Problem set 1

Math 212b

January 31, 2002 due Feb. 13

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Last semester we were very careful in insisting that the Fourier transform $\mathcal{F}$ maps the space $L_2(\mathbf{R}^n)$ (of functions of the variable x) to $L_2(\mathbf{R}_n)$ (of functions of the variable ξ) and that these were two different spaces. In this problem set we will commit the crime (somewhere between a misdemeanor and a felony) of identifying these two spaces, i.e. identifying the variable ξ and the variable x. For most, if not all of the discussion, we will restrict to the case $n = 1$, and write H for $L_2(\mathbf{R})$. Then we are thinking of $\mathcal{F}$ as a map of H into itself,

$$\mathcal{F} : \quad H \to H.$$

The Fourier inversion formula has a plus sign before the i in the exponent instead of a minus sign, which we can now translate into

$$(\mathcal{F}^2 f)(x) = f(-x)$$

which means that

$$\mathcal{F}^4 = \text{Identity}.$$

Thus $\mathcal{F}$ is a unitary operator on H whose fourth power is the identity, so its spectrum must be concentrated on the fourth roots of 1, i.e. at the points $1, i, -1$, and -1. This suggests that we look for a self-adjoint operator A whose spectrum is concentrated on the integers, and such that $\mathcal{F} = \exp(-\frac{i\pi}{2}A)$. In other words, that we find a one-parameter group of unitary transformations $Z(t)$, such that $W(\frac{\pi}{2}) = \mathcal{F}$ and such that $W'(0) = -iA$.

The goal of this problem set is to carry this out, and we shall find that the A that arises is what is called in physics the "Hamiltonian of the harmonic oscillator" - the quantum mechanical version of motion under Hooke's law. For reasons which will become apparent as we go along, it will be convenient to modify the Fourier transform slightly by defining

$$[\mathbf{F}f](\xi) = \frac{1}{\sqrt{2\pi i}} \int_{\mathbf{R}} e^{-i\xi \cdot x} f(x) dx.$$

The presence of the i under the square root sign means that $\mathbf{F}$ will be of order eight rather than four, i.e. that $\mathbf{F}^4 = -I$ and that A will have its spectrum supported on points of the form $n + \frac{1}{2}$, $n = 0, 1, \ldots$. The reason for this shift of $\frac{1}{2}$ is rather subtle, and has to do with the fact that we will be discussing a three parameter group of transformations all at once, rather than just the one parameter group associated with the harmonic oscillator. These mysterious comments will become clarified, I hope, in the course of the discussion.

1 The Fourier transform of imaginary Gaussians.

To get started, we recall some facts about the Fourier transform of Gaussians. Recall that

$$\frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} e^{-x^2/2} e^{-i\zeta x} dx = e^{-\zeta^2/2},$$

the Fourier transform of the unit Gaussian is the unit Gaussian. Set $x = \sqrt{\lambda} u$ where $\sqrt{\lambda} > 0$ is some positive real number. Then $dx = \sqrt{\lambda} du$ and, setting $\xi := \sqrt{\lambda}\zeta$ we get

$$\frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} e^{-\lambda x^2/2} e^{-i\xi u} du = \frac{1}{\sqrt{\lambda}} e^{-\xi^2/2\lambda}.$$

The left hand side of this equation makes sense as a convergent integral when λ is complex, so long as the real part of λ is positive. So does the right hand side. Both are holomorphic functions of λ providing that we interpret the square root occurring on the right hand side as being the branch obtained by choosing the positive square root on the positive real axis. But more is true: the integral on the left converges uniformly (but not absolutely) for Re $\lambda \geq 0$ (so including parts of the imaginary axis) provided we insist that $|\lambda| \geq \delta > 0$.

Assuming this fact for the moment, if we set $\lambda = -ir$, the square root of λ we must take is $r^{\frac{1}{2}} e^{-\pi i/4}$ and so

$$\frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} e^{irx^2/2} e^{-i\xi x} dx = e^{\pi i/4} r^{-\frac{1}{2}} e^{-i\xi^2/2r}. \tag{1}$$

Now to the domain of uniform convergence. I will prove something more: Suppose that c is a function which is bounded together with its first two derivatives on the entire real axis, so we are assuming that

$$|c(x)| < K, \quad |c'(x)| < K., \quad |c''(x)| < K \quad \forall\, x \in \mathbf{R}.$$

