Strongly contractive semi-groups and scattering.

Math 212b

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Throughout this lecture $\mathbf{K}$ will denote a Hilbert space and $t \mapsto S(t), t \geq 0$ a strongly continuous semi-group of contractions defined on $\mathbf{K}$ which tends strongly to 0 as $t \to \infty$ in the sense that

$$\lim_{t\to\infty} \|S(t)k\| = 0 \quad \text{for each} \quad k \in \mathbf{K}. \tag{1}$$

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1 Examples.

1.1 Translation - truncation.

Let $\mathbf{N}$ be some Hilbert space and consider the Hilbert space

$$L_2(\mathbf{R}, \mathbf{N}).$$

Let T_t denote the one parameter unitary group of right translations:

$$[T_t f](x) = f(x - t)$$

and let P denote the operator of multiplication by $\mathbf{1}_{(-\infty,0]}$ so P is projection onto the subspace $\mathbf{G}$ consisting of the f which are supported on $(-\infty, 0]$. We claim that

$$t \mapsto PT_t$$

is a semi-group acting on $\mathbf{G}$ satisfying our condition (1):

The operator PT_t is a strongly continuous contraction since it is unitary operator on $L_2(\mathbf{R}, \mathbf{N})$ followed by a projection.

Also

$$\|PT_t f\|^2 = \int_{-\infty}^{-t} |f(x)|^2 dx$$

tends strongly to zero.

We must check the semi-group property. Clearly PT_0 =Id on $\mathbf{G}$. We have

$$PT_s PT_t f = PT_s[T_t f + g] = PT_{s+t} f + PT_s g$$

where

$$g = PT_t f - T_t f$$

so

$$g \perp \mathbf{G}.$$

But

$$T_{-s} : \ \mathbf{G} \to \mathbf{G}$$

for $s \geq 0$. Hence $g \perp \mathbf{G} \Rightarrow T_s g = T^*_{-s} g \perp \mathbf{G}$ since

$$(T^*_{-s} g, \psi) = (g, T_{-s}\psi) = 0 \quad \forall\, \psi \in \mathbf{G}.$$

1.2 Incoming representations.

The last argument is quite formal. We can axiomatize it as follows: Let $\mathbf{H}$ be a Hilbert space, and $t \mapsto U(t)$ a strongly continuous group one parameter group of unitary operators on $\mathbf{H}$. A closed subspace $\mathbf{D} \subset \mathbf{H}$ is called **incoming** with respect to U if

$$U(t)\mathbf{D} \subset \mathbf{D} \quad \text{for } t \leq 0 \tag{2}$$

$$\bigcap_t U(t)\mathbf{D} = 0 \tag{3}$$

$$\overline{\bigcup_t U(t)\mathbf{D}} = \mathbf{H}. \tag{4}$$

Let $P_{\mathbf{D}} : \mathbf{H} \to \mathbf{D}$ denote orthogonal projection. The preceding argument goes over unchanged to show that S defined by

$$S(t) := P_{\mathbf{D}} U(t)$$

is a strongly continuous semi-group. We repeat the argument: The operator $S(t)$ is clearly bounded and depends strongly continuously on t. For s and $t \geq 0$ we have

$$P_{\mathbf{D}} U(s) P_{\mathbf{D}} U(t) f = P_{\mathbf{D}} U(s)[U(t)f + g] = P_{\mathbf{D}} U(t+s) f + P_{\mathbf{D}} g$$

where

$$g := P_{\mathbf{D}}U(t)f - U(t)f \in \mathbf{D}^{\perp}.$$

But $g \in \mathbf{D}^{\perp} \Rightarrow U(s)g \in |bfD^{\perp}$ for $s \geq 0$ since $U(s)g = U(-s)^{*}g$ and

$$(U(-s)^{*g,\psi)} = (g, U(-s)\psi) = 0 \quad \forall\ \psi \in \mathbf{D}$$

by (2).

