The Stone - von Neumann theorem.

Math 212

January 10, 2002

This theorem asserts the uniqueness of the "representation of the Heisenberg commutation relations in Weyl form". For a precise statement see Theorem 3.1 below. We will give two or three different proofs of this important theorem. The first proof will be in these notes.

In these notes we follow the exposition given in Lions and Vergne *The Weil representation ....* I will stick to the notation of Lions and Vergne (more or less) since transcription would probably cause more errors than could be justified by consistency in notation. In particular the 2π's in the Fourier transform are placed differently than in our conventions of last semester, and the notations for the representations of the p's and q's in Schrödinger form are opposite those standard in the physics literature!

This theorem was conjectured by Hermann Weyl soon after Heisenberg formulated his commutation relations, but was first proved independently by Stone and von Neumann in the early 1930's. It is more or less taken for granted in the physics texts and used as one of the foundation stones of quantum mechanics (although I will express some reservations about this usage later on). We will be interested in this theorem not only for its implications for physics, but also because it is equivalent to a basic theorem of Lax and Phillips in their scattering theory, and the Lax-Phillips situation is closely related to the notion of "multi-scale resolution" which is at the heart of wavelet theory.

The theorem is only true in the group theoretical form as stated below, not in the infinitesimal version which is how it is frequently stated in the physics texts. It has an "anti-symmetric" analogue - the uniqueness of the spin representation of the complex Clifford algebras. Both of these theorems are true only in finite dimensions. In infinite dimensions, the representation is not unique - it depends on the choice of "vacuum". In the anti-symmetric case, this was first realized by Dirac who predicted the existence of the positron as a "hole" in the "infinite sea" of negative energy electrons. This was one of the most amazing intellectual achievements of all time.

We will begin with some facts in linear symplectic geometry which we will also need later in our study of the wave front sets of generalized functions.

Contents

1 Elementary symplectic facts.

1.1 Symplectic vector spaces.

Let V be a (usually finite dimensional) vector space over the real numbers. A symplectic structure on V consists of an antisymmetric bilinear form

$$\omega : V \times V \to \mathbf{R}$$

which is non-degenerate. So we can think of ω as an element of $\wedge^2 V^*$ when V is finite dimensional, as we shall assume until further notice. A vector space equipped with a symplectic structure is called a symplectic vector space.

A basic example is $\mathbf{R}^2$ with

$$\omega_{\mathbf{R}^2}\left(\begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} c \\ d \end{pmatrix}\right) = \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc.$$

We will call this the standard symplectic structure on $\mathbf{R}^2$.

1.2 Special kinds of subspaces.

If W is a subspace of symplectic vector space V then $W^\perp$ denotes the symplectic orthocomplement of W:

$$W^\perp := \{v \in V |\ \omega(v, w) = 0,\ \forall w \in W\}.$$

A subspace is called

1. **symplectic** if $W \cap W^\perp = \{0\}$,
2. **isotropic** if $W \subset W^\perp$,
3. **coisotropic** if $W^\perp \subset W$, and
4. **Lagrangian** if $W = W^\perp$.

Since $(W^\perp)^\perp = W$ by the non-degeneracy of ω it follows that W is symplectic if and only if $W^\perp$ is. Also, the restriction of ω to any symplectic subspace W is non-degenerate, making W into a symplectic vector space. Conversely, to say that the restriction of ω to W is non-degenerate means precisely that $W \cap W^\perp = \{0\}$.

1.3 Normal forms.

For any non-zero $e \in V$ we can find an $f \in V$ such that $\omega(e, f) = 1$ and so the subspace W spanned by e and f is a two dimensional symplectic subspace. Furthermore the map

$$e \mapsto \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad f \mapsto \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

gives a symplectic isomorphism of W with $\mathbf{R}^2$ with its standard symplectic structure. We can apply this same construction to $W^\perp$ if $W^\perp \neq 0$. Hence by induction, we can decompose any symplectic vector space into a direct sum of two dimensional symplectic subspaces:

$$V = W_1 \oplus \cdots W_d$$

where $\dim V = 2d$ (proving that every symplectic vector space is even dimensional) and where the W_i are pairwise (symplectically) orthogonal and where each W_i is spanned by e_i, f_i with $\omega(e_i, f_i) = 1$. In particular this shows that all $2d$ dimensional symplectic vector spaces are isomorphic, and isomorphic to a direct sum of d copies of $\mathbf{R}^2$ with its standard symplectic structure.

1.4 Existence of Lagrangian subspaces.

Let us collect the $e_1, \ldots, e_d$ in the above construction and let L be the subspace they span. It is clearly isotropic. Also, $e_1, \ldots, e_n, f_1, \ldots, f_d$ form a basis of V. If $v \in V$ has the expansion

$$v = a_1 e_1 + \cdots a_d e_d + b_1 f_1 + \cdots + b_d f_d$$

in terms of this basis, then $\omega(e_i, v) = b_i$. So $v \in L^\perp \Rightarrow v \in L$. Thus L is Lagrangian. So is the subspace M spanned by the f's.