(For example the function $x \mapsto e^{i\xi x}$ has this property for any fixed ξ.) Then I claim that the integral

$$\int_{\mathbf{R}} e^{-\lambda x^2/2} c(x) dx$$

converges uniformly in λ in the region $\mathrm{Re}\lambda > 0$, $|\lambda| \geq \delta > 0$. Indeed, we must show that for any $\epsilon > 0$ we can choose R large enough so that

$$\left| \int_R^S e^{-\lambda x^2/2} c(x) dx \right| < \epsilon$$

for any $S > R$. Now

$$e^{-\lambda x^2/2} = -\frac{1}{\lambda x} \frac{d}{dx} e^{-\lambda x^2/2}$$

so long as $x \geq R > 0$. So integration by parts gives

$$\int_R^S e^{-\lambda x^2/2} c(x) dx = \frac{e^{-\lambda R^2/2} c(R)}{\lambda R} - \frac{e^{-\lambda S^2/2} c(S)}{\lambda S} + \int_R^S e^{-\lambda x^2/2} f(x) dx,$$

where

$$f(x) := \frac{d}{dx} \left(c(x)/\lambda x \right).$$

Each of the integrated terms are bounded by $K/\delta R$. The function f together with its derivative is bounded by k/x, so integrating by parts once again establishes the desired bound, since the integrand will be bounded by C/x^2 for some C. QED

2 The group $Mp(2)$.

We now introduce operators $U_d, d \geq 0$ and $V_P \ \ P \neq 0$ on H where d and P are real numbers by

$$[U_d c](x) = e^{-\pi i/4} \frac{1}{\sqrt{2\pi d}} \int e^{i(x-y)^2/2} c(y) dy \tag{2}$$

and

$$\left(V_P c\right)(x) := e^{-iPx^2/2} c(x). \tag{3}$$

The operators U_d are our old friends, the one parameter semi-group generated by the "free Hamiltonian" - the "imaginary analogue" of the fundamental solution to the heat equation. The operators V_P are the conjugates of the one parameter group corresponding to the free Hamiltonian under the Fourier transform. Since both families of operators are now acting on the same Hilbert space, we are free to multiply them. In fact, this whole problem set will be devoted to the group generated by these two types of operators - that is the class of operators one can obtain by successively multiplying operators of these two types.

1. Let $s > 0, t > 0$ and $P \neq 0$ be real numbers and define

$$\begin{aligned} A &:= 1 - sP \\ B &:= t + s - stP \\ C &:= -P \\ D &:= 1 - tP. \end{aligned}$$

Suppose that $B \neq 0$. Show that

$$(U_s \circ V_P \circ U_t)\, c(x) = e^{-\pi i/4} e^{-\pi i/4} e^{\pm \pi i/4} |2\pi B|^{-1/2} \int e^{iW(x,y)} c(y) dy \tag{4}$$

where

$$\pm = \operatorname{sgn} B$$

and

$$W(x, y) := \frac{1}{2B}[Dx^2 + Ay^2 - 2xy]. \tag{5}$$

Notice that if we take $s = t = P = 1$ we get the Fourier transform, up to an overall constant phase factor.

2. Suppose that we have real constants A, B, D with $B \neq 0$ and define C so that

$$AD - BC = 1.$$

Let $O(A, B, D)$ denote the operator given by the right hand side of (4). Show that if $O(A_1, B_1, D_1)$ and $O(A_2, B_2, D_2)$ are two such operators (where we assume that $B_1 \neq 0$ and $B_2 \neq 0$ then

$$O(A_1, B_1, D_1) \circ O(A_2, B_2, D_2) = \pm O(A_3, B_3, D_3)$$

where

$$\begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} \begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix} = \begin{pmatrix} A_3 & B_3 \\ C_3 & D_3 \end{pmatrix}$$

provided that we assume that in this matrix product $B_3 \neq 0$ as well.

Consider the map ρ defined by

$$\rho(U_d) = \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}, \quad \rho(V_P) = \begin{pmatrix} 1 & 0 \\ -P & 1 \end{pmatrix}$$

and then set ρ of any product of U's and V's to be the product of the corresponding images, so, for example,

$$\rho(U_{d_1}V_{P_1}U_{d_2}V_{P_2}) = \begin{pmatrix} 1 & d_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -P_1 & 1 \end{pmatrix} \begin{pmatrix} 1 & d_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -P_2 & 1 \end{pmatrix}$$

etc. We must show that this is well defined, i.e. depends only on the product X, and not on the expression of X as a product of U's and V's. This will follow from the next problem.

