Scattering theory according to Lax and Phillips

Math 212b

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The discussion here follows the paper "Shifts on Hilbert spaces" by Halmos in *Jour. f. Reine un Ang. Math.*(1961) pp. 102- 112, and then some excerpts from the book *Scattering theory* by Lax and Phillips

Contents

1 Invariant, reducing, non-reducible, and wandering subspaces.

In what follows $\mathbf{H}$ is a (complex) Hilbert space, and A is a bounded operator on $\mathbf{H}$.

A subspace $\mathbf{S} \subset \mathbf{H}$ is **invariant** (under A) if

$$A\mathbf{S} \subset \mathbf{S}.$$

An invariant subspace $\mathbf{S}$ is called **reducing** if $\mathbf{S}^\perp$ is also invariant. If $P = P_\mathbf{S}$ denotes orthogonal projection onto $\mathbf{S}$, then $\mathbf{S}$ is reducing (or we also say $\mathbf{S}$ reduces A) if and only if

$$PA = AP.$$

Since P is self-adjoint, this is the same as

$$PA^* = A^*P$$

which implies that $\mathbf{S}$ is invariant under A^*. Conversely, if the invariant subspace $\mathbf{S}$ is invariant under A^*, then if $x \in \mathbf{S}$ and $y \in \mathbf{S}^\perp$

$$0 = (A^*x, y) = (x, Ay)$$

so $\mathbf{S}^\perp$ is invariant under A. In short, $\mathbf{S}$ is reducing if and only if it is invariant under both A and A^* which is the same as saying that $P_\mathbf{S}$ commutes with A. If we have a collection $\{\mathbf{S}\}$ of suspaces, we will let

$$\bigvee \mathbf{S}$$

denote the closure of the sum of these subspaces. If all the $\mathbf{S}$ are invariant or reducing then so is $\bigvee \mathbf{S}$ (as is its orthogonal complement which is $\bigcap \mathbf{S}$).

An invariant subspace $\mathbf{S}$ is called **non-reducible** if it contains no proper reducing subspace. For example, the standard two dimensional Hilbert space $\mathbf{C}^2$ is non-reducible under the nilpotent matrix

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

If $\mathbf{S}$ is an invariant subspace, let $\mathbf{M}$ range over all reducing subspaces of $\mathbf{S}$ and let $\mathbf{J}$ be the orthogonal complement of $\bigvee \mathbf{M}$ in $\mathbf{S}$, so

$$\mathbf{J} := \left(\bigcap \mathbf{M}^\perp\right) \cap \mathbf{S}.$$

Then

$$\mathbf{S} = \mathbf{J} \oplus \left(\bigvee \mathbf{M}\right)$$

is an orthogonal decomposition of $\mathbf{S}$ as the direct sum of non-reducible subspace and a reducing subspace, and such a decomposition is unique.

A subspace $\mathbf{W}$ is called **wandering** if

$$\mathbf{W} \perp A^n\mathbf{W} \quad \forall\, n > 0.$$

If $\mathbf{W}$ is a wandering space, we define $\mathbf{S} = \mathbf{S}_\mathbf{W}$ by

$$\mathbf{S}_\mathbf{W} := \bigvee_{i=0}^{\infty} A^i\mathbf{W}. \tag{1}$$

Clearly

$$A\mathbf{S} = \bigvee_{i=1}^{\infty} A^i\mathbf{W}$$

so

$$A\mathbf{S} \subset \mathbf{S} \text{ and } \mathbf{S} = \mathbf{W} \oplus A\mathbf{S}$$

is a direct sum decomposition.

For example, let

$$\mathbf{W} = (A\mathbf{H})^{\perp}.$$

This is a wandering subspace since by definition $(A^n f, g) = 0$ for $n > 0$ if $g \in \mathbf{W}$ and f arbitrary. (Wandering merely requires this to be true for $f \in \mathbf{W}$.)

2 Isometries, unitaries, and pure isometries.

A linear map U is called an **isometry** if

$$(Uf, Ug) = (f, g) \quad \forall\ f, g \in \mathbf{H}.$$

This is the same as $(U^*Uf, g) = (f, g)$ for all $f, g \in \mathbf{H}$ which is the same as

$$U^*U = I.$$

An isometry preserves length, but need not be surjective.

If $\mathbf{W}$ is a wandering space for an isometry U, then

$$U^m\mathbf{W} \perp U^n\mathbf{W}, \quad m \neq n \;\; m \geq 0,\ n \geq 0. \tag{2}$$

Indeed, up to a change of notation we may assume that $m > n$ and then, with $f, g \in \mathbf{W}$ we have

$$(U^m f, U^n g) = (U^{*n}U^m f, g) = (U^{m-n} f, g) = 0.$$

We also claim that if U is an isometry and

$$\mathbf{W} = (U\mathbf{H})^{\perp}$$

then

$$\mathbf{S}_{\mathbf{W}}^{\perp} = \bigcap_{n=0}^{\infty} U^n\mathbf{H}. \tag{3}$$

Proof. The left hand side of this equation equals

$$\bigcap_{n=0}^{\infty} (U^n\mathbf{W})^{\perp}.$$

We first prove that if f belongs to this intersection then it belongs to the right hand side of (3). So we prove by induction that it belongs to all the $U^n\mathbf{H}$.

For $n = 0$ this merely asserts that $f \in \mathbf{H}$ which is trivially true. We proceed by induction: suppose we know that $f = U^n g$ for some g. By assumption $f \in (U^n\mathbf{W})^\perp$. So

$$U^n g \perp U^n \mathbf{W}$$

which implies that

$$g \perp \mathbf{W}$$

since U is an isometry. Since $\mathbf{W} = (U\mathbf{H})^\perp$, this implies that $g \in U\mathbf{H}$ so $f = U^n g \in U^{n+1}\mathbf{H}$. We have proved that the left hand side of (3) is contained in the right hand side.

