ell_1) = \tau(\ell_1, \ell_2, \ell_3). \tag{45}$$

Also, it is clearly a symplectic invariant:

$$\tau(g\ell_1, g\ell_2, g\ell_3) = \tau(\ell_1, \ell_2, \ell_3), \quad \forall g \in Sp(V). \tag{46}$$

We claim that for any four Lagrangian subspaces we have

$$\tau(\ell_1, \ell_2, \ell_3) = \tau(\ell_1, \ell_2, \ell_4) + \tau(\ell_2, \ell_3, \ell_4) + \tau(\ell_3, \ell_1, \ell_4). \tag{47}$$

To prove this, we first prove the following lemma:

Lemma 5 *Let ℓ_1 and ℓ_3 be transverse Lagrangian subspaces, so that $V = \ell_1 \oplus \ell_3$. Let p_{13} and p_{31} denote the projections onto ℓ_1 and ℓ_3 respectively, corresponding to this direct sum decomposition. Let S denote the quadratic form on ℓ_2 given by*

$$S(x) := (p_{13}x, p_{31}x), \qquad x \in \ell_2.$$

Then

$$\tau(\ell_1, \ell_2, \ell_3) = \textit{sign } S.$$

Proof of lemma. Write

$$\begin{aligned} Q(x_1 + x_2 + x_3) &= (x_1, x_2) + (x_2, x_3) + (x_3, x_1) \\ &= (x_1, p_{31}x_2) + (p_{13}x_2, x_3) + (x_3, x_1) \\ &= (p_{13}x_2, p_{31}x_2) + (x_1 - p_{13}x_2, x_3 - p_{31}x2) \\ &= (p_{13}x_2, p_{31}x_2) + (y_1, y_3) \end{aligned}$$

where

$$y_1 := x_1 - p_{31}x_2, \quad y_3 := x_3 - p_{31}x_2).$$

The map

$$(x_1 + x_2 + x_3) \mapsto y_1 + x_2 + y_3$$

is an isomorphism of $\ell_1 \oplus \ell_2 \oplus \ell_3$ onto itself. The quadratic form

$$y_1 \oplus y_3 \mapsto (y_1, y_3)$$

is totally split, hence has signature zero. Thus sign $S =$ sign Q. QED

Proof of (47). Let us first assume that ℓ_4 is transversal to ℓ_1, ℓ_2 and ℓ_3. Then $\tau(\ell_1, \ell_2, \ell_4)$ is equal to the signature of the quadratic form $(p_{14}x_2, p_{41}x_2) = (p_14, x_2, x_2)$ on ℓ_2. So the right hand sided of (47) is the signature of the quadratic form Q' on $\ell_1 \oplus \ell_2 \oplus \ell_3$ given by

$$Q'(y_1, y_2, y_3) := (p_{14}y_2, y_2) + (p_{24}y_3, y_3) + (p_{34}y_1, y_1).$$

Notice that $p_{i4}p_{j4}y_i = y_j$ and hence if we set

$$\begin{aligned} y_1 &:= \frac{1}{2}(x_1 - p_{14}x_2 + p_{14}x_3) \\ y_2 &:= \frac{1}{2}(x_2 - p_{24}x_3 + p_{24}x_1) \\ y_3 &:= \frac{1}{2}(x_3 - p_{34}x_1 + p_{34}x_2) \end{aligned}$$

then

$$\begin{aligned} x_1 &= y_1 + p_{14}y_2 \\ x_2 &= y_2 + p_{24}y_3 \\ x_3 &= y_3 + p_{31}y_1. \end{aligned}$$

We will show that the under the map $x \mapsto y$ the form Q' pulls back to Q. We have

$$(x_1, x_2) = (p_{14}y_2, y_2) + (y_1, y_2) + (y_1, p_{24}y_3) + (p_{14}y_2, p_{24}y_3).$$

We must add this expression to its two cyclic permutations. Applying that permutation $(1, 2, 3) \mapsto (2, 3, 1)$ to the third term above, and the permutation $(1, 2, 3) \mapsto (1, 3, 2)$ to the fourth term, it is enough for us to show that

$$(y_1, y_2) + (y_2, p_{34}y_1) + (p_{34}y_1, p_{14}y_2) = 0.$$

Write $y_2 = p_{14}y_2 + p_{41}y_2$. The above expression becomes

$$(y_1, p_{41}y_2) + (p_{41}y_2, p_{34}y_1) = (y_1, p_{41}y_2) + (p_{41}y_2, y_1) = 0.$$