Conversely, if L is a Lagrangian subspace of V and M is a complementary Lagrangian subspace. Then ω induces a non-degenerate linear pairing of L with M and hence any basis $e_1, \cdots e_d$ picks out a dual basis $f_1, \cdots .f_d$ of M giving a basis of the above form.

1.5 Consistent Hermitian structures.

In terms of the basis $e_1, \ldots, e_n, f_1, \ldots, f_d$ introduced above, consider the linear map

$$J: \quad e_i \mapsto -f_i, \quad f_i \mapsto e_i.$$

It satisfies

$$\begin{aligned} J^2 &= -I, & (1)\\ \omega(Ju, Jv) &= \omega(u, v), \quad \text{and} & (2)\\ \omega(Ju, v) &= \omega(Jv, u). & (3) \end{aligned}$$

Notice that any J which satisfies two of the three conditions above automatically satisfies the third. Condition (1) says that J makes V into a d-dimensional complex vector space. Condition (2) says that J is a symplectic transformation, i.e acts so as to preserve the symplectic form ω. Condition (3) says that $\omega(Ju, v)$ is a real symmetric bilinear form.

All three conditions (really any two out of the three) say that $(\ ,\ ) = (\ ,\ )_{\omega, J}$ defined by

$$(u, v) = \omega(Ju, v) + i\omega(u, v)$$

is a semi-Hermitian form whose imaginary part is ω. For the J chosen above this form is actually Hermitian, that is the real part of $(\ ,\ )$ is positive definite.

1.6 Choosing Lagrangian complements.

The results in this subsection all have to do with making choices in a "consistent" way, so as to guarantee, for example, that the choices can be made to be invariant under the action of a group.

For any a Lagrangian subspace $L \subset V$ we will need to be able to choose a complementary Lagrangian subspace L', and do so in a consistent manner, depending, perhaps, on some auxiliary data. Here is one such way, depending on the datum of a symmetric positive definite bilinear form B on V. (Here B has nothing to do with with the symplectic form.)

Let L^B be the orthogonal complement of L relative to the form B. So

$$\dim L^B = \dim L = \frac{1}{2} \dim V$$

and any subspace $W \subset V$ with

$$\dim W = \frac{1}{2} \dim V \quad \text{and } W \cap L = \{0\}$$

can be written as graph (A) where $A : L^B \to L$ is a linear map. That is, under the vector space identification

$$V = L^B \oplus L$$

the elements of W are all of the form

$$w + Aw, \quad w \in L^B.$$

We have

$$\omega(u + Au, w + Aw) = \omega(u, w) + \omega(Au, w) + \omega(u, Aw)$$

since $\omega(Au, Aw) = 0$ as L is Lagrangian. Let C be the bilinear form on L^B given by

$$C(u, w) := \omega(Au, w).$$

Thus W is Lagrangian if and only if

$$C(u, w) - C(w, u) = -\omega(u, w).$$

Now

$$\mathrm{Hom}(L^B, L) \sim L \otimes L^{B*} \sim L^{B*} \otimes L^{B*}$$

under the identification of L with L^{B*} given by ω. Thus the assignment $A \leftrightarrow C$ is a bijection, and hence the space of all Lagrangian subspaces complementary to L is in one to one correspondence with the space of all bilinear forms C on L^B which satisfy $C(u, w) - C(w, u) = -\omega(u, w)$ for all $u, w \in L^B$. An obvious choice is to take C to be $-\frac{1}{2}\omega$ restricted to L^B. In short,

Proposition 1.1 *Given a positive definite symmetric form on a symplectic vector space V, there is a consistent way of assigning a Lagrangian complement L' to every Lagrangian subspace L.*

Here the word consistent means that the choice depends only on B. This has the following implication: Suppose that T is a linear automorphism of V which preserves both the symplectic form ω and B. In other words, suppose that

$$\omega(Tu, Tv) = \omega(u, v) \quad \text{and} \quad B(Tu, Tv) = B(u, v) \quad \forall\, u, v \in V.$$

Then if $L \mapsto L'$ is the correspondence given by the proposition, then

$$TL \mapsto TL'.$$

More generally, if $T : V \to W$ is a symplectic isomorphism which is an isometry for a choice of positive definite symmetric bilinear forms on each, the above equation holds.