We now prove the reverse inclusion: If $\mathbf{M}$ is any subspace and $f \in \mathbf{M}^\perp$, and $g \in \mathbf{M}$, then $(Uf, Ug) = (f, g) = 0$. In other words,

$$U(\mathbf{M}^\perp) \subset (U\mathbf{M})^\perp.$$

So

$$U^{n+1}\mathbf{H} = U^n(U\mathbf{H}) = U^n\mathbf{W}^\perp \subset (U^n\mathbf{W})^\perp$$

so the right hand side of (3) is contained in $\bigcap_{n=0}^{\infty}(U^n\mathbf{W})^\perp$ which is the left hand side. QED

An isometry is called **unitary** if $U\mathbf{H} = \mathbf{H}$, or what amounts to the same thing, if the space $\mathbf{W} = (U\mathbf{H})^\perp = 0$. If U is unitary then it is invertible with inverse U^* so $UU^* = I$ in addition to $U^*U = I$.

Proposition 1 *If $\mathbf{W}$ is a wandering space for a unitary map U, then $\mathbf{S_W}$ is non-reducible.*

Proof. It is clear from the definition (1) that $\mathbf{S_W}$ is invariant under U. We must prove that it does not contain any non-zero subspace invariant under U^*. So it is enough to show that for any non-zero $f \in \mathbf{S_W}$ we have $(U^*)^{m+1}f = U^{-m-1}f \notin \mathbf{S_W}$ for some m. We can write

$$f = \sum_0^\infty U^n f_n$$

and at least one of the summands is non-zero. Say $f_m \neq 0$. The summand $U^m f_m$ is the projection of f onto the subspace $U^m\mathbf{W}$. So $U^{-1}f_m$ is the projection of $U^{-m-1}f$ onto $U^{-1}\mathbf{W}$. But this space is orthogonal to $\mathbf{S_W}$ as can be seen by applying U. So $U^{-m-1}f$ has a non-zero projectrion onto the orthogonal complement of $\mathbf{S_W}$. QED

We now prove that these are all the non-reducing spaces for unitaries.

Theorem 1 *If $\mathbf{M}$ is a non-reducible invariant subspace for a unitary map U, then there exists a wandering space $\mathbf{W}$ for U such that $\mathbf{M} = \mathbf{S_W}$. In fact, we may take*

$$\mathbf{W} = \mathbf{M} \cap (U\mathbf{M})^\perp.$$

Proof. Since $\mathbf{M} \supset \mathbf{S_W}$, to show that $\mathbf{M} = \mathbf{S_W}$ is the same as to show that $\mathbf{M} \cap \mathbf{S_W}^{\perp} = 0$. Since we are assuming that $\mathbf{M}$ is non-reducible, this will follow if we can prove that $\mathbf{M} \cap \mathbf{S_W}^{\perp}$ is a reducing subspace. The restriction of U to $\mathbf{M}$ is an isometry, and hence we may apply (3) to conclude that

$$\mathbf{M} \cap \mathbf{S_W}^{\perp} = \bigcap_{n=0}^{\infty} U^n \mathbf{M}$$

which shows that this space is invariant under U. But

$$U^* \left(\bigcap_{n=0}^{\infty} U^n \mathbf{M} \right) = U^{-1} \left(\bigcap_{n=0}^{\infty} U^n \mathbf{M} \right) = U^{-1}\mathbf{M} \cap \bigcap_{n=0}^{\infty} U^n \mathbf{M} \subset \bigcap_{n=0}^{\infty} U^n \mathbf{M}$$

proving that it is also invariant under U^*. QED

At the opposite extreme from the unitaries among the isometries are the **pure isometries**. An isometry is called **pure** if

$$\mathbf{S_W} = \mathbf{H} \quad \text{where} \quad \mathbf{W} = (U\mathbf{H})^{\perp}.$$

The spaces $U^m\mathbf{W}$ are perpendicular to one another for different values of $m \geq 0$ as we know, and hence the decomposition

$$\mathbf{H} = \mathbf{S_W} = \bigvee_{n=0}^{\infty} U^n \mathbf{W}$$

is an orthogonal direct sum decompostion. In other words, every $f \in \mathbf{H}$ can be written as the orthogonal series

$$f = \sum_{n=0}^{\infty} U^n f_n$$

with $f_n \in \mathbf{W}$. Then

$$Uf = \sum_{n=0}^{\infty} U^{n+1} f_n.$$

This shows that every pure isometry is isomorphic to a unilateral shift in accordance with the following definition: Let $\mathbf{W}$ be a Hilbert space and let $\mathbf{H}_+(\mathbf{W})$ denote the set of all one-sided sequences

$$g = \{g_0, g_1, g_2, \dots\}$$

with

$$\sum \|g_i\|^2 < \infty.$$

Let U_+ be the transformation

$$U_+ g = \{0, g_0, g_1, \dots\}$$

when g is given as above. If we use U^n to identify $U^n\mathbf{W}$ with $\mathbf{W}$ then the equation $\mathbf{H} = \mathbf{S_W}$ gives us an identification of $\mathbf{H}$ with $\mathbf{H}_+(\mathbf{W})$ which takes U to U_+. Conversely, the unilateral shift U_+ on a space $\mathbf{H_W}$ is a pure isometry with $(U_+\mathbf{H}_+(\mathbf{W}))^\perp$ isomorphic to $\mathbf{W}$. In short, pure isometries are the same this as unilateral shifts.

Notice that for a unilateral shift

$$U_+^*\{g_0, g_1, g_2, \dots\} = \{g_1, g_2, g_3, \dots\}. \tag{4}$$

3 Centralizers of unilateral shifts and their adjoints.

Any operator C on $\mathbf{W}$ defines an operator $C^\sharp$ on $\mathbf{H}_+(\mathbf{W})$ according to the rule

$$C^\sharp g = \{Cg_0, Cg_1, Cg_2, \dots\}.$$

The operator $C^\sharp$ clearly commutes with U_+ and U_+^*.

We claim that, conversely, any operator which commutes with both U_+ and U_+^* must be of the form $C^\sharp$. Indeed, suppose that A commutes with U_+^*. Then $\mathbf{W} = \ker U_+^*$ is invariant under A, so the restriction of A to $\mathbf{W}$ defines an operator on $\mathbf{W}$, call it C. Thus

$$A\{f_0, 0, 0, \dots\} = \{Cf_0, 0, 0, \dots\} = C^\sharp\{f_0, 0, 0, \dots\}.$$

Since A and $C^\sharp$ both commute with U_+ this implies that

$$A\{0, f_1, 0, 0, \dots\} = C^\sharp\{0, f_1, 0, 0, \dots\}$$

and more generally that A and $C^\sharp$ agree on all elements of the form

$$\{f_0, f_1, f_2, \dots, f_n, 0, 0, 0, \dots\}$$

for any n. Passing to the limit shows that they agree on all of $\mathbf{H}_+(\mathbf{W})$. QED.