We must prove that it converges strongly to zero as $t \to \infty$, i.e. that (1) holds. First observe that (2) implies that

$$U(-s)\mathbf{D} \supset U(-t)\mathbf{D} \quad \text{if} \ \ s < t$$

and since $U(-s)\mathbf{D}^{\perp} = [U(-s)\mathbf{D}]^{\perp}$ we get

$$U(-s)\mathbf{D}^{\perp} \subset U(-t)\mathbf{D}^{\perp}.$$

We claim that

$$\bigcup_{t>0} U(-t)\mathbf{D}^{\perp}$$

is dense in $\mathbf{H}$. If not, there is an $0 \neq h \in \mathbf{H}$ such that $h \in [U(-t)\mathbf{D}^{\perp}]^{\perp}$ for all $t > 0$ which says that $U(t)h \perp \mathbf{D}^{\perp}$ for all $t > 0$ or $U(t)h \in \mathbf{D}$ for all $t > 0$ or

$$h \in U(-t)\mathbf{D} \quad \text{for all } t > 0$$

contradicting (3). Therefore, if $f \in \mathbf{D}$ and $\epsilon > 0$ we can find $g \perp \mathbf{D}$ and an $s > 0$ so that

$$\|f - U(-s)g\| < \epsilon$$

or

$$\|U(s)f - g\| < \epsilon.$$

Since $g \perp \mathbf{D}$ we have $P_{\mathbf{D}}[U(s)f - g] = P_{\mathbf{D}}U(s)f$ and hence

$$\|P_{\mathbf{D}}U(s)f\| < \epsilon,$$

proving that $P_{\mathbf{D}}U(s)$ tends strongly to zero.

Comments about the axioms. Conditions (2)-(4) arise in several seemingly unrelated areas of mathematics.

- In scattering theory - either classical where the governing equation is the wave equation, or quantum mechanical where the governing equation is the Schrödinger equation - one can imagine a situation where and “obstacle” or a “potential” is limited in space, and that for any solution of the evolution equation, very little energy remains in the regions near the obstacle as $t \to -\infty$ or as $t \to +\infty$. In other words, the obstacle (or potential) has very little influence on the solution of the equation when $|t|$ is large. We can therefore imagine that for $t << 0$ the solution behaves as if it were a solution of the "free" equation, one with no obstacle or potential present. Thus the meaning of the space $\mathbf{D}$ in this case is that it represents the subspace of the space of solutions which have not yet had a chance to interact with the obstacle. The meaning of the conditions should be fairly obvious,

- In data transmission: we are all familiar with the way an image comes in over the internet; first blurry and then with an increasing level of detail. In wavelet theory we will encounter the concept of "multiresolution analysis", where the operators U provide an increasing level of detail.

- We can allow for the possibility of more general concepts of "information", for example in martingale theory where the spaces $U(t)\mathbf{D}$ represent the space of random variables available based on knowledge at time $\leq t$.

In the second example, it is more natural to allow t to range over the integers, rather than over the real numbers. But in this lecture we will deal with the continuous case rather than the discrete case. In the third example, we might want to dispense with $U(t)$ altogether, and just deal with an increasing family of subspaces.

1.3 Scattering residue.

In the scattering theory example, we want to believe that at large future times the "obstacle" has little effect and so there should be both an "incoming space" describing the situation long before the interaction with the obstacle, and also an "outgoing space" reflecting behavior long after the interaction with the obstacle. The residual behavior, i.e. the effect of the obstacle is what is of interest. For example, in elementary particle physics, this might be observed as a blip in the scattering cross-section describing a particle of a very short life-time. See the very elementary discussion of the blip arising in the Breit-Wigner formula below.

