The wave equation.

Math 212b

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Contents

We want to study various properties of the wave equation

$$u_{tt} = u_{x_1x_1} + \cdots + u_{x_nx_n} \tag{1}$$

in $(n+1)$-dimensional space. In practical applications the wave equation has the form

$$(1/c^2)u_{tt} = u_{x_1x_1} + \cdots + u_{x_nx_n} \tag{2}$$

where c is a velocity. So (1) represents (2) in an appropriate choice of coordinates. In physics (2) is satisfied by the components of the electromagnetic field (as a consequence of Maxwell's equations). It also arises in the theory of weak sound waves. Let us rapidly sketch the weak sound wave theory. "Weak" means a linear approximation to small variations from a stationary solution of

Euler's equations, which are themselves very approximate equations for motion of a gas. Euler's equations are

$$\begin{aligned}\rho_t + \sum_1^3 (\rho v^i)_{x^i} &= 0\\ \rho(v^i)_t + \rho\sum_1^3 v^j(v^i)_{x^j} &= -p_{x^i} \quad i=1,2,3\end{aligned}$$

where ρ is the density of mass an p he pressure. The first equation expresses the conservation of mass and the last three the conservation of momentum. These four equations must be supplemented by a thermodynamic relation. Assume the simplest, that p depends in a monotonically increasing fashion on ρ alone. If ρ is constant the $p_{x^i} = 0$ so $v = 0$, and $\rho = \rho_0$ a constant is a solution of (2). Now perturb this a little bit by setting

$$\begin{aligned}v &= \epsilon w\\ \rho &= \rho_0(1+\epsilon s) \text{ so}\\ p &= p_0 + \epsilon s\frac{dp}{d\rho}(\rho_0) + \cdots .\end{aligned}$$

Now $dp/d\rho$ has the dimensions of pressure/density = (force/area)/(mass/volume) and writing force =(mass)(acceleration) we see that $dp/d\rho$ has the dimensions of (velocity)^2. So let us write

$$c^2 := \frac{dp}{d\rho}(\rho_0).$$

Let us also assume that s has compact support in x and that the velocity field w is irrotational, meaning that we can write $w^i = -u_{x^i}$ for some function u. Substituting into the Euler equations and ignoring higher powers of ϵ gives

$$s_t - \sum_i u_{x^i x^i} = 0, \quad u_{tx^i} = c^2 s_{x^i}.$$

The second equation and the fact that s has compact support implies that $u_t = c^2 s + f(t)$. But adding a function of t to u does not change w, so we may assume that

$$u_t = c^2 s$$

and substituting this back into the first equation above gives the wave equation (2).

We will establish that c is the "velocity of sound" in the sense that if initially u and u_t are supported near 0 at time 0, then at time t u and u_t are supported in the ball of radius slightly larger than ct.

Let me go back to the coordinates in which $c = 1$ and formulate in vague terms the main points we want to establish in what follows: We will let

$$u(0,x) = f(x), \quad u_t(0,x) = g$$

so that the "Cauchy" or "initial value problem" consists of looking for a solution of (1) with the above initial conditions, where f and g are given. We are aiming to prove:

- **Existence and uniqueness.**The solution to the initial value problem exists and is unique.
- **Finite propagation velocity.** The value of u at (t_0, x_0) depends only on the values of f and g on the ball $|x - x_0| \leq t_0$.
- **Huygens' Principle.** If n is odd and ≥ 3 then the value of u at (t_0, x_0) depends on the values of f and g and some of their derivatives on the sphere $|x - x_0| = t_0$.
- **Propagation of the wave front.** For all dimensions, even or odd, the singularities of u depend only on the singularities of f and g on the sphere $|x - x_0| = t_0$.
- **Conservation of energy.** The "energy"

  $$E(t) := \frac{1}{2} \int \left[|u_t(t, x)|^2 + \operatorname{grad} u(t, x) \cdot \operatorname{grad} \overline{u}(t, x) \right] dx \tag{3}$$

  does not depend on t.

We will discuss these issues from several points of view.

1 Fourier analysis.

Assume $f, g \in \mathcal{S}$ and look for u such that $u(t, \cdot) \in \mathcal{S}$ for all t. Applying the Fourier transform converts (1) into

$$\hat{u}_{tt}(\xi) = -|\xi|^2 \hat{u}(\xi) \tag{4}$$

which is an ordinary differential for each fixed ξ whose solution for the given initial data is

1.1 Solution.

$$\hat{u}(t, \xi) = \cos(|\xi| t) \hat{f}(\xi) + \frac{1}{|\xi|} \sin(|\xi| t) \hat{g}(\xi). \tag{5}$$

Notice that $\cos(\theta)$ is a function of θ^2 as is $(1/\theta)\sin(\theta)$ so the right hand side is smooth as a function of ξ and t. The uniqueness theorem for ordinary differential equations thus guarantees the uniqueness of the above solution, at least if we demand that the solution belong to $\mathcal{S}$ for any fixed t.

1.2 Conservation of energy at each frequency.

The Fourier transform gives the expression of the energy (3) in terms of ξ as

$$E(t) = \int [|\hat{u}_t|^2(\xi) + |\xi|^2|\hat{u}|^2(\xi)]d\xi. \tag{6}$$

We will prove much more than the assertion that $E(t)$ is constant. We will prove that the integrand is constant. Indeed, from (5) we obtain by differentiating and by multiplying by $|\xi|$ the equations

$$\begin{aligned}\hat{u}_t(t,\xi) &= \cos(|\xi|t)\hat{g}(\xi) - \sin(|\xi|t)|\xi|\hat{f})(\xi) \\ |\xi|\hat{u}(t,\xi) &= \sin(|\xi|t)\hat{g}(\xi) + \cos(|\xi|t)|\xi|\hat{g}(\xi).\end{aligned}$$

Thus

$$\begin{pmatrix} \hat{u}_t(t,\xi) \\ |\xi|\hat{u}(t,\xi) \end{pmatrix} = \begin{pmatrix} \cos(|\xi|t) & -\sin(|\xi|t) \\ \sin(|\xi|t) & \cos(|\xi|t) \end{pmatrix} \begin{pmatrix} \hat{g}(\xi) \\ |\xi|\hat{f}(\xi) \end{pmatrix}.$$

This shows that

$$|\hat{u}_t(t,\xi)|^2 + |\xi|^2|\hat{u}(t\xi)|^2 = |\hat{g}|^2 + |\xi|^2|\hat{f}(\xi)|^2$$

is independent of t.