This proves (47) in the case that ℓ_4 is transversal to the other three Lagrangian subspaces. To prove the general case, choose a Lagrangian space ℓ_5 transverse to all four, and let us write (ijk) for $\tau(\ell_i, \ell_j, \ell_k)$. We have

$$\begin{aligned} (123) &= (125) + (235) + (315) \\ (124) &= (125) + (245) + (415) \\ (234) &= (235) + (345) + (425) \\ (314) &= (315) + (145) + (435). \end{aligned}$$

The sum of the right hand sides of the last three equations is equal to the right hand side of the first equation, proving (47) in general. QED

For any isotropic subspace, $U \subset V$, the symplectic form on V induces a symplectic form on $U^\perp/U$ (which we will continue to denote by $(\ ,\ )$.) If W is also a subspace of V then

$$U \subset (W \cap U^\perp) + U = (W + U) \cap U^\perp \subset U^\perp$$

satisfies

$$W^U := [(W \cap U^\perp) + U]^\perp = (W^\perp + U) \cap U^\perp.$$

Then

$$(W^U)^\perp = \left([(W \cap U^\perp) + U]^\perp\right) = (W^\perp)U.$$

In particular, if $\ell \subset V$ is a Lagrangian subspace, so is ℓ^U.

Proposition 5 *If*

$$U \subset (\ell_1 \cap \ell_2) + (\ell_2 \cap \ell_3) + (\ell_3 \cap \ell_1) := \ell_{123}$$

then

$$\tau(\ell_1, \ell_2, \ell_3) = \tau(\ell_1^U, \ell_2^U, \ell_3^U). \tag{48}$$

Proof. Notice that ℓ_{123} is isotropic. So if $U \subset \ell_{123}$ then $\ell_{123} \subset \ell_{123}^\perp \subset U^\perp$ and hence each of the subspaces $\ell_1 \cap \ell_2,\ \ell_2 \cap \ell_3,\ \ell_3 \cap \ell_1$ is contained in $U^\perp$. If

$$u = u_{12} + u_{23} + u_{31} \in U$$

with the obvious notation, then

$$u_{23} = u - u_{12} - u_{31} \in U + \ell_1 \cap \ell_2 + \ell_3 \cap \ell_1 = U + \ell_1 \cap U^\perp = \ell_1^U.$$

So every $u \in \ell^U$ can be written as the sum of an element of $\ell^U \cap \ell^1$ and an element of $\ell^U \cap \ell^2$. In symbols,

$$\ell_1^U = (\ell_1^U \cap \ell_1) + (\ell_1^U \cap \ell_2).$$

Also

$$\ell_1^U = (\ell_1^U \cap \ell_1) + (\ell_1^U \cap \ell_2^U).$$

We apply to these equations the following lemma:

Lemma 6 *If ℓ_1, ℓ, ℓ_2 are three Lagrangian subspaces satisfying*

$$\ell = \ell \cap \ell_1 + \ell \cap \ell_2$$

then

$$\tau(\ell_1, \ell, \ell_2) = 0.$$

Assuming the truth of the lemma for the moment, we conclude that

$$\tau(\ell_1, \ell_1^U, \ell_2) = \tau(\ell_1, \ell_1^U, \ell_2^U) = 0.$$

If we apply (47) to $\ell_1, \ell_2\ell_3\ell_1^U$ we conclude that

$$\tau(\ell_1, \ell_2, \ell_3) = \tau(\ell_1^U, \ell_2, \ell_3).$$

Then applying the same method to ℓ_2 and ℓ_1^U, ℓ_2, ℓ_3 we may replace ℓ_2 by ℓ_2^U and then finally ℓ_3 by ℓ_3^U.

We must still prove the lemma. Notice that if ℓ_1 and ℓ_2 are transversal, this follows from Lemma 5. In general, choose $Y_1 \subset \ell \cap \ell_1,\ Y_2 \subset \ell \cap \ell_2$ such that $\ell = Y_1 \oplus Y_2$. Write $x \in \ell_1 \oplus \ell_2 \oplus \ell_3$ as

$$x = x_1 + y_1 + y_2 + x_2,\ x_1 \in \ell_1, x_2 \in \ell_2, y_1 \in Y_1, y_2 \in Y_2.$$