So let $t \mapsto U(t)$ be a strongly continuous one parameter unitary group on a Hilbert space $\mathbf{H}$, let $\mathbf{D}_-$ be an incoming subspace for U and let $\mathbf{D}_+$ be an outgoing subspace (i.e. incoming for $t \mapsto U(-t)$). Suppose that

$$\mathbf{D}_- \perp \mathbf{D}_+$$

and let

$$\mathbf{K} := [\mathbf{D}_- \oplus \mathbf{D}_+]^\perp = \mathbf{D}_-^\perp \cap \mathbf{D}_+^\perp.$$

Let

$$P_\pm := \quad \text{orthogonal projection onto } \mathbf{D}_\pm^\perp.$$

Let

$$Z(t) := P_+ U(t) P_-, \qquad t \geq 0.$$

Claim:

$$Z(t) : \mathbf{K} \to \mathbf{K}.$$

Proof. Since P_+ occurs as the leftmost factor in the definition of Z, the image of $Z(t)$ is contained in $\mathbf{D}_+^\perp$. We must show that

$$x \in \mathbf{D}_-^\perp \to P_+ U(t) x \in \mathbf{D}_-^\perp$$

since $Z(t)x = P_+U(t)x$ as $P_- x = x$ if $x \in \mathbf{D}_-^\perp$. Now $U(-t) : \mathbf{D}_- \to \mathbf{D}_-$ for $t \geq 0$ is one of the conditions for incoming, and so

$$U(t) : \mathbf{D}_-^\perp \to \mathbf{D}_-^\perp.$$

So

$$U(t)x \in \mathbf{D}_-^\perp.$$

Since $\mathbf{D}_- \subset \mathbf{D}_+^\perp$ the projection P_+ is the identity on $\mathbf{D}_-$, in particular

$$P_+ : \mathbf{D}_- \to \mathbf{D}_-$$

and hence, since P_+ is self-adjoint,

$$P_+ : \mathbf{D}_-^\perp \mapsto \mathbf{D}_-^\perp.$$

Thus $P_+U(t)x \in \mathbf{D}_-^\perp$ as required. QED

By abuse of language, we will now use $Z(t)$ to denote the restriction of $Z(t)$ to $\mathbf{K}$. We claim that $t \mapsto Z(t)$ is a semi-group. Indeed, we have

$$P_+U(t)P_+x = P_+U(t)x + P_+U(t)[P_+x - x] = P_+U(t)x$$

since $[P_+x - x] \in \mathbf{D}_+$ and $U(t) : \mathbf{D}_+ \to \mathbf{D}_+$ for $t \geq 0$. Also $Z(t) = P_+U(t)$ on $\mathbf{K}$ since P_- is the identity on $\mathbf{K}$. Therefore we may drop the P_- on the right when restricting to $\mathbf{K}$ and we have

$$Z(s)Z(t) = P_+U(s)P_+U(t) = P_+U(s)U(t) = P_+U(s+t) = Z(s+t)$$

proving that Z is a semigroup.

We now show that Z is strongly contracting. For any $x \in \mathbf{H}$ and any $\epsilon > 0$ we can find a $T > 0$ and a $y \in \mathbf{D}_+$ such that

$$\|x - U(-T)y\| < \epsilon$$

since $\bigcup_{t<0} U(t)\mathbf{D}_+$ is dense. For $x \in \mathbf{K}$ we get

$$\|Z(t)x - P_+U(t)U(-T)y\| = \|P_+U(t)[x - U(-T)y]\| < \epsilon.$$

But for $t > T$

$$U(t)U(-T)y = U(t-T)y \in \mathbf{D}_+ \quad \text{so } P_+U(t)U(-T)y = 0$$

and hence

$$\|Z(t)x\| < \epsilon.$$

We have proved that Z is a strongly contractive semi-group on $\mathbf{K}$ which tends strongly to zero, i.e. that(1) holds.

2 Breit-Wigner.

The example in this section will be of primary importance to us in computations and will also motivate the Lax-Phillips representation theorem to be stated and proved in the next section.