Given L and B (and hence L') we determined the complex structure J by

$$J : L \to L', \quad \omega(u, Jv) = B(u, v) \quad u, v \in L$$

and then

$$J := -J^{-1} : L' \to L$$

and extending by linearity to all of V so that

$$J^2 = -I.$$

Then for $u, v \in L$ we have

$$\omega(u, Jv) = B(u, v) = B(v, u) = \omega(v, Ju)$$

while

$$\omega(u, JJv) = -\omega(u, v) = 0 = \omega(Jv, Ju)$$

and

$$\omega(Ju, JJv) = -\omega(Ju, v) = -\omega(Jv, u) = \omega(Jv, JJu)$$

so (3) holds for all $u, v \in V$. We should write $J_{B,L}$ for this complex structure, or J_L when B is understood

Suppose that T preserves ω and B as above. We claim that

$$J_{TL} \circ T = T \circ J_L$$

so that T is complex linear for the complex structures J_L and J_{TL}. Indeed, for $u, v \in L$ we have

$$\omega(Tu, J_{TL}Tv) = B(Tu, Tv)$$

by the definition of J_{TL}. Since B is invariant under T the right hand side equals $B(u, v) = \omega(u, J_L v) = \omega(Tu, TJ_L v)$ since ω is invariant under T. Thus

$$\omega(Tu, J_{TL}Tv) = \omega(Tu, TJ_L v)$$

showing that

$$TJ_L = J_{TL}T$$

when applied to elements of L. This also holds for elements of L'. Indeed every element of L' is of the form $J_L u$ where $u \in L$ and $TJ_L u \in TL'$ so

$$J_{TL}TJ_L u = -J_{TL}^{-1}TJ_L u = -Tu = TJ_L(J_L u).$$

QED

2 The Heisenberg algebra and group.

Let V be a symplectic vector space. So V comes equipped with a skew symmetric non-degenerate bilinear form ω. By the choice of a pair of transverse Lagrangian subspaces, and then dual bases in these subspaces, we obtain a basis

$$P_1, \ldots, P_n, Q_1, \ldots, Q_n$$

of V with

$$\begin{array}{rcl} \omega(P_i, P_j) & = & 0 \\ \omega(Q_i, Q_j) & = & 0 \\ \omega(P_i, Q_j) & = & \delta_{ij}. \end{array} \tag{4}$$

We make

$$\mathbf{h} := V \oplus R$$

into a Lie algebra by defining

$$[X, Y] := \omega(X, Y)E$$

where $E = 1 \in \mathbf{R}$ and

$$[E, E] = 0 = [E, X] \quad \forall X \in V.$$

The Lie algebra $\mathbf{h}$ is called the **Heisenberg algebra**. It is a nilpotent Lie algebra. In fact, the Lie bracket of any three elements is zero. If we write out the brackets in terms of the basis above we get

$$\begin{aligned} [P_i, Q_j] &= \delta_{ij} E \\ [P_i, P_j] &= 0 \\ [Q_i, Q_j] &= 0 \end{aligned}$$

which, together with

$$[E, P_j] = 0 = [E, Q_j]$$

are the "canonical commutation relations" up to inessential (or essential) factors such as $\hbar$ and i.

We will let N denote the simply connected Lie group with this Lie algebra. We may identify the $2n + 1$ dimensional vector space $V + \mathbf{R}$ with N via the exponential map, and with this identification the multiplication law on N reads

$$\exp(v + tE) \exp(v' + t'E) = \exp\left(v + v' + (t + t' + \frac{1}{2}\omega(v, v'))E\right). \tag{5}$$

Let dv be the Euclidean (Lebesgue) measure on V. Then the measure $dvdt$ is invariant under left and right multiplication. So the group N is unimodular. For those of you who are unfamiliar with the notion of the exponential map for Lie algebras and Lie groups, just start with (5) as a definition of multiplication, where exp is just a weird symbol.

If ℓ is a Lagrangian subspace of V, then $\ell \oplus \mathbf{R}$ is an Abelian subalgebra of $\mathbf{h}$, and in fact is maximal abelian. Similarly

$$L := \exp(\ell \oplus \mathbf{R})$$

is a maximal Abelian subgroup of N.

Define the function

$$f : N \to T^1$$
$$f(\exp(v + tE)) := e^{2\pi i t}.$$

We have

$$f\left((\exp(v + tE))(\exp(v' + t'E))\right) = e^{2\pi i (t + t' + \frac{1}{2}\omega(v, v'))}. \tag{6}$$

Therefore

$$f(h_1h_2) = f(h_1)f(h_2)$$

for

$$h_1, \quad h_2 \in L.$$

We say that the restriction of f to L is a **character** of L.

I want to consider the quotient space

$$N/L$$

which has a natural action of N (via left multiplication). In other words N/L is a homogeneous space for the Heisenberg group N. Let ℓ' be a Lagrangian subspace transverse to ℓ. Every element of N has a unique expression as

$$(\exp y)(\exp(x + sE)) \quad \text{where } y \in \ell' \quad x \in \ell.$$

This allows us to make the identification

$$N/L \sim \ell'$$

and the Euclidean measure dv' on ℓ' then becomes identified with the (unique up to scalar multiple) measure on N/L invariant under N.