As a consequence we can describe all reducing subspaces for a unilateral shift. Recall that a subspace is reducing for U_+ if and only if P, the orthogonal projection onto the subspace, commutes with U_+ and U_+^*. By the above, we must then have $P = Q^\sharp$, it is then easy to see that Q satisfies $Q^2 = Q$ and $Q^* = Q$ because P satisfies the similar equations. Hence Q is an orthogonal projection onto some subspace. In other words, the reducing subspaces for U_+ are in one to one correspondence with the subspaces of $\mathbf{W}$.

4 Bilateral shifts.

Starting with a Hilbert space $\mathbf{W}$ we can consider the space $\mathbf{H_W}$ of all bilateral sequences

$$f = \{\dots, f_{-2}, f_{-1}, (f_0), f_1, f_2, \dots\}$$

where the $\sum \|f_i\|^2$ converge. (Here the parenthesis is a notational device to locate the zero position.) We now consider the bilateral shift U defined by

$$Uf = \{\dots, f_{-3}, f_{-2}, (f_{-1}), f_0, f_1, \dots, \}.$$

The shift U is clearly unitary with inverse given by shift to the left. We can identify $\mathbf{W}$ with the subspace $\mathbf{W}_0$ consisting of those sequences with all entries zero except in the zero position. We can also indentify $\mathbf{H}_+(\mathbf{W})$ as the subspace of $\mathbf{H}_\mathbf{W}$ copnsisting of those sequences which vanish at all negative entries. The isometry U_+ is then just the restriction of U to $\mathbf{H}_\mathbf{W}$. By Prop. 1 we know that $\mathbf{H}_+(\mathbf{W})$ is a non-reducible invariant subspace for U, as is any U invariant subspace of this subspace. Since the restriction of U to $\mathbf{H}_+(\mathbf{W})$ is U_+, we see that

the U non-reducible subspaces of $\mathbf{H}_+(\mathbf{W})$ are the same as the invariant subspaces of $\mathbf{H}_+(\mathbf{W})$ under U_+.

We wish to characterize these subspaces. For this it is convenient to pass from the shift representation, in which our Hilbert space is given as the space of all ℓ_2 sequences with values in $\mathbf{W}$ to the corresponding space obtained by regarding these sequences as the coefficents of Fourier series.

5 The space $L_2(\mathbf{T}, \mathbf{W})$.

We let $\mathbf{T}$ denote the unit circle in the complex plane, i.e. the set of all complex numbers z with $|z| = 1$. Let $\mathbf{W}$ be a separable HIlbert space (i.e. one with a countable orthonormal basis). The space $L_2(\mathbf{T}, \mathbf{W})$ consists of all measurable functions on $\mathbf{T}$ with values in $\mathbf{W}$ such that

$$\frac{1}{2\pi i} \int_\mathbf{T} \|f(z)\|^2 \frac{dz}{z} < \infty.$$

If we write $z = e^{i\theta}$ then $dz = ie^{i\theta} d\theta$ so the above integral is just with respect to the measure $d\theta/(2\pi)$ which is the unique measure on the circle invariant under rotations and with total measure one.

This space $L_2(\mathbf{T}, \mathbf{W})$ is a Hilbert space under the scalar product

$$(f, g)_{L_2} := \frac{1}{2\pi i} \int_\mathbf{T} (f(z), g(z)) \frac{dz}{z}.$$

Let $L(\mathbf{W})$ denote the space of all bounded operators on $\mathbf{W}$, and let

$$F : \mathbf{T} \to L(\mathbf{W})$$

be a measurable function which is essentially bounded, meaning that the operator norm $\|F(z)\|$ is bounded off a set of measure zero. Then F defines an operator on $L_2(\mathbf{T}, \mathbf{W})$ given by

$$(F^\sharp f)(z) = F(z) f(z)$$

a.e., and this is a bounded operator, whose bound is the essential supremum of $\|F(z)\|$. If $F(z)$ is an isometry almost everywhere, then $F^\sharp$ is an isometry.

We will frequently drop the $\sharp$ in our notation for the operator $F^\sharp$. For example the operator corresponding to the function

$$z \mapsto zI_{\mathbf{W}}$$

sends any f into $zf(z)$ and we may denote it by $zI_{\mathbf{W}}$. This operator is unitary with inverse $z^{-1}I_{\mathbf{W}} = \overline{z}I_{\mathbf{W}}$.

Let $\mathbf{W}_0$ denote the constant functions with values in $\mathbf{W}$. This is a wandering subspace for the operator $zI_{\mathbf{W}}$. Indeed, if $f \equiv u \in \mathbf{W}$ and $g \equiv v \in \mathbf{W}$ then

$$((zI_{\mathbf{W}})^n f, g)_{L_2} = \frac{1}{2\pi i}\int_{\mathbf{T}} (z^n f(z), g(z))\frac{dz}{z} = (u,v)\frac{1}{2\pi i}\int_{\mathbf{T}} z^n \frac{dz}{z}$$

which equals 0 if $n \neq 0$. We claim that

$$L_2(\mathbf{T},\mathbf{W}) = \bigvee_{n=-\infty}^{\infty} (zI_{\mathbf{W}})^n \mathbf{W}_0.$$

Indeed, suppose that f is orthogonal to the right hand side, which means that

$$\frac{1}{2\pi i}\int_{\mathbf{T}} (f(z), z^n v)\frac{dz}{z} = 0 \quad \forall n \in \mathbf{Z}, \quad v \in \mathbf{W}.$$

This means that the complex valued function $(f(z), v)$ has the property that all its Fourier coefficients vanish. By the theory of Fourier series, this means that $(f(z), v)$ vanishes almost everywhere. If we let v range over an orthonormal basis (which is countable by assumption), we conclude that f vanishes almost everywhere.