1.3 Generalized solutions.

We introduce some Hilbert spaces of distributions which are convenient for the wave equation: Let

$$\langle\xi\rangle := (1 + |\xi|^2)^{\frac{1}{2}}$$

and let

$$\mathbf{H}^s := \{u \in \mathcal{S}' | \langle\xi\rangle^s \hat{u}(\xi) \in L_2(\mathbf{R}_n\} \tag{7}$$

with norm

$$\|u\|_s = \|\langle\xi\rangle^s \hat{u}\|_{L_2}. \tag{8}$$

Thus $\mathbf{H}^0 = L_2$. For example, since the Fourier transform of the δ function is a constant, and since $\langle\xi\rangle^{2s}$ is integrable if and only if $2s < -n$, we see that $\delta \in \mathbf{H}^s$ if and only if $s < -\frac{1}{2}n$. On the other hand, the Fourier transform of the constant one is a multiple of the δ function, and so does not belong to $\mathbf{H}^s$ for any s. So s describes the degree of smoothness, but to belong to $\mathbf{H}^s$ a distribution has to "vanish at infinity" sufficiently fast.

The space $\mathbf{H}^s$ is the completion of the space $\mathcal{S}$ in the $\| \ \|_s$ norm. Let us denote by $\mathbf{D}^s$ the completion of $\mathcal{S}$ in the norm

$$\|u\|_{\mathbf{D},s} := \|\operatorname{grad} u\|_{s-1}$$

($\mathbf{D}$ stands for Dirichlet). We will write simply $\mathbf{D}$ for $\mathbf{D}^1$.

So for $s = 1$ the energy is exactly

$$E(t)(u) = \frac{1}{2}\left(\|u\|^2_{\mathbf{D},1} + \|u_t\|^2_0\right).$$

The map

$$t \mapsto (u, u_t)$$

is unitary with respect to this energy norm, and the same proof (just multiplying through by a power of $\langle \xi \rangle$) shows that the map

$$t \mapsto (u, u_t)$$

extends to a unitary one parameter group on

$$\mathbf{D}^s \oplus \mathbf{H}^{s-1}.$$

We thus get distributional (weak) solutions to the wave equations for distributions which are elements of $\mathbf{H}^s$.

1.4 Cauchy's formula and Fourier integral operators.

Let us go back to equation (4) and write the solution in terms of complex exponentials instead of sines and cosines. So we write (5) as

$$\hat{u} = \frac{1}{2}[\hat{f} + \hat{g}/(i|\xi|)]e^{i\|\xi\|t} + \frac{1}{2}[\hat{f} - \hat{g}/(i|\xi|)]e^{-i\|\xi\|t}.$$

If we now apply the inverse Fourier transform, and use the explicit expression of $\hat{f}$ and $\hat{g}$ as Fourier transforms, we obtain

$$u(t,x) =$$

$$\frac{1}{2}\frac{1}{(2\pi)^n}\int\int e^{i\phi_{+,t}(x,y,\xi)}[f(y)+g(y)]d\xi dy+\frac{1}{2}\frac{1}{(2\pi)^n}\int\int e^{i\phi_{-,t}(x,y,\xi)}[f(y)-g(y)]d\xi dy \tag{9}$$

where

$$\phi_{\pm,t}(x,y,\xi) = \langle x - y, \xi \rangle \pm t|\xi|.$$

This formula was obtained by Cauchy in 1815. It suggests (for each fixed t) that we consider operators A of the form

$$(Au)(x) = \int\int e^{i\phi(x,y,\xi)}a(x,y)u(y)dy.$$

When appropriate conditions are placed on the functions ϕ and a, such an operator is called a **Fourier integral operator** and I hope to devote a substantial portion of the course to the study of such operators.

2 The energy integral method.

We have

$$2u_t\left(-u_{tt} + u_{x_1x_1} + \cdots + u_{x_nx_n}\right)$$
$$= 2(u_tu_{x_1})_{x_1} + \cdots + 2(u_tu_{x_n})_{x_n} - \left[u_t^2 + u_{x_1}^2 + \cdots u_{x_n}^2\right]_t.$$

If we define the operator Q by

$$Qu = -u_{tt} + u_{x_1x_1} + \cdots + u_{x_nx_n}$$

(so that the wave equation is $Qu = 0$) and define the vector field $V(u)$ by

$$V(u) = \begin{pmatrix} [u_t^2 + u_{x_1}^2 + \cdots u_{x_n}^2] \\ 2u_t u_{x_1} \\ \vdots \\ 2u_t u_{x_n} \end{pmatrix}$$

then we can rewrite the preceding equation as

$$2u_t Qu = \operatorname{Div} V(u), \tag{10}$$

where Div is the divergence in $n+1$ dimensions. So if u is a solution of the wave equation then the divergence of $V(u)$ vanishes identically and hence, by the divergence theorem,

$$\int_{\partial\mathcal{D}} V(u) \cdot \mathbf{n} d\sigma = 0 \tag{11}$$

for any bounded domain where $d\sigma$ is the area element on the boundary $\partial\mathcal{D}$ and $\mathbf{n}$ the unit normal field to $\partial\mathcal{D}$.

The integral of $V(u) \cdot \mathbf{n}$ over a hyperplane $t =$ constant is just $\pm 2E(t)$ (provided that this integral converges) where the sign depends on the orientation of $\mathbf{n}$. In fact, if u has compact support in $n+1$ space, then we can take $\mathcal{D}$ to be a region bounded by two such hyperplanes together with a "lateral portion" outside the support of u, and then (11) implies the conservation of energy. So (11) is in a sense a more general way of asserting the conservation of energy.