For use in the next section we record the following "commutation calculation" at the group level: Let $y \in \ell'$ and $x \in \ell$. Then

$$\exp(-x)(\exp y) = \exp(y - x - \frac{1}{2}\omega(x, y)E)$$

while

$$\exp(y)\exp(-x) = \exp(y - x - \frac{1}{2}\omega(y, x)E)$$

so, since ω is antisymmetric, we get

$$(\exp(-x))(\exp y) = (\exp y)(\exp(-x))\exp(-\omega(x, y)E). \tag{7}$$

3 The Schrodinger representation.

We continue with the notation of the preceding section. In particular, we have chosen a Lagrangian subspace ℓ, have the corresponding subgroup L and the quotient space N/L. We are going to construct a unitary representation of N which is known in group theory language as the representation of N **induced** from the character f of L.

Its definition is as follows: Consider the space of continuous functions ϕ on N which satisfy

$$\phi(nh) = f(h)^{-1}\phi(n) \quad \forall\, n \in N \quad h \in L \tag{8}$$

and which in addition have the property that the function on N/L

$$n \mapsto |\phi(n)|$$

(which is well defined on N/L on account of (8)) is square integrable on N/L. We let $H(\ell)$ denote the Hilbert space which is the completion of this space of continuous functions relative to this L_2 norm. So $\phi \in H(\ell)$ is a "function" on N satisfying (8) with norm

$$\|\phi\|^2 = \int_{N/L} |\phi|^2 d\dot{n}$$

where $d\dot{n}$ is left invariant measure on N/L.

The representation ρ_ℓ of N on $H(\ell)$ is given by left translation:

$$(\rho_\ell(m)\phi)(n) := \phi\left(m^{-1}n\right). \tag{9}$$

For the rest of this section we will keep ℓ fixed, and so may write H for $H(\ell)$ and ρ for ρ_ℓ. The dependence on ℓ will become important for us later.

Since $\exp tE$ is in the center of N, we have

$$\rho(\exp tE)\phi(n) = \phi\left((\exp -tE)n\right) = \phi\left(n(\exp -tE)\right) = e^{2\pi i t}\phi(n).$$

In other words

$$\rho\left(\exp tE\right) = e^{2\pi i t}\mathrm{Id}_H. \tag{10}$$

The Stone - von Neumann theorem (Theorem 3.1 below) characterizes all unitary representations of N which satisfy this condition.

Suppose we choose a complementary Lagrangian subspace ℓ' and then identify N/L with ℓ' as in the preceding section. Condition (8) becomes

$$\phi\left((\exp y)(\exp(x))(\exp tE)\right) = \phi(\exp y)e^{-2\pi i t}.$$

So $\phi \in H$ is completely determined by its restriction to $\exp \ell'$. In other words the map

$$\phi \mapsto \psi, \quad \psi(y) := \phi(\exp y)$$

defines a unitary isomorphism

$$R : H \to L_2(\ell')$$

and if we set

$$\sigma := R\rho R^{-1}$$

then

$$\begin{array}{lcll} [\sigma\left(\exp x\right)\psi](y) & = & e^{2\pi i\omega(x,y)}\psi(y) & x \in \ell,\ y \in \ell' \\ [\sigma(\exp u)\psi](y) & = & \psi(y-u) & y, u \in \ell' \\ \sigma(\exp(tE)) & = & e^{2\pi i t}\mathrm{Id}_{L_2(\ell')}. & \end{array} \tag{11}$$

The first of these equations follows from (7) and the definition (9) and the last two follow immediately from (9).

We define the infinitesimal version of the representation ρ by

$$\dot{\rho}(X) := \frac{d}{dt}\rho(\exp(tX))_{|t=0}$$

for $X \in \mathbf{h}$ with a similar notion and notation for σ. Under the P, Q basis (with $P_i \in \ell$ chosen above), we may identify $L_2(\ell')$ with $L_2(\mathbf{R}^n)$. Then it follows from (11) that

$$\begin{array}{rcl} \dot{\sigma}(P_j) & = & 2\pi i x_j \\ \dot{\sigma}(Q_j) & = & -\frac{\partial}{\partial x_j} \\ \dot{\sigma}(E) & = & 2\pi i \mathrm{Id} \end{array} \qquad (12)$$

This is the Schrodinger version of the Heisenberg commutation relations. So we can regard (11) as an "integrated version" of the Heisenberg commutation relations. The Stone-von Neumann theorem asserts that the representation σ, and hence the representation ρ is irreducible and is the unique irreducible representation (up to isomorphism) satisfying (11). In fact, to be more precise, the theorem asserts that any unitary representation of N such that

$$\exp(tE) \mapsto e^{2\pi i t}\mathrm{Id}$$

must be isomorphic to a **multiple** of ρ in the following sense:

Let H_1 and H_2 be Hilbert spaces. We can form their tensor product as vector spaces, and this tensor product inherits a scalar product determined by

$$(u \otimes v, x \otimes y) = (u, x)(v, y).$$

The completion of this (algebraic) tensor product with respect to this scalar product will be denoted by $H_1 \otimes H_2$ and will be called the (Hilbert space) tensor product of H_1 and H_2. If we have a representation τ of a group G on H_1 we get a representation

$$g \mapsto \tau(g) \otimes \mathrm{Id}_{H_2}$$

on $H_1 \otimes H_2$ which we call a multiple of the representation τ. We can now state:

Theorem 3.1 [The Stone-von-Neumann theorem.] *The representation $\rho(\ell)$ of N is irreducible, and any representation such that* $\exp(tE) \mapsto e^{2\pi i t}\mathrm{Id}$ *is isomorphic to a multiple of $\rho(\ell)$.*

We will spend the next few sections proving this basic theorem.

4 The group algebra.

If G is a locally compact Hausdorff topological group with a given choice of Haar measure, we defined the convolution of two continuous functions of compact support on G by

$$(\phi_1 \star \phi_2)(g) := \int_G \phi_1(u)\phi_2(u^{-1}g)du.$$

If ψ is another continuous function on G we have

$$\int_G (\phi_1 \star \phi_2)(g)\psi(g)dg = \int_{G\times G} \phi_1(u)\phi_2(h)\psi(uh)dudh.$$

This right hand side makes sense if ϕ_1 and ϕ_2 are distributions of compact support and ψ is smooth. Also the left hand side makes sense if ϕ_1 and ϕ_2 belong to $L_1(G)$ and ψ is bounded, etc.

If we have a continuous unitary representation τ of G on a Hilbert space H, we can define

$$\tau(\phi) := \int_G \phi(g)\tau(g)dg$$

which means that for u and $v \in H$

$$(\tau(\phi)u, v) = \int_G \phi(g)(\tau(g)u, v)dg.$$

This integral makes sense if ϕ is continuous and of compact support, or if G is a Lie group, if u is a C^∞ vector in the sense that $\tau(g)u$ is a C^∞ function of g and ϕ is a distribution. In either case we have

$$\tau(\phi_1 \star \phi_2) = \tau(\phi_1)\tau(\phi_2).$$

If the left invariant measure is also invariant under the map $g \mapsto g^{-1}$ and so right invariant, and if we define

$$\phi^*(g) := \overline{\phi(g^{-1})}$$

then

$$\tau(\phi^*) = \tau(\phi)^*.$$

We gave a more general definition of ϕ^* valid in the non-unimodular case in the notes on Haar measure. The preceding formula holds in general, but we will stick to the unimodular case at hand.

5 The Weyl transform.

Let τ be a representation of N satisfying our condition

$$\tau(tE) = e^{2\pi it}\mathrm{Id}.$$

Then τ descends to a representation of

$$B := N/\exp(\mathbf{Z}E)$$

since $\tau(\exp(kE)) = \mathrm{Id}$ for $k \in \mathbf{Z}$.

Let Φ denote the collection of continuous functions on N which satisfy

$$\phi(n \exp tE) = e^{-2\pi i t}\phi(n).$$

Every $\phi \in \Phi$ can be considered as a function on B, and every $n \in B$ has a unique expression as $n = (\exp v)(\exp tE)$ with $v \in V$ and $t \in \mathbf{R}/\mathbf{Z}$. We take as our left invariant measure on B the measure $dvdt$ where dv is Lebesgue measure on V and dt is the invariant measure on the circle with total measure one. The set of elements of Φ are then determined by their restriction to $\exp(V)$. Then for $\phi_1, \phi_2 \in \Phi$ of compact support (as functions on B) we have (with $\star$ denoting convolution on B)

$$(\phi_1 \star \phi_2)(\exp v)$$

$$\begin{aligned}
&= \int_V \int_T \phi_1((\exp u)(\exp tE))\phi_2((-\exp u)(\exp(-tE))(\exp v))dudt \\
&= \int_V \phi_1(\exp u)\phi_2((\exp -u)(\exp v))du \\
&= \int_V \phi_1(\exp u)\phi_2(\exp(v-u)\exp(-\frac{1}{2}\omega(u,v)E))du \\
&= \int_V \phi_1(\exp u)\phi_2(\exp(v-u))e^{\pi i \omega(u,v)}du.
\end{aligned}$$

So if we use the notation

$$\psi(u) = \phi(\exp u)$$

and $\psi_1 \star \psi_2$ for the ψ corresponding to $\phi_1 \star \phi_2$ we have

$$(\psi_1 \star \psi_2)(v) = \int_V \psi_1(u)\psi_2(v-u)e^{\pi i \omega(u,v)}du. \tag{13}$$

We thus get a "twisted" convolution on V.