Then

$$Q(x) = (x_1, y_2) + (y_1, x_2) + (x_2, x_1) = (x_2 - y_2, x_1 - y_1).$$

So Q is the pullback of the quadratic form

$$\tilde{Q}(u_1, u_2) = (u_2, u_1)$$

on $\ell_1 \oplus \ell_2$ via the map $x \mapsto x_1 - y_1 \oplus x_2 - y_2$. But the quadratic form $\tilde{Q}$ has signature zero. Indeed, the kernel of $\tilde{Q}$ is $\ell_{12} = \ell_1 \cap \ell_2$ and $\tilde{Q}$ is the pullback of the corresponding form on $(\ell_1/\ell_{12}) \oplus (\ell_2/\ell_{12})$ which is split, since ℓ_1/ℓ_{12} and $\ell_2 \cap \ell_{12}$ are transverse Lagrangian subspaces of $\ell_{12}^{\perp}/\ell_{12}$. This completes the proof of the lemma and hence of (48). QED

Notice that each of the spaces ℓ_i^U/U is a Lagrangian subspace of $U^{\perp}/U$, and that $U \oplus U \oplus U$ lies in the kernel of the quadratic form Q on $\ell_1^U \oplus \ell_2^U \oplus \ell_3^U$. Hence (48) implies

$$\tau(\ell_1, \ell_2, \ell_3) = \tau(\ell_1/U, \ell_2/U/\ell_3/U) \qquad (49)$$

where the right hand side is computed in the symplectic vector space $U^{\perp}/U$.

In particular, if we take $U = \ell_1 \cap \ell_2 + \ell_2 \cap \ell_3 + \ell_3 \cap \ell_1$ we can reduce the computation of the Maslov index to the transversal case.

14 Intertwinings and the Maslov index.

Let ℓ_1, ℓ_2, ℓ_3 be three real Lagrangian subspaces, and $\mathcal{H}_1,\ \mathcal{H}_2,\ \mathcal{H}_3$ the corresponding models of the unique irreducible given by the Stone von Neumann theorem. Let $N_i := \exp(\ell_i + \mathbf{R}), i = 1, 2, 3$. The function σ_j defined on N_j by

$$\sigma_j(\exp(v + tE)) = e^{2\pi t}$$

is a character on N_j and the space $\mathcal{H}_j$ is the induced representation space from this character. The canonical intertwining operator $I_{kj} : \mathcal{H}_j \to \mathcal{H}_k$ is given on functions by

$$[I_{kj}\phi](n) = \int_{N_k/N_k \cap N_j} \phi(nh_2)\sigma_2(h_2)dh_2,$$

where the invariant measures are chosen so as to make this unitary. (In terms or an adjusted basis, this amounts to a partial Fourier transform.) The complex scalar $\mu(1,2,3)$ of absolute value one is defined by

$$I_{13} \circ I_{32} \circ I_{21} = \mu(1,2,3)I.$$

The main formula of this section is

$$\mu = e^{-\frac{\pi i}{4}\tau} \tag{50}$$

where

$$\mu = \mu(1,2,3), \quad \tau = \tau(1,2,3) = \tau(\ell_1, \ell_2\ell_3).$$

We first prove this formula when $\dim V = 2$. In case all three Lagrangian subspaces coincide, the Maslov index vanishes, and all three I_{jk} are the identity. In case $\ell_1 = \ell_3 \neq \ell_2$, say, then the Maslov index still vanishes, I_{21} is the Fourier transform, $I_{32} = I_{12}^{-1}$ and I_{31} is the identity, and again the formula holds. So we need only check the formula (for two dimensional V) when the Lagrangian subspaces are pairwise transverse. We may then choose a symplectic basis so that $\ell_1 = \mathbf{R}P$, $\ell_2 = \mathbf{R}Q$ and $\ell_3 = \mathbf{R}(P \pm Q)$. It is enough to examine the case $\ell_3 = \mathbf{R}(P+Q)$, the other case being similar (alternatively reducing to it by interchange of Lagrangian subspaces). Since $\tau(\ell_1, \ell_3, \ell_1) = 1$, we have $\tau(1,2,3) = -1$ and we must prove

$$I_{13} \circ I_{32} \circ I_{21} = e^{\frac{\pi i}{4}} I, \tag{51}$$

In doing this computation, we regard ℓ_2 as a complementary subspace to ℓ_1, while we regard ℓ_1 as a complementary subspace to ℓ_2 and ℓ_3. This means that we get identifications R_1, R_2, R_3 of $\mathcal{H}_i,\ i = 1,2,3$ with $L^2(\mathbf{R})$ by

$$[R_1\phi](x) = \phi(\exp xQ)$$

while

$$[R_j\phi](x) = \phi(\exp xP), \quad j = 2,3.$$

Let us write

$$F_{jk} := R_j \circ I_{jk} \circ I_k^{-1}.$$

By definition,

$$[I_{21}\phi](\exp xP) = \int_{N_2/(N_1\cap N_2)} \phi(\exp xP \exp \xi Q) d\xi.$$