Suppose that $\mathbf{K}$ is one dimensional, and that

$$Z(t)d = e^{-\mu t}d$$

for $d \in \mathbf{K}$ where

$$\operatorname{Re}\mu > 0.$$

This is obviously a strongly contractive semi-group in our sense. Consider the space $L_2(\mathbf{R}, \mathbf{N})$ where $\mathbf{N}$ is a copy of $\mathbf{K}$ but with the scalar product whose norm is

$$||d||_N^2 = 2\operatorname{Re}\mu ||d||^2.$$

Let

$$f_d(t) = \begin{cases} e^{\mu t}d & t \leq 0 \\ 0 & t > 0. \end{cases} .$$

Then

$$||f_d||^2 = \int_{-\infty}^{0} e^{2\operatorname{Re}\mu t}(2\operatorname{Re}\mu)||d||^2 dt = ||d||^2$$

so the map

$$R : \mathbf{K} \to L_2(\mathbf{R}, \mathbf{N}) \quad d \mapsto f_d$$

is an isometry. Also

$$P(T_t f_d)(s) = Pf_d(s - t) = e^{-\mu t} f_d(s)$$

so

$$(PT_t) \circ R = R \circ Z(t).$$

This is an example of the representation theorem in the next section.

If we take the Fourier transform of f_d we obtain the function

$$\sigma \mapsto \frac{1}{\sqrt{2\pi}} \frac{1}{\mu - i\sigma} d$$

whose norm as a function of σ is proportional to the Breit-Wigner function

$$\frac{1}{\mu^2 + \sigma^2}.$$

It is this "bump" appearing in graph of a scattering experiment which signifies a "resonance", i.e. an "unstable particle" whose lifetime is inversely proportional to the width of the bump.

3 The representation theorem for strongly contractive semi-groups.

Let $t \mapsto S(t)$ be a strongly contractive semi-group on a Hilbert space $\mathbf{K}$. We want to prove that the pair $\mathbf{K}, S$ is isomorphic to a restriction of Example 1.

Theorem 1 **[Lax-Phillips.]** *There exists a Hilbert space* $\mathbf{N}$ *and an isometric map* R *of* $\mathbf{K}$ *onto a subspace of* $PL_2(\mathbf{R}, \mathbf{N})$ *such that*

$$S(t) = R^{-1} P T_t R$$

for all $t \geq 0$.

Proof. Let B be the infinitesimal generator of S, and let $D(B)$ denote the domain of B. The sesquilinear form $f, g \mapsto$

$$-(Bf, g) - (f, Bg)$$

is non-negative definite since B satisfies

$$\text{Re } (Bf, f) \leq 0.$$

Dividing out by the null vectors and completing gives us a Hilbert space $\mathbf{N}$ whose scalar product we will denote by $(\ ,\ )_{\mathbf{N}}$. If $k \in D(B)$ so is $S(t)k$ for every $t \geq 0$. Let us define

$$f_k(-t) = [S(t)k]$$

where $[S(t)k]$ denotes the element of $\mathbf{N}$ corresponding to $S(t)k$. For simplicity of notation we will drop the brackets and simply write

$$f_k(-t) = S(t)k$$

and think of f as a map from $(-\infty, 0]$ to $\mathbf{N}$. We have

$$\|f(-t)\|^2_{\mathbf{N}} = \|S(t)k\|^2_N = -2\text{Re } (BS(t)k, k)_N = -\frac{d}{dt}\|S(t)k\|^2.$$