This last result implies that we may identify $L_2(\mathbf{T},\mathbf{W})$ with the shift space $\mathbf{H}(\mathbf{W})$ so that $zI_{\mathbf{W}}$ becomes identified with the shift operator U, and then $\mathbf{H}_+(\mathbf{W})$ becomes identified with the subspace consisting of all functions whose negative Fourier coefficients vanish. We will make this identification and write $H = \mathbf{H}(\mathbf{W}) = L_2(\mathbf{T},\mathbf{W})$. The following two lemmas will be useful:

Lemma 1 *If $f, g \in H$ and*

$$(U^n f, g)_{L_2} = 0 \quad \forall n \neq 0$$

then $(f(z), g(z))$ is a constant almost everywhere.

Proof. The hypothesis says that

$$\frac{1}{2\pi i}\int_{\mathbf{T}} z^n (f(z), g(z))\frac{dz}{z} = 0 \quad \forall\, n \neq 0$$

which says that all the non-zero Fourier coefficients of the complex valued essentially bounded function $(f(z), g(z))$ vanish. This implies that the function is a constant almost everywhere. QED

Lemma 2 *If* $\mathbf{V}$ *is a wandering subspace for* U *then dim* $\mathbf{V} \leq$ *dim* $\mathbf{W}$.

Proof. Let f_i be an orthonomal set in $\mathbf{V}$. this set is finite or countable since H is separable. We have $(U^n f_i, f_j) = 0$ for $n \neq 0$, so (f_i, f_j) is a constant almost everywhere, and since we are assuming that

$$\frac{1}{2\pi i}\int_{\mathbf{T}} (f_i(z), f_j(z))\frac{dz}{z} = \delta_{ij}$$

we conclude that $(f_i(z), f_j(z)) = \delta_{ij}$ for almost all z. So throwing away at most countably many sets of measure zero, we conclude that there is at least one value of z such that the $f_i(z)$ form an orthonormal set in $\mathbf{W}$. This implies that the cardinality of the set $\{f_i\}$ is at most the dimension of $\mathbf{W}$. QED

6 Invariant subspaces.

The operator valued function which is constant almost everywhere, say $C(z) \equiv C$ a.e. goes over into the operator $C^\sharp$ in our previous notation. In other words, our new use of the notation $\sharp$ is consistent under the identification of $H = \mathbf{H}(\mathbf{W})$ with $L_2(\mathbf{T}, \mathbf{W})$.

The operator $U = zI_{\mathbf{W}}$ commutes with all operators of the form $F^\sharp$ by its very definition. This has the following consequence: Suppose that $F(z)$ is an isometry for almost all z. Then if $u, v \in \mathbf{W}_0$ and we set $f(z) := F(z)u,\ g(z) := F(z)v$. Then $(U^n f, g) =$

$$\frac{1}{2\pi i}\int_{\mathbf{T}} z^n (F(z)u, F(z)v)\frac{dz}{z} = \frac{1}{2\pi i}\int_{\mathbf{T}} z^n (u,v)\frac{dz}{z} = (u,v)\frac{1}{2\pi i}\int_{\mathbf{T}} z^n\frac{dz}{z} = 0$$

if $n \neq 0$. In other words, the space $F^\sharp \mathbf{W}_0$ is a wandering space for U. The crucial step that made this work was the equation

$$(F(z)u, F(z)v) = (u,v)$$

a.e. which followed from the fact that F is an isometry a.e. Suppose that F satisfies the following weaker condition:

$$\text{There is a subspace } \mathbf{U} \subset \mathbf{W} \text{ such that } F_{|\mathbf{U}} \text{ is an isometry and } F_{|\mathbf{U}^\perp} = 0. \text{ a.e..} \tag{5}$$

Then we can still conclude that $F^\sharp\mathbf{W}$ is a wandering subspace for U. Conversely, suppose that $\mathbf{B}$ is a wandering space for U. We know that dim $\mathbf{B} \leq$ dim $\mathbf{W}$. Hence there is a subspace $\mathbf{U}$ of $\mathbf{W}$ which has the same dimension as $\mathbf{B}$. Let $A : \mathbf{W} \to \mathbf{B}$ be an isometry of $\mathbf{U}$ onto $\mathbf{W}$ and vanish on $\mathbf{U}^\perp$. Define the operator valued function $F(z) \in L(\mathbf{W})$ by

$$F(z)(v) := (Av)(z).$$

Now

$$(U^n Av, Av) = 0$$

for $n > 0$ since $\mathbf{B}$ is wandering, and also therefore for $n < 0$ since U is unitary. Therefore by Lemma 1 $\|F(z)v\|$ is a constant a.e., and in particular is bounded. If $v \in \mathbf{U}$ then

$$\|v\|^2 = \|Av\|^2 = \frac{1}{2\pi i}\int_{\mathbf{T}} \|Av(z)\|^2 \frac{dz}{z} = \frac{1}{2\pi i}\int_{\mathbf{T}} \|F(z)v\|^2 \frac{dz}{z}$$

so that $\|F(z)v\| = \|v\|$ a.e. By applying this result to an orthonormal basis of $\mathbf{U}$ (which is countable or finite) we conclude that the restriction of $F(z)$ to $\mathbf{U}$ is an isometry for almost all z, and similarly the restriction of $F(z)$ to $\mathbf{U}^\perp$ vanishes a.e. If $f \in \mathbf{W}_0$ so $f(z) = v$ a.e., then $(F^\sharp f)(z) = F(z)v = (Av)(z)$ a.e. so $F^\sharp \mathbf{W} subspace \mathbf{B}$. If $g \in \mathbf{B}$ then $g = (Av)(z) = F(z)v$ a.e.. So we have shown that $\mathbf{B} = F^\sharp \mathbf{W}_0$ for an F satisfying (5). To summarize:

Proposition 2 *A subspace of H is wandering for U if and only if it is of the form $F^\sharp \mathbf{W}_0$ for an F satisfying (5).*