We identify $N_2/(N_1 \cap N_2)$ with ℓ_2 and use

$$\phi(\exp xP \exp \xi Q) = \phi(\exp xP \exp \xi Q \exp(-xP) \exp xP) = e^{-2\pi i x\xi} \phi(\exp xiQ).$$

Thus, writing $f = R_1\phi$,

$$[F_{21}f](x) = \int e^{2\pi i x\xi} f(\xi) d\xi.$$

In other words, $F_{12} = F$, the Fourier transform.

Next let us compute F_{32}: We have

$$[I_{32}\phi](\exp xP) = \int \phi\left((\exp xP)(\exp \xi(P+Q)\right) d\xi$$

and

$$(\exp xP)(\exp \xi(P+Q)) = (\exp(x+\xi)P)(\exp \xi Q)(\exp -\frac{\xi^2}{2}E)$$

so

$$[F_{32}f](x) = \int f(x+\xi)e^{\pi i\xi^2}d\xi.$$

Replace ξ by $\xi - x$ in this integral gives

$$[F_{32}f](x) = e^{\pi ix^2}\int f(\xi)e^{\pi i\xi^2}e^{-2\pi ix\xi}d\xi.$$

In other words,

$$F_{32} = e^{\pi ix^2}Fe^{\pi ix^2},$$

where, in this equation, we have used the symbol $e^{\pi ix^2}$ to denote the operation of multiplication by the function $e^{\pi ix^2}$.

Finally,

$$[I_{13}\phi](\exp xQ) = \int \phi\left((\exp xQ)(\exp \xi P)\right) d\xi$$

and

$$(\exp xQ)(\exp \xi P) = (\exp(\xi - x)P)(\exp x(P+Q))\exp(-x\xi + \frac{x^2}{2})E$$

so

$$[F_{13}f](x) = \int f(\xi - x)e^{2\pi ix\xi}e^{-\pi ix^2}d\xi = e^{\pi ix^2}\int f(\xi)e^{2\pi ix\xi}d\xi$$

where we made the change of variable $\xi \mapsto \xi + x$ in passing from the first integral to the second. So

$$F_{13} = e^{\pi ix^2}F^{-1}.$$

To summarize:

$$F_{21} = F, \quad F_{32} = e^{\pi ix^2}Fe^{\pi ix^2}, \quad F_{13} = e^{\pi ix^2}F^{-1}.$$

So to prove (51) we must prove

$$F_{32} \circ F_{21} = e^{\frac{\pi i}{4}}Fe^{-\pi ix^2}. \tag{52}$$

We first do a Gaussian integral: For Im $z > 0$, let $\left(\frac{z}{i}\right)^{-\frac{1}{2}}$ denote that determination of the square root which takes the value 1 at $z = i$. We claim that

$$\int e^{-2\pi ix\xi}e^{\pi iz\xi^2}d\xi = \left(\frac{z}{i}\right)^{-\frac{1}{2}}e^{-\frac{\pi i}{z}x^2}.$$

Indeed, both sides are holomorphic in x and z for Im$z > 0$, so it is enough to prove the formula for $z = iy,\ y > 0,\ x = iu, u$ real. The left hand side then becomes

$$\int e^{2\pi u\xi} e^{-\pi y\xi^2} d\xi.$$

Making the change of variables $\xi \mapsto y^{-1/2}\xi$ and completing the square proves the formula. Now for any $f \in L^2$, let $\hat{f} = Ff$ denote its Fourier transform. For Im $z > 0$, we may apply the Plancherel formula to conclude that

$$\int \hat{f} e^{\pi iz\xi^2} d\xi = \int f(\xi) F\left(e^{\pi iz\xi^2}\right) d\xi = \left(\frac{z}{i}\right)^{-\frac{1}{2}} \int f(\xi) e^{-z^{-1}\pi i\xi^2} d\xi$$

be the preceding result. Both sides are continuous in z up to the real axis, and so we may pass to the limit $z = 1$ to conclude that

$$\int \hat{f}(\xi) e^{\pi i\xi^2} d\xi = e^{\frac{\pi i}{4}} \int f(\xi) e^{-\pi i\xi^2} d\xi.$$

We have

$$[F_{32} \circ F_{21} f](x) = \int \hat{f}(x+\xi) e^{\pi i\xi^2} d\xi.$$

but

$$\hat{f}(x+\xi) = [F(e^{-2\pi i\xi u} f(u)](x)$$

so

$$[F_{32} \circ F_{21} f](x) = e^{\frac{\pi i}{4}} \int e^{-2\pi i\xi u} f(u) e^{-\pi iu^2} du$$

which is exactly (52). We have now proved (50) when V is two dimensional.