Integrating this from 0 to r gives

$$\int_{-r}^{0} \|f(s)\|^2_N = \|k\|^2 - \|S(r)k\|^2.$$

By hypothesis, the second term on the right tends to zero as $r \to \infty$. This shows that the map

$$R : k \mapsto f_k$$

is an isometry of $D(B)$ into $L_2((-\infty, 0], \mathbf{N})$, and since $D(B)$ is dense in $\mathbf{K}$, we conclude that it extends to an isometry of $\mathbf{D}$ with a subspace of $PL_2(\mathbf{R}, \mathbf{N})$ (by extension by zero, say). Also

$$RS(t)k = f_{S(t)k}$$

is given by

$$f_{S(t)k}(s) = S(-s)S(t)k = S(t-s)k = S(-(s-t)k) = f_k(s-t)$$

for $s < 0$, and $t > 0$ so

$$RS(t)k = PT_t Rk.$$

Thus $R(\mathbf{K})$ is an invariant subspace of $PL_2(\mathbf{R}, N)$ and the intertwining equation of the theorem holds. QED

We can strengthen the conclusion of the theorem for elements of $D(B)$:

Proposition 1 *If $k \in D(B)$ then f_k is continuous in the* $\mathbf{N}$ *norm for $t \leq 0$.*

Proof. For $s, t, > 0$ we have

$$\|f_k(-s) - f_k(-t)\|_N^2 = -2\text{Re }(B[S(t) - S(s)]k, [S(t) - S(s)]k)$$

$$\leq 2\|[S(t) - S(s)]Bk\| \; \|[S(t) - S(s)]k\|$$

by the Cauchy-Schwartz inequality. Since S is strongly continuous the result follows. QED

Let us apply this construction to the semi-group associated to an incoming space $\mathbf{D}$ for a unitary group U on a Hilbert space $\mathbf{H}$. Let $d \in \mathbf{D}$ and $f_d = Rd$ as above. We know that $U(-r)d \in \mathbf{D}$ for $r > 0$ by (2). Notice also that

$$S(r)U(-r)d = PU(r)U(-r)d = Pd = d$$

for $d \in \mathbf{D}$. Then for $t \leq -r$ we have, by definition,

$$f_{U(-r)d}(t) = S(-t)U(-r)d = PU(-t)U(-r)d = S(-t-r)d = f_d(t+r)$$

and so by the Lax-Phillips theorem,

$$\|U(-r)d\|_{\mathbf{D}}^2 = \int_{-\infty}^0 \|f_{U(-r)}d\|_{\mathbf{N}}^2 dx \geq \int_{-\infty}^{-r} \|f_{U(-r)d}(x)\|_{\mathbf{N}}^2 dx$$

$$= \int_{-\infty}^0 |f_d(x)|_{\mathbf{N}}^2 dx = \|d\|^2.$$

Since $U(-r)$ is unitary, we have equality throughout which implies that

$$f_{U(-r)d}(t) = 0 \quad \text{if } t > -r.$$

We have thus proved that if

$$r > 0$$

then

$$f_{U(-r)d}(t) = \begin{cases} f_d(t+r) & \text{if} \quad t \leq -r \\ 0 & \text{if} \quad -r < t \leq 0 \end{cases}. \tag{5}$$

4 The Sinai representation theorem.

This says that

Theorem 2 *If* $\mathbf{D}$ *is an incoming subspace for a unitary one parameter group,* $t \mapsto U(t)$ *acting on a Hilbert space* $\mathbf{H}$ *then there is a Hilbert space* $\mathbf{N}$*, a unitary isomorphism*

$$R: \ \mathbf{H} \to L_2(\mathbf{R}, \mathbf{N})$$

such that

$$RU(t)R^{-1} = T_t$$

and

$$R(\mathbf{D}) = PL_2(\mathbf{R}, \mathbf{N}),$$

where P *is projection onto the subspace consisting of functions which vanish on* $(0, \infty]$ *a.e.*