Let us use (11) to prove the finite velocity of propagation for the solution to the wave equation. Let us write

$$\mathbf{n} = \begin{pmatrix} \nu_0 \\ \nu_1 \\ \vdots \\ \nu_n \end{pmatrix}$$

and

$$u_0 := u_t, \quad u_i := u_{x_i}, \ i = 1, \ldots, n.$$

Then

$$\nu_0 V(u) \cdot \mathbf{n} = 2u_0u_1\nu_0\nu_1 + \cdots 2u_0u_n\nu_0\nu_n - \nu_0^2(u_0^2 + u_1^2 + \cdots u_n^2)$$

while

$$(\nu_0 u_i - u_0\nu_i)^2 = u_i\nu_0^2 - 2u_0u_i\nu_0\nu_i + u_0^2\nu_i^2$$

so summing yields

$$-\nu_0 V(u) \cdot \mathbf{n} = \sum_{i=1}^n (\nu_0 u_i - u_0\nu_i)^2 + u_0^2[\nu_0^2 - \nu_1^2 - \cdots - \nu_n^2].$$

So if S is a surface along which the normal vector at every point has the property that $\nu_0 \neq 0$ we can write

$$\int_S V(u)\cdot \mathbf{n} d\sigma = -\int_S \frac{1}{\nu_0}\left[\sum_{i=1}^{n}(u_i\nu_0 - u_0\nu_i)^2 + u_0^2[\nu_0^2 - \nu_1^2 - \cdots - \nu_n^2]\right] d\sigma. \quad (12)$$

Let us now apply (11) to the truncated cone

$$C_r := \{(t,x) || x-a| \leq R-t, \quad 0 \leq t \leq r < R\}.$$

The base of this cone is the ball of radius R in the $t=0$ hyperplane. Then (11) becomes

$$-2E_{R-r,a}(r) + \int_S V(u)\cdot \mathbf{n} d\sigma + 2E_{R,a}(0) = 0, \quad (13)$$

where I have used the shorthand

$$E_{s,a}(t) = \frac{1}{2}\int_{|x-a|\leq s}(u_t^2(t,x) + u_{x_1}^2(t,x) + \cdots + u_{x_n}^2(t,x))dx$$

for that portion of the energy located in the ball of radius s about a.

On S we have $\nu_0 \equiv \frac{1}{\sqrt{2}}$ and $\nu_0^2 - \nu_1^2 - \cdots - \nu_n^2 \equiv 0$ so

$$\int_S V(u)\cdot \mathbf{n} d\sigma \leq 0.$$

So if $u(0,x) = 0$ and $u_t(0,x) = 0$ for $|x-a| \leq R$ so that $E_{R,a} = 0$, it follows from (13) that $E_{R-r,a} \leq 0$ which is impossible unless $E_{R-r,a}(r) = 0$ which means that $u(t,x)$ and $u_t(t,x)$ vanish for $|x-a| \leq R-r$. Since this is true for all $0 < r < R$ we conclude

Theorem 1 [**Finite propagation speed.**] *If $u(0,x)$ and $u_t(0,x)$ vanish on $|x-a| < R$ and u is a solution of the wave equation, then u vanishes on the cone*

$$C := \{(t,x) || x-a| \leq R-t, \quad 0 \leq t \leq R\}.$$

If $u(0,x)$ and $u_t(0,x)$ have compact support, then so do $u(t,x)$ and $u_t(t,x)$ for any $t > 0$ and hence choosing R large enough so that the integral over S vanishes, (13) implies the conservation of energy.

2.1 The wave equation in an exterior domain.

To show the advantage of this method of proof over the one using the Fourier transform, consider the wave equation on an open set G of $\mathbf{R}^n$ whose boundary ∂G is piecewise smooth and bounded, and we consider the wave equation (1) supplemented by the boundary condition

$$u(t,x) = 0 \quad \text{for } x \in \partial G \quad \text{and all } t. \quad (14)$$

Suppose we choose R sufficiently large that ∂G is contained in the ball of radius R and also in the ball of radius $R-r$ for a certain $r < R$. So the cylinder $[0,r]\times\partial G$ is completely contained in the truncated cone, C_r introduced above, and we can consider the intersection of $\mathbf{R}\times G$ with C_r, that is, the region $\mathcal{D}$ inside the truncated cone but exterior to ∂G. In addition to the top, bottom and lateral portion S of the boundary of the cone, we now have $[0,r]\times\partial G$ as a part of $\partial\mathcal{D}$ to consider when applying (11), but this makes no contribution, since $\mathbf{n}$ has no t component (i.e.$\nu_0 = 0$) and $u_t \equiv 0$ on ∂G by our boundary condition. Hence (13) holds, where now

$$E_{s,a}(t) = \frac{1}{2}\int_{\{|x-a|\leq s\}\cap G}(u_t^2(t,x) + u_{x_1}^2(t,x) + \cdots + u_{x_n}^2(t,x))dx.$$

Thus we conclude that

$$E_{R-r}(r) \leq E_R(0).$$

Also we conclude that if the initial data for u vanish in the ball $|x| \leq R$ then u vanishes in the cone $|x| \leq R-t$. Reversing the direction of time allows us to get the reverse inequality $E_R(T) \geq E_{R-T}(0)$. So we have proved

Proposition 1 *If u is a solution of (1) and (14) defined on all of $\mathbf{R}\times G$, and if the initial energy*

$$E(u)(0) = \frac{1}{2}\int_G [u_t^2(t,x) + u_{x_1}^2(t,x) + \cdots + u_{x_n}^2(t,x)]dx_{|t=0}$$

is finite, then $E(u)(t) = E(u)(0)$ for all t.

We have not yet established the existence or uniqueness of the solution to the initial value problem with boundary condition. but the proposition says that if we do establish these facts, then we would get a unitary group $U(t)$ (taking initial values at time zero into initial values at time t) on the Hilbert space

$$H = \mathbf{D}\oplus\mathbf{H}^0$$

where these spaces are defined as in Section 1.3 except that the integration is over G instead of $\mathbf{R}^n$. In particular, $\mathbf{D}$ is the closure in the norm

$$\|\phi\|_{\mathbf{D}}^2 = \frac{1}{2}\int_G |\operatorname{grad}\phi|^2 dx$$

of the space of C^∞ functions of compact support in G.