We prove (50) in general by induction on the dimension of V. If ℓ_1, ℓ_2, ℓ_3 are pairwise transverse we can find a basis $P_1, \ldots, P_n$ of ℓ_1, a dual basis $Q_1, \ldots, Q_n$ of ℓ_2 such that $P_1 \pm Q_1, \ldots, P_n \pm Q_n$ is a basis of ℓ_3. Then the formula reduces to the two dimensional case.

Next, we prove (50) in another special case:

Lemma 7 *Suppose that $\ell = \ell \cap \ell_1 + \ell \cap \ell_2$. Then*

$$I_{\ell_2\ell_1} = I_{\ell_2\ell} \circ I_{\ell\ell_1}. \tag{53}$$

Proof. Let Z_1 be a complementary subspace to $\ell \cap \ell_1 \cap \ell_2 = \ell_1 \cap \ell_2$ in $\ell \cap \ell_2$ and let Z_2 be a complementary subspace to $\ell \cap \ell_2$ in ℓ_2 so that

$$\ell_2 = Z_2 \oplus \ell_\cap \ell_2, \quad \ell = Z_1 \oplus \ell \cap \ell_1.$$

We have

$$[I_{\ell_2\ell} \circ I_{\ell\ell_1}\phi](g) \quad = \quad \int_{\ell_2/(\ell\cap\ell_2)} [I_{\ell\ell_1}\phi](g \exp x) dx$$

$$
\begin{aligned}
&= \int_{x\in\ell_2/(\ell_2\cap\ell)}\int_{y\in\ell/(\ell_1\cap\ell)} \phi(g\exp x\exp y)dxdy\\
&= \int_{Z_2}\int_{Z_1}\phi(g\exp z_2\exp z_1)\\
&= \int_{Z_1\oplus Z_2}\phi(z_1+z_2)dz_1dz_2\\
&= \int_{\ell_2/(\ell_1\cap\ell_2)}\phi(g\exp u)du\\
&= [I_{\ell_2\ell_1}\phi](g). \quad \text{QED}
\end{aligned}
$$

We were allowed to replace $(\exp z_2)(\exp z_1)$ by $\exp(z_1+z_2)$ since $Z_1+Z_2\subset\ell$. Also $Z_1\oplus Z_2$ is a complement to $\ell_1\cap\ell_2$ in ℓ_2.

We now turn to the general case. Suppose that ℓ_1,ℓ_2,ℓ_3 are not pairwise transverse, for example that

$$U := \ell_1\cap\ell_2 \neq \{0\},$$

so that

$$\tau(\ell_1,\ell_2,\ell_3)=\tau(\ell_1,\ell_2,\ell_3^U).$$

We have $\ell_3^U := (\ell_3\cap U^\perp)+U \subset (\ell_3\cap\ell_3^U)+\ell_3\cap\ell_1$ so ℓ_1,ℓ_3^U,ℓ_3 satisfy the conditions of the lemma as do ℓ_2,ℓ_3^U,ℓ_3 hence

$$
\begin{aligned}
I_{\ell_3,\ell_2} &= I_{\ell_3\ell_3^U}\circ I_{\ell_3^U\ell_1}\\
I_{\ell_1\ell_3} &= I_{\ell_1\ell_3^U}\circ I_{\ell_3^U\ell_3}, \quad \text{therefore}\\
I_{\ell_1\ell_3}\circ I\ell_2\ell_3\circ I_{\ell_2\ell_1} &= I_{\ell_1\ell_3^U}\circ I_{\ell_3^U\ell_3}\circ I_{\ell_3^U\ell_3}\circ I_{\ell_3\ell_2}\circ I_{\ell_2\ell_1}\\
&= I_{\ell_1\ell_3^U}\circ I_{\ell_3^U\ell_2}\circ I_{\ell_2\ell_1}.
\end{aligned}
$$

We may replace ℓ_3 by ℓ_3^U in the proof of the formula. Now all three subspaces lie in $U^\perp$ which is a proper subspace of V. All three subspaces contain U, and hence all the operators involve integrations over quotients spaces which eliminate $\exp U$. Hence both sides of (50) involve formulas which are the same when we replace ℓ_i by ℓ_i^U/U, and this we know to be true by induction, since $U^\perp/U$ is a symplectic vector space of smaller dimension than V. QED