Suppose we now ask when is a subspace of $H_+ := \mathbf{H}_+(\mathbf{W})$ an invariant subspace for U_+. We know from the proof of Theorem 1 that any non-zero invariant subspace of H_+ is non-reducible for U, and then from Theorem 1 that it is of the form $\mathbf{S_B}$ for some wandering subspace $\mathbf{B}$ of H. We know from Proposition 2 that $\mathbf{B}$ is of the form $F^\sharp \mathbf{W}_0$ for an F satisfying (5). So the question boils down to this: When is $\mathbf{B} \subset \mathbf{H}_+$, or what is the same thing, when does $F^\sharp$ map $\mathbf{W}_0$ into H_+? This amounts to the condition that $(F(z)u, v)$ have only non-negative non-vanishing Fourier coefficients for all $u, v \in \mathbf{W}$. We say that such an operator valued function is of the Hardy class. Roughly speaking it says that F is the boundary value of an analytic $L(\mathbf{W})$ valued function defined on the interior of the unit disk. Suppose that F is such a function. Then we know that any invariant suspace of H_+ is of the form $\mathbf{S_B}$ where $\mathbf{B} = F^\sharp(\mathbf{W}_0)$ where F is of Hardy class and satisfies (5). But then

$$\mathbf{S_B} = \bigvee_0^\infty U^n(F^\sharp(\mathbf{W}_0)) = F^\sharp \left(\bigvee_0^\infty U^n \mathbf{W}_0\right) = F^\sharp H_+.$$

We have proved

Theorem 2 [**Beurling-Lax.**] *A subspace $\mathbf{M}$ of H_+ is invariant under U_+ if and only if it is of the form*

$$\mathbf{M} = F^\sharp H_+$$

where F is of Hardy class and satisfies (5).

7 Uniqueness.

To what extent does the invariant subspace $\mathbf{M}$ determine the function F? Let us begin our investigation of this subject by asking: Suppose that F_1 and F_2

are operator valued functions on $\mathbf{T}$ satisfying (5), with associated subspaces $\mathbf{U}_1$ and $\mathbf{U}_2$ of $\mathbf{W}_0$. When is

$$F_1^\sharp H_+ \subset F_2^\sharp H_+?$$

Clearly a necessary condition is

$$F_1^\sharp \mathbf{W}_0 \subset F_2^\sharp H_+$$

which asserts that for every $v \in \mathbf{W}$ there is an $f \in H_+$ such that

$$F_1(z)v = F_2(z)f(z)$$

a.e. Suppose we replace $f(z)$ by its projection onto $\mathbf{U}_2$ for each z. This does not change the above equation, but now f is a $\mathbf{U}_2$ valued function. Since $F_2(z)$ is an isometry (for almost all z) this implies that $f(z)$ is uniquely determined a.e. by $F_2(z)f(z)$, and hence there is an operator valued function $G(z)$ defined a.e. such that

$$G(z)v = f(z)$$

a.e.. If $v \in \mathbf{U}_1$

$$\|v\| = \|F_1(z)v\| = \|F_2(z)f(z)\| = \|G(z)v\|$$

so G is an isometry a.e.. If $v \perp \mathbf{U}_1$ then $f(z) = 0$ and so $G(z)v = 0$ a.e.. Thus G satisfies (5) with initial space $\mathbf{U}_1$ and the image of $G(z)$ is contained in $\mathbf{U}_2$ for almost all z. Also, since the image of $G(z)$ lies in H_+ a.e., G is of Hardy class. In other words

$$F_1 = F_2 G$$

with G satsiying (5) and of Hardy class. We say that F_1 is **divisible** by F_2. Conversely, if the F_1 is divisible by F_2 then

$$F_1^\sharp H_+ = F_2^\sharp G^\sharp H_+ \subset F_2^\sharp H_+.$$

We have proved

Proposition 3 *If F_1 and F_2 satisfy (5) then*

$$F_1^\sharp H_+ \subset F_2^\sharp H_+$$

if and only if F_1 is divisible by F_2.

Clearly every F is divisible by itself and divisibility is a transitive relation. Suppose that $F_1 = F_2 G_2$ and $F_2 = F_1 G_1$ where G_1 is a function of Hardy class satisfying (5) with initial space $\mathbf{U}_2$ and image contained in $\mathbf{U}_1$, and where G_2 is of Hardy class, and satifies (5) with initial space U_1 and image contained in $\mathbf{U}_2$. For $v \in \mathbf{W}$ and almost all z we have

$$F_1(z)v = F_2(z)G_2(z)v = F_1(z)G_1(z)G_2(z)v.$$

If $v \in \mathbf{U}_1$ this implies that $G_1(z)G_2(z)v = v$, while if $v \perp \mathbf{U}_1$ then $G_2(z)v = 0$ a.e. so also $G_1(z)G_2(z)v = 0$ a.e.. In short, for a.e. z, $G_1G_2(z)$ is orthogonal projection onto $\mathbf{U}_1$.

I claim that for a.e. z

$$G_1(z)G_1(z)^* \text{ is also orthogonal projection onto } \mathbf{U}_1.$$

(I don't see a completely trivial proof of this, which Halmos treats as an obvious triviality, but here is a proof:) Write G_1 instead of $G_1(z)$ for the purposes of this proof. Since $G_1\mathbf{W} = G_1(\mathbf{U}_2) \subset \mathbf{U}_1$, if $v \perp \mathbf{U}_1$ then $(w, G_1^*v) = (G_1w, v) = 0$ for any $w \in \mathbf{W}$. So $G_1G_1^*v = 0$ when $v \perp \mathbf{U}_1$. Similarly, since (interchanging 1 and 2 above, we know that G_2G_1 is orthogonal projection onto $\mathbf{U}_2$ and since $G_1^*G_2^* = (G_1G_2)^* = G_1G_2$ we conclude that the image of G_1^* is $\mathbf{U}_2$. Since the image of G_1G_2 is $\mathbf{U}_1$, we conclude that $G_1G_1^*\mathbf{U}_1 = \mathbf{U}_1$. So write any $v \in \mathbf{U}_1$ as

$$v = G_1G_1^*w, \quad w \in \mathbf{U}_1.$$

Then

$$G_1G_1^*v = G_1G_1^*G_1G_1^*w = G_1G_1^*w = v$$

since $G_1^*G_1u = u$ on $\mathbf{U}_2$ because G_1 is an isometry there. This proves the above equality.

In any event, we have proved that

$$G_1G_1^* = G_1G_2.$$

Since the images of G_1^* and G_2 are $\mathbf{U}_2$ and since $G_1(z)$ is an isometry there a.e., we conclude that $G_1(z)^* = G_2(z)$ a.e. on $\mathbf{U}_1$ and hence on all of $\mathbf{W}$ since both vanish on $\mathbf{U}_1^\perp$.