The proposition does suggest that we be able to use Stone's theorem to construct $U(t)$. If we had $U(t)$ sending initial data

$$\begin{pmatrix} f \\ g \end{pmatrix} \mapsto U(t)\begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} u(t,x) \\ u_t(t,x) \end{pmatrix}$$

then differentiating with respect to t and setting $t = 0$ suggests that the infinitesimal generator be given by

$$A = \begin{pmatrix} 0 & I \\ -\Delta & 0 \end{pmatrix} \tag{15}$$

where, we recall, our convention is

$$-\Delta\phi = \phi_{x_1x_1} + \cdots + \phi_{x_nx_n}.$$

So let us define the domain $D(A) \subset H$ to consist of all

$$\begin{pmatrix} f \\ g \end{pmatrix} \quad \text{such that } -\Delta f \in L_2(G) \text{ and } g \in L_2(G) \cap \mathbf{D}^1.$$

Lemma 1 *The operator A is skew-adjoint.*

The domain $D(A)$ is dense in H since it contains all data of compact support. To check that A is skew-symmetric observe that if f_2 and g_2 are smooth and have compact support then

$$\left(\begin{pmatrix} 0 & I \\ -\Delta & 0 \end{pmatrix} \begin{pmatrix} f_1 \\ g_1 \end{pmatrix}, \begin{pmatrix} f_2 \\ g_2 \end{pmatrix} \right) = \int (\operatorname{grad} g_1) \cdot \overline{\operatorname{grad} f_2} dx - \int \Delta f_1 \overline{g_2} dx$$

$$= \int (\operatorname{grad} g_1) \cdot \overline{\operatorname{grad} f_2} dx - \int (\operatorname{grad} f_1) \cdot \overline{\operatorname{grad} g_2} dx.$$

If

$$\begin{pmatrix} f_2 \\ g_2 \end{pmatrix} \in D(A),$$

we can find approximating sequences which are smooth and have compact support and whose second component converges to g_2 in the $\mathbf{D}$ norm. So the above relation holds for all pairs of elements of $D(A)$ proving that A is skew-symmetric.

Now let us show that the domain of A^* is contained in $D(A)$. To say that

$$\begin{pmatrix} f_2 \\ g_2 \end{pmatrix} \in D(A^*) \quad \text{and} \quad \begin{pmatrix} 0 & I \\ -\Delta & 0 \end{pmatrix} \begin{pmatrix} f_2 \\ g_2 \end{pmatrix} = \begin{pmatrix} f_3 \\ g_3 \end{pmatrix}$$

means that for all

$$\begin{pmatrix} f_1 \\ g_1 \end{pmatrix} \in D(A)$$

we have

$$\left(A \begin{pmatrix} f_1 \\ g_1 \end{pmatrix}, \begin{pmatrix} f_2 \\ g_2 \end{pmatrix} \right) = \left(\begin{pmatrix} f_1 \\ g_1 \end{pmatrix}, \begin{pmatrix} f_3 \\ g_3 \end{pmatrix} \right).$$

Suppose we choose $f_1 = 0$ and g_1 smooth and of compact support. The above equation then says

$$(g_1, f_2)_{\mathbf{D}} = (g_1, g_3)_{L_2(G)}.$$

Integrating the left hand side by parts gives

$$(\Delta g_1, f_2)_{L_2} = (g_1, g_3)_{L_2}$$

which says that

$$\Delta f_2 = g_3$$

as distributions. Next choose $f_2 = 0$ giving

$$-(\Delta f_1, g_2)_0 = (f_1, f_3)_{\mathbf{D}}.$$

Let ϕ be a smooth function with compact support G'. If $\psi \in \mathbf{D}$ then (when $n > 2$)

$$\int_{G'} |\psi|^2 dx \leq \text{const.})\|\psi\|^2_{\mathbf{D}} \tag{16}$$

where the constant depends on G'. (For the proof see below.) Thus

$$|(\phi, \psi)_{L_2}| \leq (\text{const.})\|\psi\|_{\mathbf{D}}$$

where the constant depends on ϕ. Thus the linear function $\psi \mapsto (\psi, \phi)_{L_2}$ is bounded in the $\| \ \|_{\mathbf{D}}$ norm, so by the Riesz representation theorem there is an $f_1 \in \mathbf{D}$ such that

$$(f_1, \psi)_{\mathbf{D}} = (\phi, \psi)_{L_2} \quad \forall \ \psi \in \mathbf{D}.$$

If ψ is smooth with compact support this gives, after integrating by parts,

$$(f_1, \Delta\psi)_{L_2} = (\phi, \psi)_{L_2}$$

so

$$\Delta f_1 = \phi$$

as distributions. Substituting this into the equation $-(\Delta f_1, g_2)_0 = (f_1, f_3)_{\mathbf{D}}$ gives

$$-(\phi, g_2)_{L_2} = (\phi, f_3)_{L_2}$$

so

$$-g_2 = f_3.$$

We have verified that $D(A^*) \subset D(A)$ (modulo the proof of (16)).

Theorem 2 *We may now apply Stone's theorem to conclude that there exists a unitary one parameter group on H with generator A.*

Proof of (16). For any ψ of compact support extend ψ by zero if necessary so as to be defined on all of $\mathbf{R}^n$. Then write

$$\psi(x) = -\int_{|x|}^{\infty} \partial_r \psi\left(r\frac{x}{|x|}\right) dr.$$

