Problem set 2

Math 212b

February 14, 2001 due Feb. 26

Contents

1 The L_2 Euler operator

This is a continuation of the preceding problem set. There are three types of one-parameter subgroups of $Sl(2,\mathbf{R})$, and we have only studied two out of the three. A one parameter subgroup of $Sl(2,\mathbf{R})$ is determined by its infinitesimal generator, that is by the two by two matrix A such that $t \mapsto \exp tA$ is the one parameter group in question. Since $\exp tM$ is to have determinant one, this means (by differentiation and setting $t = 0$) that tr $M = 0$. Multiplying M by a scalar only means that we are changing the parameter t by that scalar. So there are three possibilities (other than the trivial case where $M = 0$):

- The eigenvalues of M are both zero. Then M is a nilpotent matrix and is conjugate to
$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$
and we in fact studied the one-parameter groups generated by two such matrices,
$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix},$$
- the matrix M can have eigenvalues $\pm i$, and we studied one such one-parameter group - the rotations
$$\begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$$
(with rotation through ninety degrees corresponding to the Fourier transform).

- There is also the possibility that M has eigenvalues ± 1, for example

$$M = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

which generates the one parameter group

$$\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}.$$

1. What is the one parameter group of unitary transformations in $Mp(2)$ which corresponds to this last one-parameter subgroup of $Sl(2)$? Notice that for all elements of this one parameter group the upper right hand corner is zero, so our "generic formula" valid for $B \neq 0$ will not apply. But there is a very simple expression for the elements of the one parameter group in $Mp(2)$ covering the above one parameter subgroup in $Sp(2)$. What is it? [Hint: Observe that

$$\left[\begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

to get the infinitesimal generator of the desired one parameter group, and then stare at this infinitesimal generator to guess and then prove what is the desired one parameter group in $Mp(2)$.]

2 The group $Sp(2n)$.

I want to pass from 2 to $2n$ dimensions. Please reread the sections in the handouts on symplectic vector spaces. Accoding to one of the theorems proved there, all syplectic vector spaces are even dimensional, and any two of the same dimension are isomorphi.

So we may assume that our symplectic vector space is $\mathbf{R}^n \oplus \mathbf{R}^n$ and we write the typical vector as

$$u = \begin{pmatrix} q \\ p \end{pmatrix} \quad \text{where } q, p \in \mathbf{R}^n$$

and where the symplectic form ω is given by

$$\omega(u, u') = p \cdot q' - p' \cdot q$$

with $\cdot$ denoting the usual scalar product on $\mathbf{R}^n$. If we introduce the standard positive definite scalar product $\bullet$ on

$$\mathbf{R}^n \oplus \mathbf{R}^n = \mathbf{R}^{2n}$$

$$u \bullet u' = q \cdot q' + p \cdot p'$$

then

$$\omega(u, v) = u' \bullet Ju$$

where

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

where I is the $n \times n$ identity matrix. A $2n \times 2n$ matrix T is called symplectic if $\omega(Tu, Tu') = \omega(u, u')$ for all $u, u' \in \mathbf{R}^n$ which amounts to the condition

$$T^t JTu \bullet u' = Ju \bullet u' \quad \forall \;\; u, u' \in \mathbf{R}^n$$

where T^t denotes the transpose of T. So the condition on T is

$$T^t JT = J.$$

2. Write T in "block form" as

$$T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where now $A, B, C,$ and D are $n \times n$ matrices. Express the above condition on T in terms of A, B, C, D.

The purpose of the next few problems is to prove that every element of the group $Sp(2n)$ of symplectic matrices can be written as a finite product of matrices of the following two types

$$\begin{pmatrix} I & dI \\ 0 & I \end{pmatrix} \qquad d \geq 0$$

and

$$\begin{pmatrix} I & 0 \\ -P & I \end{pmatrix} \quad \text{where} \quad P = P^t$$

is a symmetric $n \times n$ matrix.

3. Obtain the matrix

$$\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

as the product of three such matrices.

4. Obtain the matrix

$$\begin{pmatrix} I & P \\ 0 & I \end{pmatrix}$$

by conjugating

$$\begin{pmatrix} I & 0 \\ -P & 0 \end{pmatrix}$$

by the cube of the preceding matrix.