Proof. We apply the results of the last section. For each $d \in \mathbf{D}$ we have obtained a function $f_d \in L_2((-\infty, 0], \mathbf{N})$ and we extend f_d to all of $\mathbf{R}$ by setting $f_d(s) = 0$ for $s > 0$. We thus defined an isometric map R from $\mathbf{D}$ onto a subspace of $L_2(\mathbf{R}, \mathbf{N})$. Now extend R to the space $U(r)\mathbf{D}$ by setting

$$R(U(r)d)(t) = f_d(t - r).$$

Equation (5) assures us that this definition is consistent in that if d is such that $U(r)d \in \mathbf{D}$ then this new definition agrees with the old one. We have thus extended the map R to $\bigcup U(t)\mathbf{D}$ as an isometry satisfying

$$RU(t) = T_t R.$$

Since $\bigcup U(t)\mathbf{D}$ is dense in $\mathbf{H}$ the map R extends to all of $\mathbf{H}$. Also by construction

$$R \circ P_{\mathbf{D}} = P \circ R$$

where P is projection onto the space of functions supported in $(-\infty, 0]$ as in the statement of the theorem.

We must still show that R is surjective. For this it is enough to show that we can approximate any simple function with values in $\mathbf{N}$ by an element of the image of R. Recall that the elements of the domain of B, the infinitesimal generator of $P_{\mathbf{D}}U(t)$, are dense in $\mathbf{N}$, and for $d \in D(B)$ the function f_d is continuous, satisfies $f(t) = 0$ for $t > 0$, and $f(0) = n$ where n is the image of d in $\mathbf{N}$. Hence

$$(I - P)U(\delta)d$$

is mapped by R into a function which is approximately equal to n on $[0, \delta]$ and zero elsewhere. Since the image of R is translation invariant, we see that we can approximate any simple function by an element of the image of R, and since R is an isometry, the image of R must be all of $L_2(\mathbf{R}, \mathbf{N})$.

5 The Stone - von Neumann theorem.

Let us show that the Sinai representation theorem implies a version (for $n = 1$) of the Stone - von Neumann theorem:

Theorem 3 *Let $\{U(t)\}$ be a one parameter group of unitary operators, and let B be a self-adjoint operator on a HIlbert space $\mathbf{H}$. Suppose that*

$$U(t)BU(-t) = B - tI. \tag{6}$$

Then we can find a unitary isomorphism R of $\mathbf{H}$ with $L_2(\mathbf{R}, \mathbf{N})$ such that

$$RU(t)R^{-1} = T_t$$

and

$$RBR^{-1} = m_x,$$

where m_x is multiplication by the independent variable x.

Remark. If iA denotes the infinitesimal generator of U, then differentiating (6) with respect to t and setting $t = 0$ gives

$$[A, B] = iI$$

which is a version of the Heisenberg commutation relations. So (6) is a "partially integrated" version of these commutation relations, and the theorem asserts that (6) determines the form of U and B up to the possible "multiplicity" given by the dimension of $\mathbf{N}$.

Proof. By the spectral theorem, write

$$B = \int \lambda dE_\lambda$$

where $\{E_\lambda\}$ is the spectral resolution of B, and so we obtain the spectral resolutions

$$U(t)BU(-t) = \int \lambda d[U(t)E_\lambda U(-t)]$$

and

$$B - tI = \int (\lambda - t)dE_\lambda = \int \lambda dE_{\lambda+t}$$

by a change of variables.

We thus obtain

$$U(t)E_\lambda U(-t) = E_{\lambda+t}.$$

Remember that E_λ is orthogonal projection onto the subspace associated to $(-\infty, \lambda]$ by the spectral measure associated to B. Let $\mathbf{D}$ denote the image of E_0. Then the preceding equation says that $U(t)\mathbf{D}$ is the image of the projection E_t. The standard properties of the spectral measure - that the image of E_t increase with t, tend to the whole space as $t \to \infty$ and tend to $\{0\}$ as $t \to -\infty$ are exactly the conditions that $\mathbf{D}$ be incoming for $U(t)$. Hence the Sinai representation theorem is equivalent to the Stone - von -Neumann theorem in the above form. QED

HIstorically, Sinai proved his representation theorem from the Stone - von Neumann theorem. Here, following Lax and Phillips, we are proceeding in the reverse direction.