By Cauchy-Schwarz we have

$$|\psi(x)|^2 \leq \left(\int_{|x|}^{\infty} r^{1-n} dr\right) \left(\int_{|x|}^{\infty} |\partial_r \psi|^2 r^{n-1} dr\right)$$

$$= \frac{|x|^{2-n}}{n-2} \int_{|x|}^{\infty} |\partial_r \psi|^2 r^{n-1} dr.$$

Integrating with respect to $\omega = \frac{x}{|x|}$ on the unit sphere gives

$$\int \psi(R\omega) d\omega \leq \frac{R^{2-n}}{n-2} \|\psi\|_{\mathbf{D}}.$$

Multiplying by R^{n-1} and integrating gives

$$\int_{|x| \leq R} |\psi|^2 dx \leq \frac{R^2}{2(n-2)} \|\psi\|_{\mathbf{D}}$$

which implies (16). QED

3 The Radon transform.

Parameterize the hyperplanes in $\mathbf{R}^n$ in a two to one fashion by the pairs $(s, \omega) \in \mathbf{R} \times \mathbf{S}^{n-1}$. Here the hyperplane

$$H_{(s,\omega)} := \{z | z \cdot \omega = s\}$$

is the hyperplane orthogonal to the direction ω and at signed distance s from the origin, where the sign of s is chosen so that $s\omega$ belongs to the hyperplane. The points (s, ω) and $(-s, -\omega)$ parameterize the same hyperplane. On the hyperplane H we will use dH_z to denote the measure induced by the Euclidean metric. The Radon transform R maps functions on $\mathbf{R}^n$ (say elements of C_0^∞) to functions on $\mathbf{R} \times \mathbf{S}^{n-1}$ by averaging over hyperplanes:

$$Ru(s, \omega) = \frac{1}{(2\pi)^{\frac{n-1}{2}}} \int_{H_{(s,\omega}} u(z) dH_z. \tag{17}$$

We have

$$Ru(-s, -\omega) = Ru(s, \omega).$$

The Radon transform is related to the Fourier transform as follows:

$$\hat{u}(r\omega) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbf{R}^n} e^{-ir\omega \cdot z} u(z) dz = \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{\mathbf{R}} e^{-irs} Ru(s, \omega) ds.$$

In other words

$$\hat{u}(r\omega) = \mathcal{F}_s Ru(\cdot, \omega). \tag{18}$$

3.1 The inversion formula.

The Fourier inversion formula says

$$u(z) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbf{S}^{n-1}} \int_0^\infty e^{iz\cdot r\omega} \hat{u}(r\omega) r^{n-1} dr d\omega$$

where we have used polar coordinates. Substituting (18) gives

$$\frac{1}{(2\pi)^{\frac{n+1}{2}}} \int_{\mathbf{S}^{n-1}} \int_0^\infty e^{i(z\cdot\omega)r} \int_{-\infty}^\infty Ru(s,\omega) e^{-isr} r^{n-1} ds dr d\omega.$$

In this integral r is positive so we may write $|r|^{n-1}$ instead of r^n. If we then make the replacements

$$r \mapsto -r,\ s \mapsto -s,\ \omega \mapsto -\omega$$

the remaining integrands do not change. So we can rewrite the last integral as an integral over all r, i.e.

$$\frac{1}{2} \frac{1}{(2\pi)^{\frac{n+1}{2}}} \int_{\mathbf{S}^{n-1}} \int_{-\infty}^\infty e^{i(z\cdot\omega)r} \int_{-\infty}^\infty Ru(s,\omega) e^{-isr} r^{n-1} ds dr d\omega.$$

We can then apply the Fourier inversion formula. For any function of the variable s define the operator $|D_s|^{n-1}$ by

$$|D_s|^{n-1} = \mathcal{F}^{-1} \circ m_{|\sigma|^{n-1}} \circ \mathcal{F}$$

where $m_{|\sigma|^{n-1}}$ denotes multiplication by $|\sigma|^{n-1}$. If n is odd $|D_s|^{n-1} = D_s^{n-1}$. We get the *inversion formula for the Radon transform*:

$$u(z) = \frac{1}{2} \frac{1}{(2\pi)^{\frac{n-1}{2}}} \int_{\mathbf{S}^{n-1}} (|D_s|^{n-1} R)(z\cdot\omega, \omega) d\omega. \tag{19}$$

3.2 The Plancherel formula for the Radon transform.

If we apply the same method to the Plancherel formula for the Fourier transform

$$\frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbf{R}^n} |u(z)|^2 dz = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbf{R}^n} |\hat{u}(\zeta)|^2 d\zeta$$

we obtain

$$\int_{\mathbf{R}^n} |u(z)|^2 dz = \frac{1}{2\pi} \int_{\mathbf{R}} \int_{\mathbf{S}^{n-1}} ||D_s|^{(n-1)/2} |Ru(s,\omega)|^2 ds d\omega. \tag{20}$$

If we apply Δ to (19) we get

$$(\Delta u) = \frac{1}{2} \frac{1}{(2\pi)^{\frac{n-1}{2}}} \int_{\mathbf{S}^{n-1}} (D_s^2 |D_s|^{n-1} R)(z\cdot\omega, \omega) d\omega.$$

Applying the Radon transform gives the important formula

$$R\Delta = D_s^2 R. \tag{21}$$

The Radon transform takes the n-dimensional Laplacian into the one dimensional Laplacian. We can use this fact to write down a formula for the solution and verify Huygens' principle in odd dimensions > 1.

3.3 Huygens' principle for odd $n \geq 3$.

For $n = 1$ the solution to the initial value problem for the wave equation is given by

$$u(t,x) = \frac{1}{2}[f(x+t) + f(x-t)] + \frac{1}{2}\int_{x-t}^{x+t} g(\tau)d\tau \tag{22}$$

as can be checked by inspection. Notice that the dependence on g extends to the entire interval $[x-t, x+t]$, but the values of $\partial_x u$ or $\partial_t u$ depend only on the values of f' and g at $x \pm t$.

Now let us consider the wave equation in odd dimensions greater than one. Applying the Radon transform in the x variables to u and to the initial conditions, and using (21) we get

$$Ru(t,s,\omega) = \frac{1}{2}\left[Rf(s+t,\omega) + Rf(s-t,\omega) + \int_{s-t}^{s+t} Rg(\tau,\omega)d\tau\right].$$

If we now apply the inverse Radon transform, we see, because of the presence of the D_s^{n-1} in (19) that $u(t,x)$ depends only on the values of f, g and their derivatives on the sphere $|y-x| = t$.