So far we have not used the fact that G_1 and G_2 are of Hardy class. But if G_1 and G_1^* are both of Hardy class, then G_1 has vanishing positive and negative Fourier coefficients and hence must be a constant. (In more detail, if $G = G_1$ or G_2 and $u, v \in \mathbf{W}$ we have

$$\frac{1}{2\pi i}\int_{\mathbf{T}} (G(z)u, v)\overline{z}^n \frac{dz}{z} = 0$$

and

$$\frac{1}{2\pi i}\int_{\mathbf{T}} (G(z)^*v, u)\overline{z}^n \frac{dz}{z} = 0 = \overline{\frac{1}{2\pi i}\int_{\mathbf{T}} (G(z)u, v)z^n \frac{dz}{z}}.$$

Hence by Lemma 1 we conclude that $(G(z)u, v)$ is a constant a.e. and hence by the separability of $\mathbf{W}$, G is a constant a.e. So we have proved

Theorem 3 **[Halmos.]** *If F is of Hardy class and satuisfies (5) then FH_+ determines F up to multiplication on the right by a constant partial isometry, i.e. a constant G satisfying (5).*

8 The continuous version.

We have characterized subspaces of $\mathbf{H}_+(\mathbf{W})$ which are invariant under the shift U_+ and where all the action is taking place over the integers. I. e. we are considering the space of ℓ_2 maps from the integers to $\mathbf{W}$ with H_+ the subspace consisting of those sequences which vanish at the negative integers.

Suppose we consider $L_2(\mathbf{R}, \mathbf{W})$, the translation operator T_t given by

$$(T_t f)(x) := f(x - t)$$

and the subspace $L_{2,+}(\mathbf{R}, \mathbf{W})$ consisting of functions which vanish for $x < 0$. This subspace is carried into itself by $T_t,\ t > 0$ and we can ask for a characterization of those subspaces of $L_{2,+}(\mathbf{R}, \mathbf{W})$ which are invariant under $T_t,\ t > 0$.

We can reduce this problem to the preceding one by means of the Cayley transform which we review in a more general setting. So let H be a HIlbert space, $V(t)$ a one parameter group of unitary transformations and $D \subset H$ a subspace. Let

$$A := \lim_{t \to 0+} \frac{1}{t}[V(t) - I].$$

Here the limit is taken in the strong sense and is defined on the domain $D(A)$ consisting of those elements x such that $\lim_{t \to 0+} \frac{1}{t}[V(t) - I]x$ exists. By Stone's theorem, A is a skew adjoint operator. In our case,

$$A = \frac{\partial}{\partial x}$$

where the right hand side is defined to be the strong limit as above with $V(t) = T_t$.

The general theory of self-adjoint operators or of semi-groups tells us that the right half plane, in particular the point $\lambda = 1$ is in the resolvent set of A, and hence

$$(I - A)H = H$$

and $(I - A)$ has a bounded inverse on H. Similarly the image of $I + A$ is all of H and has a bounded inverse. We set

$$U := (I + A)(I - A)^{-1}$$

which is a bounded mapping with domain and image equal to all of H and is unitary. Indeed, write any $x \in H$ as

$$s = y - Ay$$

so that

$$Ux = y + Ay$$

where $y \in D(A)$. Then

$$\|Ux\|^2 x\| = (y + Ay, y + Ay) = (y - Ay, y - Ay) = \|x\|^2$$

since $(Ay, y) + (y, Ay) = 0$. The unitary operator U is called the Cayley transform of the skew adjoint operator A.

From the definition it follows that the Cayley transform of $-A$ is U^{-1}.

Proposition 4 *$V(t)D \subset D$ for all $t > 0$ if and only if $UD \subset D$.*

Proof. Using our usual notation

$$R(z, A) = (zI - A)^{-1} = \int_0^\infty e^{-zt} V(t) dt$$

we have

$$U = R(1, A) + AR(1, A) = 2R(1, A) - I$$

so

$$Ux = 2\int_0^\infty e^{-t} V(t) x dt - x.$$

So if $V(t)x \in D$ for all $t \geq 0$ we conclude that $Ux \in D$.

Conversely, the resolvent expansion shows that if z_0 is any point in the right hand plane where $R(z_0, A)D \subset D$, then the same holds in any open disk centered at z_0 and contained in the right hand plane. From $U = 2R(1, A) - I$ we conclude that if $UD \subset D$ then $R(1, A)D \subset D$ and hence that $R(\lambda, A)D \subset D$ for all positive real λ. Thus, if $x \in D$ and $y \perp D$

$$0 = (R(\lambda, A)x, y) = \int_0^\infty e^{-\lambda t}(V(t)x, y) dt.$$

This says that the Laplace transform of the continuous bounded function $t \mapsto (V(t)x, y))$ is zero, and hence the function itself is zero by the uniqueness properties of the Laplace transform. Thus $V(t)D \subset D$. QED

We may apply the proposition to the translation group T_t. The interior of the unit circle goes over into the left half plane, and so an invariant subspace is characterized by bing the image of multiplication by a function which is the boundary value of a operator valued function on the left half plane.

In the applications to scattering theory it is useful to reformulate this result slightly: We want to consider an invariant subspace of $L_2((-\infty, 0], N)$ which is invariant under right translations - implying that its orthogonal complement is invariant under left translation. We also want to put in an i so that the infinitesimal generator of our unitary group is given by multiplication by $i\xi$, in other words we are considering the Fourier transform of the representation of our Hilbert space as $L_2(\mathbf{R}, N)$.