If $\phi \in \Phi$ and if we define ϕ^* as above, then $\phi^* \in \Phi$ and the corresponding transformation on the ψ's is

$$\phi^*(\exp v) = \overline{\psi(-v)}.$$

We now define

$$W_\tau(\psi) = \tau(\phi) = \int_B \phi(b)\tau(b)db = \int_V \psi(v)\tau(\exp v)dv.$$

The last equation holds because of the opposite transformation properties of τ and $\phi \in \Phi$.

If $\phi \in \Phi$ then $\delta_m \star \phi$ is given by

$$(\delta_m \star \phi)(n) = \phi(m^{-1}n)$$

which belongs to Φ if ϕ does and if $m = \exp(w)$ then

$$(\delta_m \star \phi)(\exp u) = e^{\pi i \omega(w,u)}\psi(u - w).$$

Similarly,

$$(\phi \star \delta_m)(\exp u) = e^{-\pi i \omega(w,u)}\psi(u - w).$$

Let us write $w \star \psi$ for the function on V corresponding to $\delta_m \star \phi$ under our correspondence between elements of Φ and functions on V.

Then the facts that we have proved such as

$$\tau(\phi_1 \star \phi_2) = \tau(\phi_1)\tau(\phi_2)$$

translate into

$$\begin{aligned} W_\tau(\psi_1 \star \psi_2) &= W_\tau(\psi_1)W_\tau(\psi_2) \quad (14)\\ W_\tau(\psi^*) &= W_\tau(\psi)^* \quad (15)\\ W_\tau(w \star \psi) &= \tau(\exp w)W_\tau(\psi) \quad (16)\\ W_\tau(\psi \star w) &= W_\tau(\psi)\tau(\exp w). \quad (17) \end{aligned}$$

6 Hilbert-Schmidt Operators.

Let H be a separable Hilbert space. An operator A on H is called **Hilbert-Schmidt** if in terms of some orthonormal basis $\{e_i\}$ we have

$$\sum \|Ae_i\|^2 < \infty.$$

Since

$$Ae_i = \sum (Ae_i, e_j)e_j$$

this is the same as the condition

$$\sum_{ij} |(Ae_i, e_j)|^2 < \infty$$

or

$$\sum |a_{ij}|^2 < \infty$$

where

$$a_{ij} := (Ae_i, e_j)$$

is the matrix of A relative to the orthonormal basis. This condition and sum does not depend on the orthonormal basis and is denoted by

$$\|A\|_{HS}^2.$$

This norm comes from the scalar product

$$(A, B)_{HS} = \operatorname{tr} B^*A = \sum (B^*Ae_i, e_i) = \sum (Ae_i, Be_i).$$

Indeed,

$$\begin{aligned}
(A^*Ae_i, e_i) &= (Ae_i, Ae_i) \\
&= \left(\sum_j (Ae_i, e_j)e_j, Ae_i \right) \\
&= \sum_j (Ae_i, e_j)(e_j, Ae_i) \\
&= \sum_j a_{ij}\overline{a_{ij}} \\
&= \sum_j |a_{ij}|^2,
\end{aligned}$$

and summing over i gives $\|A\|^2_{HS}$.

The rank one elements

$$E_{ij}, \qquad E_{ij}(x) := (x, e_j)e_i$$

form an orthonormal basis of the space of Hilbert-Schmidt operators. We can identify the space of Hilbert-Schmidt operators with the tensor product $H \otimes \overline{H}$ where $\overline{H}$ is the space H with scalar multiplication and product given by the complex conjugate, e.g multiplication by $c \in \mathbf{C}$ is given by multiplication by $\overline{c}$ in H.

If $H = L_2(V, dy)$ (where V can be any measure space with measure dy, but we will be interested in our case) we can describe the space of Hilbert-Schmidt operators as follows: Let $\{e_i\}$ be an orthonormal basis of $H = L_2(V)$ and consider the rank one operators E_{ij} introduced above. Then

$$(E_{ij}\psi)(x) = (\psi, e_j)e_i(x) = \int_V \psi(y)\overline{e_j(y)}e_i(x)dy$$

$$= \int_Y K_{ij}(x, y)\psi(y)dy$$

where

$$K_{ij}(x, y) = e_i(x)\overline{e_j(y)}.$$

This has norm one in $L_2(V \times V)$ and hence the most general Hilbert-Schmidt operator A is given by the $L_2(V \times V)$ kernel

$$K = \sum a_{ij}K_{ij}$$

with a_{ij} the matrix of A as above.