15 Lattices.

A discrete subgroup, r of a real vector space V is called a lattice if V/r is compact. Then there exists a basis $e_1,\dots,e_d$ of V such that $r=\mathbf{Z}e_1\oplus\cdots\oplus\mathbf{Z}e_d$. If V is symplectic, the dual lattice to r consists of all $v\in V$such that $(v,u)\in\mathbf{Z}$, $\forall u\in r$. A lattice is **self dual** if $r^*=r$. We let $B:=N/(\exp\mathbf{Z}E)$. So any Stone von Neumann representation of N is in fact a representation of B. We may identify B with $V\times\mathbf{R}/\mathbf{Z}$ with elements $(v,e^{2\pi it})$ and multiplication law

$$(u,z)\cdot(v,w)=(u+v,zwe^{\pi i(u,v)}).$$

Let $\overline{R}$ be the image in B of $R := \exp(r + \mathbf{R}E)$ in N. Then $\overline{R}$ is a maximal commutative subgroup of B:

$$(u, z) \cdot (v, w) = (v, w) \cdot (u, w) \Leftrightarrow \ (u, v) \in \mathbf{Z}.$$

Let χ be a character of $\overline{R}$ whose restriction to the set of all $(0, z)$ is given by $\chi(0, z) = z$. We may consider χ as a character of R satisfying $\chi(\exp tE) = e^{2\pi it}$. We may then form the induced representation of N (or of B) corresponding to this character. This representation is by left translation on the space $\mathcal{H}_{r,\chi}$ (L^2 completion of) space of all functions ϕ on N which satisfy

$$\phi(n\gamma) = \chi(\gamma)^{-1}\phi(b) \ \ \forall n \in N, \gamma \in R \tag{54}$$

and

$$\int_{N/R} |\phi|^2 < \infty.$$

(since $N/R = V/r$ is compact, it has a canonical left invariant measure with total volume equal to 1.)

Our goal is to show that the representation of N on $\mathcal{H}_{r,\chi}$ is again a model of the Stone von Neumann irreducible representation, and to study its intertwining properties with the Schrodinger models. We first discuss the dependence on χ.

Proposition 6 *Let χ and χ' be two characters of R which satisfy*

$$\chi(\exp tE) = \chi'(\exp tE) = e^{2\pi it}.$$

Then there exists an $n_0 = \exp v \in N$ such that

$$\chi'(\gamma) = \chi(n_0\gamma n_0^{-1}) \ \ \forall \gamma \in R.$$

In particular, the representations of N by left translation on $\mathcal{H}_{r,\chi}$ and $\mathcal{H}_{r,\chi'}$ are isomorphic.

Proof. χ'/χ is a character of the discrete commutative group r so there exists some $v \in V$ such that

$$(\chi'/\chi)(\exp \beta) = e^{2\pi i(v,\beta)}, \quad \forall \beta \in r.$$

If $\gamma = \exp \beta$ and $n_0 := \exp v$ this becomes the equation in the proposition. If ϕ satisfies (54) relative to χ' then the function $n \mapsto \phi(nn_0)$ satisfies

$$\phi(n\gamma n_0) = \phi(nn_0n_0^{-1}\gamma n_0) = \phi(nn_0)\chi(\gamma)$$

and so satisfies (54) for χ. Since right multiplication commutes with left multiplication, this induces an isomorphism of modules. QED

Next we need to describe bases adjusted to a self dual lattice:

Lemma 8 *Let r be a self dual lattice and m and isotropic subspace such that $r \cap m$ generates m as a subspace. Then there exists a Darboux basis*

$$P_1, \ldots, P_n, Q_1, \ldots, Q_n$$

of V such that

$$r = \mathbf{Z}P_1 \oplus \cdots \oplus \mathbf{Z}P_n \oplus \mathbf{Z}Q_1 \oplus \cdots \oplus \mathbf{Z}Q_n$$

and

$$m = \mathbf{R}P_1 \oplus \cdots \oplus \mathbf{R}P_k, \quad k \leq n.$$

Proof. We will be interested in the case $m = 0$, but for the proof of the lemma, we may assume that $m \neq 0$, for example by choosing $m = \mathbf{R}\beta,\ \beta \in r$. We may also assume that the lemma is known for V of smaller dimension, and will proceed to prove the lemma by induction. Choose $0 \neq x_1 \in ar \cap m$. Since r is self dual, $(x_1, \beta) \in \mathbf{Z}$ for all $\beta \in r$ and hence there is an integer, N such that $(x_1, r) = N\mathbf{Z}$. Then $(1/N)x_1 \in r^*$, and since r is self dual it actually belongs to r. So let $P_1 := (1/N)x_1 \in r$. Then $(P_1, r) = \mathbf{Z}$, hence there is a $Q_1 \in r$ with $(P_1, Q_1) = 1$. Decompose

$$V = \mathbf{R}P_1 \oplus \mathbf{R}Q_1 \oplus V_0, \qquad V_0 := (\mathbf{R}P_1 \oplus \mathbf{R}Q_1)^{\perp}.$$