3.4 The Lax-Phillips transform.

Let $U_0(t)$ be the unitary group acting on the space $H = \mathbf{D} \oplus \mathbf{H}^0$ as considered above, but in the special case where $G = \mathbf{R}^n$, i.e. there are no boundary conditions. Let us denote an element of H by

$$\mathbf{u} = \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}.$$

Consider the map $\mathcal{L}$ defined by

$$\mathcal{L}\mathbf{u} = (2\pi)^{\frac{1}{2}}\left(D_s^{\frac{1}{2}(n+1)}Ru_0 - D_s^{\frac{1}{2}(n-1)}Ru_1\right). \tag{23}$$

Initially this is defined on smooth elements of H of compact support or even on elements such that u_0 and u_1 belong to $\mathcal{S}$. Notice that the two terms on the right have opposite parities under the map $(s,\omega) \mapsto (-s,-\omega)$ (because the powers of D_s differ by one). So $\mathcal{L}$ is injective.

Let

$$\mathbf{N} := L_2(\mathbf{S}^{n-1}).$$

Theorem 3 [Lax-Phillips.] *$\mathcal{L}$ extends to a unitary map of H onto $L_2(\mathbf{R},\mathbf{N})$ and*

$$\mathcal{L}U_0(t)\mathcal{L}^{-1} = T_t \tag{24}$$

where

$$(T_t k)(s) = k(s-t)$$

is the operator of right translation by t on $L_2(\mathbf{R},\mathbf{N})$.

Proof. Consider $\mathbf{v}_{\sigma,\omega}$ defined by

$$\mathbf{v}_{\sigma,\omega}(x) := \frac{(-i\sigma)^{\frac{1}{2}(n-3)}}{(2\pi)^{n/2}} e^{-i\sigma\omega\cdot x} \begin{pmatrix} 1 \\ i\sigma \end{pmatrix}.$$

It does not belong to H but it does satisfy

$$A\mathbf{v}_{\sigma,\omega} = i\sigma\mathbf{v}_{\sigma,\omega}$$

where A is the operator given by (15). Also, when (the components) of $\mathbf{u}$ belong to $\mathcal{S}$, the scalar product $(\mathbf{u}, \mathbf{v}_{\sigma,\omega})_H$ is defined. Define

$$\mathcal{P}\mathbf{u}(\sigma,\omega) := (\mathbf{u}, \mathbf{v}_{\sigma,\omega})_H. \tag{25}$$

Again, this is defined (initially) for $\mathbf{u} \in \mathcal{S}$. We have

$$\mathcal{P}A\mathbf{u} = (A\mathbf{u}, \mathbf{v}_{\sigma,\omega})_H = -(u, A\mathbf{v}_{\sigma,\omega})_H = i\sigma\mathcal{P}\mathbf{u}. \tag{26}$$

Also, writing $\mathbf{v}$ for $\mathbf{v}_{\sigma,\omega}$, integration by parts gives

$$(\mathbf{u},\mathbf{v})_H = \frac{1}{2}\int \left(u_0\overline{\Delta v_0} + u_1\overline{v_1}\right) dx =$$

$$\frac{1}{2}\left[-(-i\sigma)^{\frac{1}{2}(n+1)}[\mathcal{F}^{-1}u_0](\sigma\omega) + (i\sigma)^{\frac{1}{2}(n-1)}[\mathcal{F}^{-1}u_1](\sigma\omega)\right].$$

Here $\mathcal{F}^{-1}$ is the inverse Fourier transform. Because of the difference in powers of σ in the two terms, they have opposite parity with respect to $(\sigma,\omega) \mapsto (-\sigma,-\omega)$ and hence are orthogonal in $L_2(\mathbf{R}\times\mathbf{S}^{n-1})$ and hence

$$\|\mathcal{P}\mathbf{u}\|^2 = \frac{1}{4}\left[\|\sigma^{\frac{1}{2}(n+1)}\mathcal{F}^{-1}u_0\|^2 + \|\sigma^{\frac{1}{2}(n-1)}\mathcal{F}^{-1}u_1\|^2\right].$$

Since both $\mathcal{F}^{-1}u_0$ and $\mathcal{F}^{-1}u_1$ are even, we have

$$\begin{array}{lllll} \|\sigma^{\frac{1}{2}(n+1)}\mathcal{F}^{-1}u_0\|^2 & = & 2\int_{\mathbf{S}^{n-1}}\int_0^\infty |\mathcal{F}^{-1}u_0|^2\sigma^{\frac{1}{2}(n+1}d\sigma d\omega & = & 2\|u_0\|^2_{\mathbf{D}} \\ \|\sigma^{\frac{1}{2}(n-1)}\mathcal{F}^{-1}u_1\|^2 & = & 2\int_{\mathbf{S}^{n-1}}\int_0^\infty \sigma^{\frac{1}{2}(n-1)}d\omega d\sigma & = & 2\|u_1\|^2_{L_2} \end{array}$$

where we have used (the Fourier) Plancherel theorem. This shows that $\mathcal{P}$ is an isometry. From the orthogonality in the decomposition and applying the Fourier transform to invert $\mathcal{F}^{-1}$, we see that all elements of $L_2(\mathbf{R}\times\mathbf{S}^{n-1})$ which are smooth, and which vanish near $\sigma = 0$ and ∞ are in the image of $\mathcal{P}$. Since this set is dense in $L_2(\mathbf{R}\times\mathbf{S}^{n-1})$ we have shown that

Proposition 2 *$\mathcal{P}$ is an isometry of H onto $L_2(\mathbf{R},\mathbf{N})$ and $\mathcal{P}\circ A\circ\mathcal{P}^{-1}$ is multiplication by $i\sigma$.*

It follows that

Proposition 3 *$\mathcal{P}\circ U(t)\circ\mathcal{P}^{-1}$ is multiplication by $e^{i\sigma t}$.*

Since we can express the Fourier transform in terms of the Radon transform as given by (18), we see that

$$\mathcal{P}\mathbf{u} = \mathcal{F}_s^{-1}\circ\mathcal{L}.$$

This completes the proof of the theorem since T_t is carried into multiplication by $e^{i\sigma t}$ by the one dimensional Fourier transform.