7 Proof of the irreducibility of $\rho(\ell)$.

Let us consider the case where $\tau = \rho = \rho(\ell)$. I claim that the map W_ρ defined on the elements of Φ of compact support extends to an isomorphism from $L_2(V)$ to the space of Hilbert-Schmidt operators on $H(\ell)$. Indeed, write

$$W_\rho(\psi) = \int_V \psi(v)\rho(\exp v)dV$$

and decompose

$$V = \ell \oplus \ell'$$

$$v = y + x, \quad s \in \ell, \; y \in \ell'$$

so

$$\exp(y + x) = \exp(y)\exp(x)\exp(-\frac{1}{2}\omega(y, x))$$

so

$$\rho(\exp(y + x)) = \rho(y)\rho(x)e^{-i\pi\omega(y,x)}$$

and hence

$$W_\rho(\psi) = \int\int \psi(y + x)\rho(\exp y)\rho(\exp x)e^{-\pi i\omega(y,x)}dxdy.$$

So far the above would be true for any τ, not necessarily ρ. Now let us use the explicit realization of ρ as σ on $L_2(\mathbf{R}^n)$ in the form given in (11).

We obtain

$$[W_\sigma(\psi)(f)](\xi) = \int\int e^{-\pi i\omega(y,x)}\psi(y + x)e^{2\pi i\omega(x,\xi-y)}f(\xi - y)dxdy.$$

Making the change of variables $y \mapsto \xi - y$ this becomes

$$\int\int e^{-\pi i\omega(\xi-y,x)}e^{2\pi i\omega(x,y)}\psi(\xi - y + x)f(y)dy.$$

so if we define

$$K_\psi(\xi, y) := \int e^{\pi i\omega(x,y+\xi)}\psi(\xi - y + x)dx$$

we have

$$[W_\sigma(\psi)f)](\xi) = \int K_\psi(\xi, y)f(y)dy.$$

Here we have identified ℓ' with $\mathbf{R}^n$ and $V = \ell' + \ell$ where ℓ is the dual space of ℓ' under ω. So if we consider the partial Fourier transform

$$\mathcal{F}_x : L_2(\ell' \oplus \ell) \to L_2(\ell' \oplus \ell')$$

$$(\mathcal{F}_x\psi)(y, \xi) = \int e^{-2\pi i\omega(x,\xi)}\psi(y + x)dx$$

(which is an isomorphism) we have

$$K_\psi(\xi, y) = (\mathcal{F}_x\psi)(\xi - y, -\frac{1}{2}(y+\xi)).$$

We thus see that the set of all K_ϕ is the set of all Hilbert-Schmidt operators on $L_2(\mathbf{R}^n)$.

Now if a bounded operator C commutes with all Hilbert-Schmidt operators on a Hilbert space, then $CE_{ij} = E_{ij}C$ implies that $c_{ij} = c\delta_{ij}$, i.e. $C = c\mathrm{Id}$. So we have proved that every bounded operator that commutes with all the $\rho_\ell(n)$ must be a constant. Thus $\rho(\ell)$ is irreducible.

8 Completion of the proof.

We fix ℓ, ℓ' as above, and have the representation ρ realized as σ on $L_2(\ell')$ which is identified with $L_2(\mathbf{R}^n)$ all as above. We want to prove that any representation τ satisfying (10) is isomorphic to a multiple of σ.

We consider the "twisted convolution" (13) on the space of Schwartz functions $\mathcal{S}(\mathcal{V})$. If $\psi \in \mathcal{S}(V)$ then its Weyl kernel $K_\psi(\xi, y)$ is a rapidly decreasing function of (ξ, y) and we get all operators with rapidly decreasing kernels as such images of the Weyl transform W_σ sending ψ into the kernel giving $\sigma(\phi)$.

Consider some function $u \in \mathcal{S}(\ell')$ with

$$\|u\|_{L_2(\ell')} = 1.$$

Let P_1 be the projection onto the line through u, so P_1 is given by the kernel

$$p_1(x, y) = \overline{u(y)}u(x).$$

We know that it is given as

$$p_1 = W_\sigma(\psi) \quad \text{for some} \quad \psi \in \mathcal{S}(V).$$

We have $P_1^2 = P_1, P_1^* = P_1$ and

$$P_1\sigma(n)P_1 = \alpha(n)P_1 \quad \text{with} \quad \alpha(n) = (\sigma(n)u, u).$$

Recall that $\phi \mapsto \sigma(\phi)$ takes convolution into multiplication, and that K_ψ is the kernel giving the operator $W_\sigma(\psi) = \sigma(\phi)$ where $\phi \in \Phi$ corresponds to $\psi \in \mathcal{S}(V)$. Then in terms of our twisted convolution $\star$ given by (13) the above three equations involving P_1 get translated into

$$\psi \star \psi = \psi, \quad \psi^* = \psi, \quad \psi \star n \star \psi = \alpha(n)\psi, \tag{18}$$

Now let τ be any unitary representation of N on a Hilbert space H satisfying (10). We can form $W_\tau(\psi)$.