We will be done by induction if we prove

$$r = (\mathbf{Z}P_1 \oplus \mathbf{Z}Q_1) \oplus r \cap V_0, \qquad m = \mathbf{R}P_1 \oplus m \cap V_0.$$

Suppose $\beta \in r$. Then $(\beta, P_1) = m_1,\ (\beta, Q_1) = n_1$ with $m_1, n_1 \in \mathbf{Z}$. Hence $\beta - n_1 P_1 + m_1 Q_1 \in r \cap V_0$. Similarly, if $v \in m$ with $(v, Q_1) = x$ then $v - xP_1 \in m \cap V_0$. The intersection $r \cap V_0$ is a self dual lattice in V_0 and $m \cap V_0$ is an isotropic subspace. QED

In particular, if r is a self-dual lattice we may choose a basis as in the lemma and then define

$$\ell := \mathbf{R}P_1 \oplus \cdots \oplus \mathbf{R}P_n, \quad \ell' := \mathbf{R}Q_1 \oplus \cdots \oplus \mathbf{R}Q_n.$$

Then ℓ and ℓ' are complementary Lagrangian subspaces with

$$r = r \cap \ell \oplus r \cap \ell'.$$

Every element of R can be written uniquely as

$$\gamma = (\exp u)(\exp v)(\exp tE), \quad u \in r \cap \ell,\ v \in r \cap \ell'$$

and then

$$\chi_{\ell,\ell'}(\gamma) := e^{2\pi i t}$$

defines a character on R which satisfies $\chi_{\ell\ell'}(\exp v) = 1, \quad v \in \ell$.

So we start with the following data

- ℓ a Lagrangian subspace,
- r a self dual lattice such that $r \cap \ell$ generates ℓ as a vector space,
- χ a character of R such that $\chi(\exp \beta) = 1, \quad \beta \in r \cap \ell$.

We wish to define an intertwining operator

$$I_{(r,\chi),\ell} : \mathcal{H}_\ell \to \mathcal{H}_{r,\chi}$$

by

$$[I_{(r,\chi)\ell}\phi](n) = \sum_{u \in r/r\cap\ell} \chi(\exp u)\phi(n \exp u). \tag{55}$$

Notice that this sum makes sense formally, since ϕ satisfies $\phi(n \exp y) = \phi(n), \ y \in \ell$ and hence for $y \in r \cap \ell$. Notice also that the right hand side formally belongs to $\mathcal{H}_{r,\chi}$ and that the operator $I_{(r,\chi)\ell}$ intertwines the (left) action of N. We must show that it is well defined, and that for an appropriate choice of invariant measure defining $\mathcal{H}_\ell$ it is unitary.

Let us choose a basis as in the lemma, and then a complementary Lagrangian subspace ℓ' spanned by $Q_1, \dots, Q_n$. We then have an identification

$$R_\ell : \mathcal{H}_\ell \to L^2(\mathbf{R}^n, dy), \quad [R\phi](y) = \phi(\exp y \cdot Q).$$

We may choose the $u \cdot Q$, $u = (u_1, \dots, u_n) \in \mathbf{Z}^n$ as representatives of $r/r \cap \ell$ so

$$[I_{(r,\chi)\ell}\phi](\exp(x \cdot P + y \cdot Q)) = \sum_{u \in \mathbf{Z}^n} \phi\left(\exp(x \cdot P + y \cdot Q)(\exp u \cdot Q)\right)\chi(\exp u \cdot Q).$$

We have

$$\exp(x \cdot P + y \cdot Q)\exp(u \cdot Q) = \exp((y+u)\cdot Q)\exp(x \cdot P)\exp(\frac{1}{2}x \cdot y + x \cdot u)E.$$