3. Suppose that X is written as a product of U's and V's as above, and that the product of the corresponding $\rho(U)$ and $\rho(V)$ is

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

with $B \neq 0$. Show, by induction on the number factors, that

$$(Xc) = i^{\#}e^{-\pi i/4}|B|^{-1/2}(2\pi)^{-1/2}\int e^{iW(x,y)}c(y)dy$$

where W is given by (5). Suppose that we have two decompositions of X into a product of U's and V's where the product of corresponding matrices each have $B \neq 0$. Conclude that these two products are the same. By multiplying on the right by U_t with t small, conclude that this constraint is unnecessary, and that $\rho(X)$ is well defined, independent its decomposition as a product of U's and V's. Conclude that X is determined by $\rho(X)$ up to a sign.

3 The group $Sp(2) = Sl(2)$.

4. Show that any real two by two matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad \text{with} \quad AD - BC = 1$$

can be written as a product of matrices of the form

$$\begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ -P & 1 \end{pmatrix}.$$

Indeed, show that if $C \neq 0$ then we can find s and t so that

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ C & 1 \end{pmatrix} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

so three matrices are enough, while if $C = 0$ it can be done with four matrices. Here we did not impose the restriction that $s > 0$ and $t > 0$. But show that we can get the matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

using positive values of s and t, and hence its cube and from that we can get

$$\begin{pmatrix} 1 & -s \\ 0 & 1 \end{pmatrix}$$

and hence all two by two matrices of determinant one.

Conclude that the set of all products of U's and V's is a group, call it $Mp(2)$ and that ρ is a two to one homomorphism of $Mp(2)$ onto the group of $Sl(2)$ of all two by two real matrices of determinant one. (The symbol Mp does not stand for "motion picture" as in MP3 but rather for metaplectic, which is a very bad pun introduced by André Weil, on top of an equally atrocious choice of name "symplectic" introduced by Hermann Weyl as a name for a class of groups.)

4 The infinitesimal generators.

If $Z(t)$ is a one-parameter subgroup of $Mp(2)$, then $\rho(Z(t))$ is a one parameter group of matrices. The fact that ρ is two to one, implies that $\rho(Z(t))$ determines $Z(t)$ since out of the two possible inverse images of the identity matrix under ρ, we pick then one which is the identity operator, and then, by continuity, this fixes $Z(t)$. This means that there is a one-to-one correspondence between infinitesimal generators of one parameter subgroups of $Mp(2)$, and infinitesimal generators of the group of two by two matrices of determinant one. For example, we have the correspondence

$$\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \longleftrightarrow U_t.$$

Differentiating the matrix with respect to t at $t = 0$ gives

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

while the infinitesimal generator of U_t is $\frac{1}{2}i\frac{d^2}{dx^2}$. We thus have the correspondence

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \longleftrightarrow \frac{1}{2}i\frac{d^2}{dx^2}.$$

Similarly, we have the correspondence

$$\begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \longleftrightarrow -\frac{1}{2}ix^2$$

where the operator on the right means multiplication by $-\frac{1}{2}ix^2$.

The sum of these two matrices must correspond to the sum of these two operators (why?) and so

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \longleftrightarrow \frac{1}{2}i\left(\frac{d^2}{dx^2} - x^2\right).$$

Now the matrix on the left is the infinitesimal generator (= derivative at $t = 0$) of the group of (clockwise) rotations

$$\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}.$$

For $t \neq k\pi$ we know that the corresponding operator is of the form

$$(Z(t)f)(x) = i^{\#}e^{-\pi i/}4|\sin t|^{-1/2}(2\pi)^{-1/2}\int \exp\left[\frac{i}{2\sin t}\left(\cos t[x^2+y^2]-2xy\right)\right]f(y)dy,$$

(and we know that $\# = 0$ for small $t > 0$). (The fact that ρ is a double cover manifests itself here by the fact that when $t = 2\pi$ we do not come back to I but rather to $-I$.) When $t = \pi/2$ we get (a phase factor times) the Fourier transform. So the Fourier transform corresponds to rotation through ninety degrees.

5 The spectrum of the harmonic oscillator.

As we mentioned previously, the conventions in physics are to write $Z(t) = \exp(-iAt)$ so that in our case we have

$$A = -\frac{1}{2}\left(\frac{d^2}{dx^2} - x^2\right).$$

The operator A is called the **harmonic oscillator**. We will now find its spectrum.