Lemma 8.1 *The set of linear combinations of the elements*

$$\tau(n)W_\tau(\psi)x, \quad x \in H, \quad n \in N$$

is dense in H.

Proof. Suppose that $y \in H$ is orthogonal to all such elements and set $n = \exp w$. Then for any $x \in H$

$$0 = (y, \tau(n)W_\tau(\psi)\tau(n)^{-1}x) = \int_V (y, \tau(\exp w)\tau(\exp(v)\tau(\exp(-w)\psi(v)dv$$

$$= \int_V (y, \tau(\exp(v + \omega(w, v)E)x)\psi(v)dv = \int_V (y, \tau(\exp v)x)e^{-2\pi i\omega(w,v)}\psi(v)dv$$

$$= \mathcal{F}[(y, \tau(\exp v)x)\psi].$$

The function is square brackets whose Fourier transform is being taken is continuous and rapidly vanishing. Indeed, x and y are fixed elements of H and τ is unitary, the expression $(y, \tau(\exp v)x)$ is bounded by $\|y\|\|x\|$ and is continuous, and ψ is a rapidly decreasing functions of v. Since the Fourier transform of the function

$$v \mapsto (y, \tau(\exp(v))x)\psi(v)$$

vanishes, the function itself must vanish. Since ψ does not vanish everywhere, there is some value v_0 with $\psi(v_0) \neq 0$, and hence

$$(y, \tau(\exp v_0)x) = 0 \quad \forall x \in H.$$

Writing $x = \tau(\exp v_0)^{-1}z$ we see that y is orthogonal to all of H and hence $y = 0$. QED

Now from the first two equations in (18) we see that $W_\tau(\psi)$ is an orthogonal projection onto a subspace, call it H_1 of H. We are going to show that H is isomorphic to $H(\ell) \otimes H_1$ as a Hilbert space and as a representation of N.

We wish to define

$$I : H(\ell) \otimes H_1 \to H, \quad \rho(n)u \otimes b \mapsto \tau(n)b$$

where $b \in H_1$.

We first check that if

$$b_1 = W_\tau(\psi)x_1 \quad \text{and} \quad b_2 = W_\tau(\psi)x_2$$

then for any $n_1, n_2 \in N$ we have

$$(\tau(n_1)W_\tau(\psi)x_1, \tau(n_2)W_\tau(\psi)x_2)_H = (\rho(n_1)u, \rho(n_2)u)_{H(\ell)} \cdot (b_1, b_2)_{H_1}. \qquad (19)$$

Proof. Since $\tau(n)$ is unitary and $W_\tau(\psi)$ is self-adjoint, we can write the left hand side of (19) as

$$(\tau(n_1)W_\tau(\psi)x_1, \tau(n_2)W_\tau(\psi)x_2)_H = (W_\tau(\psi)\tau(n_2^{-1}n_1)W_\tau(\psi)x_1, x_2)_H$$

and by the last equation in (18) this equals

$$= \alpha(n_2^{-1}n_1)(W_\tau(\psi)x_1, x_2)_H.$$

From the definition of α we have

$$\alpha(n_2^{-1}n_1) = (\rho(n_2^{-1}n_1)u, u)_{H(\ell)} = (\rho(n_1)u, \rho(n_2)u)_\ell$$

since $\rho(n_2)$ is unitary. This is the first factor on the right hand side of (19).

Since $W_\tau(\psi)$ is a projection we have

$$(W_\tau(\psi)x_1, x_2)_H = (W_\tau(\psi)x_1, W_\tau(\psi)x_2)_H = (b_1, b_2)_{H_1},$$

which is the second factor on the right hand side of (19). We have thus proved (19).

Now define

$$I: \sum_{i=1}^{N} \rho(n_i)u \otimes b_i \mapsto \sum \tau(n_i)b_i.$$

This map is well defined, for if

$$\sum_{i=1}^{N} \rho(n_i)u \otimes b_i = 0$$

then

$$\|\sum_{i=1}^{N} \rho(n_i)u \otimes b_i\|_{H(\ell)\otimes H_1} = 0$$

and (19) then implies that

$$\|\sum_{i=1}^{N} \rho(n_i)u \otimes b_i\|_{H(\ell)\otimes H_1} = \|\sum_{i=1}^{N} \tau(n_i)b_i\|_H = 0.$$

Equation (19) also implies that the map I is an isometry where defined. Since ρ is irreducible, the elements $\sum_{i=1}^{N} \rho(n_i)u$ are dense in $H(\ell)$, and so I extends to an isometry from $H(\ell) \otimes H_1$ to H. By Lemma 8.1 this map is surjective. Hence I extends to a unitary isomorphism (which clearly is also a morphism of N modules) between $H(\ell) \otimes H_1$ and H. This completes the proof of the Stone - von Neumann Theorem.