4 The propagator when $n = 3$.

Let us go back to the formula (5) for the solution of the wave equation in Fourier space. Take $f = 0$ and $g = 1$, and let $\hat{K}_t(\xi)$ be the corresponding solution. Thus

$$K_t(x) = \mathcal{F}^{-1}\left(\frac{\sin|\xi|t}{|\xi|}\right) \tag{27}$$

has the property that it is a distributional solution of the wave equation with the initial conditions

$$K_0 = 0, \quad \frac{d}{dt}(K_t)_{|t=0} = (2\pi)^{n/2}\delta$$

where δ is the delta function in x-space. From the formula (5) we can now conclude that the general expression for the solution to the wave equation with initial conditions $u(0,x) = f(x), u_t(0,x) = g(x)$ can be written as

$$u = \partial_t(K_t \star f) + K_t \star g \tag{28}$$

since the Fourier transform takes convolution into multiplication. So it would be useful to get a more explicit expression for K_t in $x-$space. Rather than directly evaluating the inverse Fourier transform of the right hand side of (27) I will proceed in a roundabout way in three spatial dimensions where the formula for K_t is particularly nice. The exposition here follows Rauch *Partial Differential Equations* pages 152-161. For the formula for K in general, see Folland *Introduction to Partial Differential Equations* pages 223 (for odd n) and 226 (for even n).

4.1 The Laplacian for functions of r.

If $f(x) = \phi(r)$ where $r = |x|$ then

$$-\Delta f(x) = \phi''(r) + \frac{n-1}{r}\phi'(r). \tag{29}$$

Proof.

$$\begin{aligned}
\Delta f &= \sum_{j=1}^{n} \partial_{x_j}\left(\phi'(r)\cdot\frac{x_j}{r}\right) \\
&= \sum\left[\phi''(r)\cdot\frac{x_j^2}{r^2} + \phi'(r)\frac{1}{r} - \phi'(r)\cdot\frac{x_j^2}{r^3}\right] \\
&= \phi''(r) + \frac{n-1}{r}\phi'(r). \quad \text{QED}
\end{aligned}$$

Taking $n = 3$ and multiplying by r gives

$$-r\Delta f = r\phi'' + 2\phi' = (r\phi)''.$$

4.2 Smooth rotationally invariant solutions of the wave equation.

So if u is a smooth rotationally invariant solution of the wave equation in three spatial dimensions, then $v = ru$ is a solution of the one dimensional wave equation for $r > 0$ and conversely. If u is smooth at $r = 0$ it is extendable as an even function of r to negative values of r. Hence

Proposition 4 *The map $v(t,r) \leftrightarrow ru(t,r)$ is a bijection between the smooth rotationally invariant solutions of the wave equation in three dimensions and the odd solutions of the wave equation in one dimension.*

The most general solution of the wave equation in one dimension is

$$\phi(x+t) + \psi(x-t).$$

If v is of the above form and odd, it must be equal to the odd part of the above so

$$2v(t,x) = \phi(x+t) - \phi(-x+t) + \psi(x-t) - \psi(t+x) = 2[F(x+t) - F(t-x)]$$

where $2F = \phi - \psi$. So the most general odd solution of the wave equation is

$$v(t,x) = F(t+x) - F(t-x).$$

The left hand side is unchanged if we add a constant to F and the above equation then defines a bijection between $C^\infty(\mathbf{R})/\mathbf{R}$ and smooth odd solutions of the wave equation. Therefore

Theorem 4 *The map $F \leftrightarrow u$*

$$u(t,x) := \begin{cases} \frac{F(t+|x|)-F(t-|x|)}{|x|}, & x \neq 0 \\ 2F'(t) & x = 0 \end{cases} \tag{30}$$

defines a bijection between $C^\infty(\mathbf{R})/\mathbf{R}$ and smooth rotationally invariant solutions of the wave equation in three spatial dimensions.

Let $f(r) = ru(0,r)$ and $g(r) = ru_t(0,r)$ so the Cauchy data for u give

$$F(r) - F(-r) = f(r), \quad F'(r) - F'(-r) = g(r).$$

Differentiating the first equation and adding to the second gives

$$F' = f' + g.$$

As F is only determined up to an additive constant, we may choose

$$F(r) = \int_{-\infty}^{r} f'(s)ds + \int_{-\infty}^{r} g(s)ds.$$

Suppose that the Cauchy data at $t = 0$ are supported in the ball of radius ρ. Then the first integral above vanishes for $r < \rho$ or $r > \rho$ since f vanishes there. The second integral also vanishes when $r > \rho$ since g is odd. Hence F is supported in $[-\rho, \rho]$.

Let us examine the behavior of the solution in the physical region where $r \geq 0$. For large positive t we have $r + t > \rho$ and so $F(t + r) = 0$. Then

$$u(t, x) = -\frac{F(t-r)}{r}.$$

This is an outgoing wave supported in the region $-\rho < t - r < \rho$ which is the same as $t - \rho < r < t + \rho$. In particular, we see a special case of Huyghens' principle. There is an attenuation factor of $1/r$ in the shape of the wave. In the distant past, where $t << 0$ we have $t - r < -\rho$ so $F(t - \rho) = 0$ and

$$u(t, x) = \frac{F(t+r)}{r}.$$

So the entire history consists of an incoming spherical wave which emerges as an outgoing spherical wave (with a change in sign).

Now let us choose a function j which is smooth with support contained in the unit ball, and such that

$$j \geq 0, \qquad \int j dx = (2\pi)^{\frac{3}{2}}$$

and set

$$j_\epsilon := \epsilon^{-3} j\left(\frac{x}{\epsilon}\right).$$

We have

$$j_\epsilon \to (2\pi)^{\frac{3}{2}} \delta$$

in $\mathbf{H}^s$ for all $s < -\frac{3}{2}$ and hence the solution

$$u_\epsilon(t, x)$$

to the wave equation with initial values $f_\epsilon = 0, g_\epsilon = j_\epsilon$ will converge in $\mathbf{H}^s$ to our desired K, the solution to the wave equation with initial condition $K(0, x) = 0, \partial_t K(0, x) = (2\pi)^{\frac{3}{2}} \delta$.