5. Show that $A(x^n e^{-x^2/2}) = (n+\frac{1}{2})x^n e^{-x^2/2} + p(x)e^{-x^2/2}$ where p is a polynomial of degree less than n. Conclude from this that $(n+\frac{1}{2})$ is an eigenvalue of A, and in fact that there exists a polynomial (unique up to multiplication by a constant) H_n such that $H_n(x)e^{-x^2/2}$ is an eigenvector with eigenvalue $\frac{1}{2}$. Show directly [hint; by induction - prove and use the fact that the operator $x - \frac{d}{dx}$ commutes with A] that each of these functions is an eigenvector of the old Fourier transform $\mathcal{F}$ (with what eigenvalue?).

The functions $H_n(x)e^{-x^2/2}$ form an orthogonal basis of $L_2(\mathbf{R})$ as can be seen by the following argument: We know that they are orthogonal to one another since they are eigenvectors of A corresponding to different eigenvalues. So we must show that if $f \in L_2(\mathbf{R})$ is orthogonal to all the $H_n(x)e^{-x^2/2}$ then it must be zero. If f is orthogonal to all the $H_n(x)e^{-x^2/2}$ then it is orthogonal to any polynomial times $e^{-x^2/2}$. In particular, for any real number a we have

$$0 = e^{-a^2/2}\sum_0^n (f, \frac{(ax)^n}{n!}e^{-x^2/2}).$$

Passing to the limit we get

$$0 = (f, e^{-(x-a)^2/2}) = \sqrt{2\pi}(f \star e^{-x^2/2})(a).$$

Taking the Fourier transform, this implies that

$$\hat{f}(x)e^{-x^2/2} \equiv 0$$

so $\hat{f} = 0$ and hence $f = 0$. QED

6 Hamilton's point characteristic.

I should try to explain a bit about the relationship between the quadratic function W as given by (5) and the matrix

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

We can think of this matrix as a linear map from $\mathbf{R}^2$ to $\mathbf{R}^2$:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} q_1 \\ p_1 \end{pmatrix} = \begin{pmatrix} q_2 \\ p_2 \end{pmatrix}.$$

We can think of the graph of M as a two dimensional subspace of the space $\mathbf{R}^4$ with a general vector of $\mathbf{R}^4 = \mathbf{R}^2 \oplus \mathbf{R}^2$ written as

$$\begin{pmatrix} q_1 \\ p_1 \\ q_2 \\ p_2 \end{pmatrix}.$$

Any two dimensional subspace of $\mathbf{R}^4$ can be considered as the graph of some linear transformation of the above form, provided that its projection on $\mathbf{R}^2$ under the map

$$\begin{pmatrix} q_1 \\ p_1 \\ q_2 \\ p_2 \end{pmatrix} \mapsto \begin{pmatrix} q_1 \\ p_1 \end{pmatrix}$$

is surjective. Because then every $\begin{pmatrix} q_1 \\ p_1 \end{pmatrix}$ is the image of a unique point in the two dimensional subspace of $\mathbf{R}^4$ whose last two coordinates give the image of $\begin{pmatrix} q_1 \\ p_1 \end{pmatrix}$ under the linear transformation. But suppose that the image of this subspace under the projection

$$\begin{pmatrix} q_1 \\ p_1 \\ q_2 \\ p_2 \end{pmatrix} \mapsto \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$$

is the entire two dimensional space of all $\begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$. Then we can regard this subspace as a map from $\begin{pmatrix} q_1 \\ q_2 \end{pmatrix}$ to $\begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$. To be explicit, if $B \neq 0$, then we can solve the equations

$$\begin{aligned} q_2 &= Aq_1 + Bp_1 \\ p_2 &= Cq_1 + Dp_1 \end{aligned}$$

for p_1 and p_2 in terms of q_1 and q_2.

6. Show that the expression for p_1 and p_2 in terms of q_1 and q_2 is given by

$$p_1 = -\frac{\partial W}{\partial q_1}, \quad p_2 = \frac{\partial W}{\partial q_2}$$

where W is given by (5), and that once we have solved for the p's in terms of the q's the expression for W takes on the more suggestive form

$$W(q_1, q_2) = \frac{1}{2}(p_2 q_2 - p_1 q_1).$$

I am going to stop here, and refer you to an advanced course in classical mechanics for the proper setting for these computations.