In other words, we want to transform the interior of the unit disk onto the upper half plane. We accomplish this by the fractional linear transformation

$$z = i\frac{1-w}{1+w}$$

which sends

$$1 \mapsto 0, \quad -1 \mapsto \infty, \quad i \mapsto 1$$

and has inverse

$$w = \frac{1+iz}{1-iz}.$$

For a general fractional linear transformation

$$w = \frac{az+b}{cz+d}$$

we have

$$dw = \frac{(ad-bc)ds}{cz+d)^2}$$

so in our case we have, on the unit circle,

$$e^{itheta} \mapsto i\frac{1-e^{itheta}}{1+e^{itheta}} := \sigma$$

and

$$d\theta = \frac{1}{i}\frac{dw}{w} = \frac{2i}{i}\frac{1-iz}{1+iz}\frac{dz}{(1-iz)^2} = \frac{2}{1+\sigma^2}d\sigma.$$

So the map

$$L_2(\mathbf{T},\mathbf{N}) \to L_2(\mathbf{R},\mathbf{N})$$

given by

$$g(e^{itheta}) \mapsto f(\sigma) := \pi^{-\frac{1}{2}}\frac{1}{1-i\sigma}g\left(\frac{1+i\sigma}{1-i\sigma}\right)$$

is unitary, and carries multiplication by $e^{i\theta}$ into multiplication by

$$\frac{1+i\sigma}{1-i\sigma}.$$

The functions which are in $L_2(\mathbf{T},\mathbf{R})$ and which are boundary values of functions which are holomorphic in the interior of then unit disk go over into those elements of $L_2(\mathbf{R},\mathbf{N})$ which are are boundary values of functions which are holomorphic in the upper half plane are whose integrals over the lines Im $z =$ constant > 0 are square integrable (since these lines are the images of the concentric circles in the unit disk). With minor cosmetic changes we can formulate the continuous version of the Beurling-Lax theorem as follows:

Theorem 4 *Let* $\mathbf{K}$ *be the orthogonal complement of a left translation invariant subspace of* $L_2((-\infty,0],\mathbf{N})$. *Then there exists an operator valued function* $z \mapsto S(z)$ *on the lower half plane, where* $S(z)$ *is an operator of norm less than or equal to one, and extends almost everywhere by continuity to an (a.e. defined) isometric operator valued function on the real axis, such that the Fourier transform of* $\mathbf{K}$ *is the orthogonal complement of* $S^\sharp A_-$ *in* A_- *where* A_- *denotes the subspace of* $L_2(\mathbf{R},\mathbf{N})$ *consisting of boundary values of functions holomorphic in the lower half plane (i.e. the Fourier transform of* $L_2((\infty,0],\mathbf{N})$.

9 Strongly contractive semi-groups.

Suppose we start with a strongly contractive semi-group $Z(t)$, for example what we called the scattering residue for a pair of incoming and outgoing subspaces of a unitary group. We then have the Lax-Phillips representation theorem which says that there exists a Hilbert space $\mathbf{N}$ and an isometric map R of $\mathbf{K}$ onto a subspace of $PL_2(\mathbf{R},\mathbf{N})$ such that

$$S(t) = R^{-1}PT_tR$$

for all $t \geq 0$. The subspace $\mathbf{K}$ will be the orthogonal complement of a subspace of $L_2((-\infty,0],\mathbf{N})$ which is invariant under left translations, and hence by the preceding theorem we get an associated operator valued function $S(z)$ holomorphic in the lower half plane. Here S is related to the function F in Theorem 2 by

$$S(z) = F\left(\frac{1+iz}{1-iz}\right).$$

Let B be the infinitesimal generator of Z. It is dissipative, and hence the right hand plane is contained in its resolvent set. There is a relation between the spectrum of B (in the left hand plane) and the null spaces of $S(z)$ in the lower half plane:

Theorem 5 *If* Re $\mu < 0$ *then* μ *belongs to the point spectrum of* B *if and only if* $S(i\overline{\mu})$ *has a non-trivial null space.*

Proof. Let x be an eignenvector of B, so

$$Z(t)x = e^{\mu t}x.$$

This means that in the Lax-Phillips representation x corresponds to a function f satisfying

$$f(s-t) = e^{\mu t}f(s), \quad s \leq 0 \leq t$$

and we have verified that f is continuous and hence that

$$f(s) = \begin{cases} e^{-\mu s}n & (s \leq 0 \\ 0 & (s > 0) \end{cases}$$

for some non-zero $n \in \mathbf{N}$. Applying the inverse Fourier transform takes f into the function

$$h(\sigma) = \frac{n}{i\sigma - \mu}.$$

We have characterized the space $\mathbf{K}$ as the orthogonal complement of $S^\sharp A_-$ in A_-. We know that $h \in A_-$ since the denominator vanishes at only one point which is in the upper half-plane. To say that h is orthogonal to $S^\sharp A_-$ is the same as to say that $(S^\sharp A_-)^*h$ is orhtogonal to A_- and hence lise in A_+, the space of boundary values of functiosn holmorphic in the upper half plane and square integrable along horizontal lines. Now $(S^\sharp)^* = (S^*)^\sharp$ and the function

$$\sigma \mapsto S(\sigma)^*h(\sigma)$$

has a unique meromorphic extension to the upper half-plane, namely

$$z \mapsto \frac{S(\overline{z})^* n}{iz - n}.$$

This will be holomorphic if and only if the numerator vanishes at $z = -i\mu$ which says that

$$S(i\overline{\mu})^* n = 0.$$

QED

A more careful argument, which I will not reproduce here (see lax-Phillips page 70) yields

Theorem 6 *If* Re $\mu < 0$ *thenμ belongs to the resolvent set of B if and only if $S(i\overline{\mu})$ is bounded and surjective with bounded inverse. If μ is purely imaginary, then μ belongs to the resolvent set of B if and only if $S(z)$ can be continued analytically across the real axis in a neighborhood of $\sigma = i\overline{\mu}$.*

10 The Lax-Phillips Scattering "Matrix".

Let $V(t)$ be a unitary one parameter group acting on a Hilbert space H, and suppose that $D_+[D_-]$ is an outgoing [incoming] space for $V(t)$. Let us make the fundamental assumption that

$$D_+ \perp D_-.$$

By the Sinai representation theorem we have unitary isomorphisms

$$Y_\pm H \to L_2(\mathbf{R} < \mathbf{N})$$

with

$$Y_+(D_+) = L_2((0,\infty].\mathbf{N}), \quad Y_-(D_-) = L_2([0,\infty), \mathbf{N}).$$