By definition $\phi \in \mathcal{H}_\ell$ is invariant under right multiplication by elements of $\exp \ell$ and so, if we write $f = R\phi$ then

$$[[I_{(r,\chi)\ell} \circ R^{-1} f](\exp x \cdot P + y \cdot Q) = \sum_{u \in \mathbf{Z}^n} \chi(\exp u \cdot Q) e^{-2\pi i x \cdot u} e^{-\pi i x \cdot y} f(y+u). \tag{56}$$

This series converges for $f \in \mathcal{S}(\mathbf{R}^n)$, and we want to think of the left hand side as a function on $\mathbf{R}^{2n}/\mathbf{Z}^{2n}$. The functions $x \mapsto e^{-2\pi i x \cdot u}, \quad u \in \mathbf{Z}$ form an orthonormal basis of $L^2(\mathbf{R}^n/\mathbf{Z}^n, dx)$ and hence

$$\begin{aligned} \|[I_{(r,\chi)\ell}\phi]\|^2 \quad &:= \int 0 \le y_i < 1 \int 0 \le x_i < 1 |e^{-ix\cdot y} \sum_u f(y+u)\chi(\exp u \cdot Q)e^{-2\pi i x \cdot u}|^2 dx \\ &= \int_{0 \le y_i < 1} \int_{0 \le x_i < 1} |\sum_u f(y+u)\chi(\exp u \cdot Q) e^{-2\pi i x \cdot u}|^2 dx \end{aligned}$$

$$
\begin{aligned}
&= \int_{0\le y_i<1} \sum_u |f(y+u)|^2 dy \\
&= \sum_{u\in \mathbf{Z}^n} \int_{0\le y_i<1} |f(y+u)|^2 dy \\
&= \int_{\mathbf{R}^n} |f(y)|^2 dy \\
&= \|\phi\|^2.
\end{aligned}
$$

We see that $I_{(r,\chi)\ell}$ extends from the Schwartz functions to be an isometry.

We next construct the inverse operator

$$I_{\ell(r,\chi)} : \mathcal{H}_{(r,\chi)} \to \mathcal{H}_\ell$$

by

$$[I_{\ell,(r,\chi)}\psi](n) := \sum_{\ell/r\cap\ell} \psi(n \exp y) dy. \tag{57}$$

We will show that

$$I_{(r,\chi)\ell} \circ I_{\ell(r,\chi)} = \mathrm{Id}.$$

This will show that $I_{(r,\chi)\ell}$ is surjective, hence that the representation of N on $\mathcal{H}_{r,\chi}$ is irreducible, and hence that $I_{(r,\chi)\ell}$ is bijective. We have

$$[I_{(r,\chi)\ell} \circ I_{\ell(r,\chi)}\psi)](\exp(x\cdot P + y\cdot Q))$$

$$
\begin{aligned}
&= \sum_u \chi(\exp u\cdot Q) e^{-\pi i x\cdot U} e^{-\pi i x\cdot y} [I_{\ell,(r,\chi)}\psi](\exp(y\cdot Q + u\cdot Q)) \\
&= \sum_u \chi(\exp u\cdot Q) e^{-\pi i x\cdot U} e^{-\pi i x\cdot y} \int_{\ell/(r\cap\ell)} \psi\left((\exp y\cdot Q)(\exp u\cdot Q)(\exp w\cdot P)\right) dw \\
&= \sum_u \chi(\exp u\cdot Q) e^{-\pi i x\cdot U} e^{-\pi i x\cdot y} \int_{\ell/(r\cap\ell)} \psi\left((\exp y\cdot Q)(\exp w\cdot P)(\exp u\cdot Q)\right) e^{2\pi i w\cdot u} dw \\
&= e^{-\pi i x\cdot y} \sum_u \int_{\mathbf{R}^n/\mathbf{Z}^n} \psi\left((\exp y\cdot Q)(\exp w\cdot P)\right) e^{2\pi i u\cdot(w-x)} dw \\
&= e^{-\pi i x\cdot y} \sum_u \int_{\mathbf{R}^n/\mathbf{Z}^n} \psi\left((\exp y\cdot Q)(\exp(w+x)\cdot P)\right) dw.
\end{aligned}
$$

The function $w \mapsto \alpha(w) := \psi\left((\exp y\cdot Q)(\exp(w+x)\cdot P)\right)$ is periodic, so we the last sum and integral is just the sum of its Fourier coefficients which evaluates to $\alpha(0)$. Thus the last expression becomes

$$e^{-\pi i x\cdot y}\psi\left((\exp y\cdot Q)(\exp x\cdot P)\right) = \psi\left(\exp(x\cdot P + y\cdot Q)\right). \quad \text{QED}$$