We know that u_ϵ has the form

$$u_\epsilon(t, x) := \begin{cases} \frac{F_\epsilon(t+|x|) - F_\epsilon(t-|x|)}{|x|}, & x \neq 0 \\ 2F'_\epsilon(t) & x = 0 \end{cases}.$$

Since $u_\epsilon(0, x) = 0$ we set that $F_\epsilon(r) - F_\epsilon(-r) = 0$, in other words F is an even function of r. Hence F'_ϵ is an odd function. Computing $\partial_t u_\epsilon(0, x)$ gives

$$F'_\epsilon = \frac{1}{2} r j_\epsilon.$$

Since F_ϵ is only determined up to additive constant, we may choose

$$F_\epsilon(r) = \frac{1}{2}\int_{-\infty}^{r} sj_\epsilon(s)ds.$$

Since supp j_ϵ is contained in the ball of radius ϵ we know that $u_\epsilon(t,x)$ is supported in a shell of width ϵ about the ball of radius $|t|$. So we expect that in the limit we will get a distribution supported on the sphere of radius t. Also,

$$\partial_t^2 \int u_\epsilon(t,x)dx = \int \partial_t^2 u_\epsilon(t,x)dx = \int -\Delta u_\epsilon(t,x)dx = 0$$

where we have used the fact that u_ϵ is a solution of the wave equation, and then integration by parts and the fact that u_ϵ has compact support. Thus

$$\int u_\epsilon(t,x)dx = at + b$$

for suitable constants a and b. Evaluating at $t = 0$ gives $b = 0$ and differentiating with respect to t and setting $t = 0$ gives $a = (2\pi)^{\frac{3}{2}}$.

We also claim that

$$u_\epsilon(t,x) \geq 0 \quad \text{for} \quad t \geq 0.$$

Indeed

$$ru_\epsilon(t,x) = F_\epsilon(t+r) - F_\epsilon(t-r) = \frac{1}{2}\int sj_\epsilon(s)ds.$$

If $t - r \geq 0$ this is non-negative because j_ϵ. For $t - r < 0$ the integral from $t - r$ to $r - t$ vanishes because we extended rj_ϵ to be odd. The remaining portion of the integral, from $r - t$ to $r + t$ has a non-negative integrand. For $t < 0$ we have

$$u_\epsilon(t,x) \leq 0$$

since $-u_\epsilon(-t,x)$ is a solution of the wave equation with the same initial conditions as u_ϵ and so $u_\epsilon(-t,x) = -u_\epsilon(t,x)$. So we can write

$$(\text{sgn}t)u_\epsilon(t,x) \geq 0$$

for all t.

The above facts will be enough to allows us to conclude that

$$\langle K(t), \psi\rangle = \frac{(2\pi)^{\frac{3}{2}}}{4\pi t}\int_{|x|=|t|} \psi d\sigma_{|t|} \tag{31}$$

when

$$t \neq 0$$

where $d\sigma_{|t|}$ is the area element on the sphere of radius $|t|$. Indeed, using polar coordinates we have

$$\begin{aligned}
\langle u_\epsilon(t), \psi\rangle &= \int_{\mathbf{S}^2}\int_0^\infty u_\epsilon(t,r)\psi(r\omega)r^2 drd\omega \\
&= \int_{\mathbf{S}^2}\int_0^\infty u_\epsilon(t,r)\psi(t\omega)r^2 drd\omega + \int_{\mathbf{S}^2}\int_0^\infty u_\epsilon(t,r)[\psi(r\omega) - \psi(t\omega)]r^2 drd\text{\o}mega \\
&=: I_1 + I_2.
\end{aligned}$$

We can estimate I_2 as follows: we have $|r\omega - t\omega| < \epsilon$ on supp $u_\epsilon(t,\cdot)$ so $|\psi(r\omega) - \psi(t\omega)| < C\epsilon$ where C is the supremum of the partial derivatives of ψ so $C = \|\operatorname{grad}\psi\|_\infty$. Then

$$|I_2| \le \epsilon C \int_{\mathbf{S}^2}\int_0^\infty |u_\epsilon|(t,r)r^2 dr d\omega.$$

If $t > 0$ we can remove the absolute value sign since $u_\epsilon \ge 0$, and then use our evaluation of the integral to obtain

$$|I_2| \le \|\operatorname{grad}\psi\|_\infty (2\pi)^{\frac{3}{2}} t.$$

For a compact range of t and for ψ with bounded $\|\operatorname{grad}\psi\|_\infty$ this tends uniformly to zero.

To evaluate I_1 we use Fubini:

$$\int_{\mathbf{S}^2}\int_0^\infty u_\epsilon(t,r)\psi(t\omega)r^2 dr d\omega = \int_{\mathbf{S}^2}\psi(t\omega)d\omega \int_0^\infty u_\epsilon(t,r)r^2 dr.$$

We can write

$$\int_{\mathbf{S}^2}\psi(t\omega)d\omega = \frac{1}{t^2}\int_{|x|=t}\psi d\sigma$$

since the surface area form of the sphere of radius t is $d\sigma = t^2 d\omega$. On the other hand

$$\int_0^\infty u_\epsilon(t,r)r^2 dr = \frac{1}{4\pi}\int_{\mathbf{S}^2}\int_0^\infty u_\epsilon(t,r)r^2 dr d\omega = \frac{(2\pi)^{\frac{3}{2}}}{4\pi}.$$

We have proved (31) for $t > 0$. It follows for $t < 0$ since both sides of (31) are odd functions of t.

If we now insert our formula for K into (28) we get

$$u(t,x) = \frac{1}{4\pi t}\int_{|y-x|=|t|} g(y)d\sigma(y) + \frac{\partial}{\partial t}\left(\frac{1}{4\pi t}\int_{|y-x|=|t|} f(y)d\sigma(y)\right), \tag{32}$$

This formula makes Huyghens' principle very explicit (and gives another proof of it in three dimensions).

4.3 δ-functions along submanifolds.

We can think of the formula (31) for $K(t)$ as a "δ-function associated to the sphere of radius t". More generally, if Y is a submanifold of a manifold X, we may consider a δ-density along Y as a linear function ℓ on the space of C^∞ functions of compact support on X, where $\langle \ell, \psi\rangle$ is obtained by first restricting ψ to Y and then integrating over Y (relative to some smooth measure on Y). We will spend some time studying the functorial behavior of such objects.