5. If P is symmetric and non-singular, show how to obtain

$$\begin{pmatrix} 0 & P^{-1} \\ p & 0 \end{pmatrix}$$

and then

$$\begin{pmatrix} P & 0 \\ 0 & P^{-1} \end{pmatrix}$$

as products of the matrices already constructed.

We now need a little lemma which says that *any non-singular $n \times n$ matrix A can be written as the product of three symmetric matrices.*

6. Prove this directly for the case of a 2×2 matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

by first showing that A can be written as the product of a diagonal matrix and a symmetric matrix if $bc \neq 0$ and then show that if $bc = 0$, so $ad \neq 0$ multiplying by

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

reduces to the previous case.

7. Recall from linear algebra that any non-singular $n \times n$ matrix can be written as $A = PO$ where P is symmetric positive definite and O is orthogonal. So we must show that every orthogonal matrix can be written as the product of two symmetric matrices. Recall that any orthogonal matrix can be "block diagonalized", i.e can be written as $O = RAR^{-1}$ where R is orthogonal and A is block diagonal where the blocks are either two by two orthogonal matrices with non-zero off diagonal matrices or are one-by-one blocks. Conclude the lemma in general.

We have now obtained all matrices of the form

$$\begin{pmatrix} A & 0 \\ 0 & (A^t)^{-1} \end{pmatrix}$$

with A non-singular.

8. Show that we can obtain any symplectic matrix of the form

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

with A non-singular. [Hint: multiply on the left by

$$\begin{pmatrix} I & 0 \\ -E & I \end{pmatrix}$$

and choose E appropriately.]

We now complete the proof: If A is singular, we can (by row and column reduction) find invertible $n \times n$ matrices L and M such that

$$LAM = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$$

where I_r is the $r \times r$ identity matrix with r = rank of A. So pre- and post multiplying by

$$\begin{pmatrix} L & 0 \\ 0 & (L^t)^{-1} \end{pmatrix} \text{ and } \begin{pmatrix} M & 0 \\ 0 & (M^t)^{-1} \end{pmatrix}$$

we may assume that our matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is of the form

$$A = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}.$$

Write

$$C = \begin{pmatrix} C_1 & C_2 \\ C_3 & C_4 \end{pmatrix}$$

where C_1 is the upper left $r \times r$ block of C etc. Then

$$A^t C = \begin{pmatrix} C_1 & C_2 \\ 0 & 0 \end{pmatrix}.$$

9. Conclude from one of the conditions that T be symplectic that $C_2 = 0$ and C_1 is symmetric.

Multiply $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ on the left by

$$\begin{pmatrix} I & E \\ 0 & I \end{pmatrix}$$

where

$$E = \begin{pmatrix} 0 & 0 \\ 0 & I_{n-r} \end{pmatrix}$$

to obtain a matrix

$$\begin{pmatrix} A' & B' \\ C & D \end{pmatrix}$$

where

$$A' = \begin{pmatrix} I_r & 0 \\ C_3 & C_4 \end{pmatrix}.$$

10. Show that C_4 and hence A' is non-singular, completing the proof that the matrices

$$\begin{pmatrix} I & dI \\ 0 & I \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} I & 0 \\ -P & 0 \end{pmatrix}$$

generate $Sp(2n)$.

3 The metaplectic representation.

Whew! A lot of algebra! But we are now in position to generalize our construction of $Mp(2)$ to $Mp(2n)$. Just define the operators U_d and V_P on $L_2(\mathbf{R}^n$ by

$$V_P \text{ is multiplication by } \exp(-iPx \cdot x/2)$$

and

$$(U_d f)(x) = \exp(-\pi in/4) d^{-n/2} (2\pi)^{-n/2} \int \exp[i(x-y)\cdot(x-y)/2d] f(y) dy.$$

We take the product of these operators and prove just as before that there is a two to one map of the group generated by these operators onto the symplectic group. You are probably wiped out by now so I won't press the details, as I will give a much more abstract description of $Mp(2n)$ in terms of the Stone-von-Neumann theorem in class.