Define

$$S := Y_+ \circ Y_-^{-1}.$$

Then S is unitary, and since

$$Y_\pm V(t) Y_\pm^{-1} = T_t,$$

we see that

$$S \circ T_t = T_t circ S$$

for all t. If

$$k_- := Y_- f \in L_2((-\infty, 0], \mathbf{N})$$

then

$$f \in D_-$$

so $f \perp D_+$ impying that $k_+ := Y_+ f \perp L_2((0,\infty], \mathbf{N})$ so $Y_+ f \in L_2((-\infty, 0], \mathbf{N})$. In other words,

$$S : L_2((-\infty, 0], \mathbf{N}) \to L_2((-\infty, 0], \mathbf{N}).$$

We may now replace S by $\mathcal{S} := \mathcal{F}^{-1} \circ S \circ \mathcal{F}$ where $\mathcal{F}$ is the Fourier transform. The operator $\mathcal{S}$ commutes with all multiplications by bounded functions, since S commutes with all translations. Hence there is an operator valued function

$$\sigma \mapsto \mathbf{S}(\sigma)$$

such that

$$\mathcal{S} = \mathbf{S}^\sharp.$$

On the other hand, from $S(D_+) = L_2([0,\infty), \mathbf{N})$ we have

$$Y_+(D_+^\perp) = L_2((-\infty,0], \mathbf{N}).$$

Hence, since $\mathbf{K} = D_- \perp \cap D_+^\perp$, we see that $Y_+(\mathbf{K}$ is the orthogonal complement in $L_2((-\infty,0], \mathbf{N})$ of $S(L_2((-\infty,0], \mathbf{N}))$. Passing to the Fourier transform. we see that $\mathcal{S}$ is exactly the operator defining the subspace as is given by the Beurling-Lax and the Lax-Phillips theorems!

I now quote directly from Lax-Phillips page 53:

> Physisicst customarily define the scattering operator in terms of a perturbed and unperturbed group of unitary operators , $\{U(t)\}$ and $\{U_0(t)\}$ respectively. They begin with the wave operators
>
> $$W_\pm = \text{strong} \lim_{t\to\pm\infty} U(-t)U_0(t)$$
>
> and then definethe scattering operaotr S in terms of these as
>
> $$S = W_+^{-1} W_-.$$
>
> We shall show that given any group of operators $\{U(t)\}$ and a pair of orthogonal incoming and outgoing spaces D_- and D_+ we can define another group of unitary operators $\{U_0(t)\}$ so that if we regard $U(t)$ as a perturbation of $U_0(t)$ then the classical scattering operator $W_+W_-^{-1}$ coincides with (actually differs by a trivial factor from) the scattering operator which we introduced in the first part of this section.
>
> We define $U_0(t)$ to be right translation on $H_0 = L_2(\mathbf{R}, \mathbf{N})$. To connect $U_0(t)$ with $U(t)$ we make the following identifications in the Hilbert spaces on which they act: The subspace $L_2((-\infty,-1), \mathbf{N})$ of H_0 is identified with D_- in the incoming representation of $\{U(t)\}$ (after translating one unit to the left); the subspace $L_2((1,\infty), \mathbf{N})$ of H_0 is identified with D_+ in the outgoing translation representation of $\{U(t)\}$ (after translating one unit to the right); the leftover portion $L_2([-1,1], \mathbf{N})$ of H_0 is identified with $H \ominus [D_+ \oplus D_-]$ in a unitary but otherwise arbitrary fashion.
>
> We can now describe the wave operators: It is clear that when the support of $h \in H_0$ is bounded from below, then for t sufficiently

large, say $T > t$, the function $U(-t)U_0(t)$ is *independent of* t and consequently the limit W_+h exists and is an isometry. Since the set of functions h with support bounded from below is dense in H_0, it follows that W_+ exists for all $h \in H_0$. similarly if $f \in H$ has the property has the property that for some T, $U(T)f \in D_+$, then f will belong to the range of W_+. By assumption the set of such f is dense in H, so the range of W_+ is dense. The operator W_+ is therefore an isometry.

The above analysis shows that if $f = W_+h$, then h is the outgoing representer of f, shifted to the right by one unit. Similarly, the other wave operator W_- provides the incoming representation shifted to the left by one unit. Hence the scattering operator

$$S = W_+ \circ W_-^{-1}$$

does indeed relate the incoming and outgoing translation representations of the functions in H. The extraneous shifts introduce a harmless exponential factor into the corresponding scattering matrix.

The perturbation considered in this example has the special feature that there exist substantial subspaces, D_- and D_+, over which $U_0(t)$ and $U(t)$ act in the same way for t negative and positive, respectively. In applications to potential scattering this corresponds to the fact that our method applies only to potentials with bounded support . Also, the existence of incoming and outgoing subspaces implies , as the representation theorem shows, that the spectrum of the infinitesimal generator has uniform multiplicity over the whole real axis. This places another limitation on the kind of scattering problem which can be treated directly by our approach.

In the applications to potential and obstacle scattering, the unperturbed group $\{U(t)\}$, which was here introduced quite artificially, corresponds to wave propagation in free space. We shall in this case explicitly construct a translation representation which is both incoming and outgoing with respect to D_- and D_+. The unperturbed outgoing representer of f can be taken as the perturbed outgoing representer of f for all $f \in D_+$, and as the perturbed incoming representer of f for all $f \in D_-$. Since the translates of $D_+[D_-]$ are dense in H, this procedure leads to the construction of incoming and outgoing representations for $\{U(t)\}$ without appeal to the general representation theorem.

In chapter V, Lax and Phillips study the space of solutions of the wave equation in odd dimensions ≥ 3 in the exterior of an obstacle on whose boundary the solutions are required to be zero. The spaces D_- and D_+ consist of all initial data for which the solution vanishes idenitcally in some spherical neighbrorhod (consisting of a ball of radius ρ) of the obstacle for all times in the past $t < 0$

[future $t > 0$]. They show that each of these subspaces satisfies the axiom for incoming [outgoing] . They then study the associated contractive semigroup $\{Z(t)\}$ and its infinitesimal generator B. They show that $Z(2\rho)R(\kappa, B)$ is compact for suitable κ and conclude from this that the spectrum of B is discrete. They show from this that the scattering operat or has the form: Identity plus an integral operator whose kernel is the asymptotic value at infinity of the scattered plane waves. The entries of the scattering matrix are just the values of this kernel and these quantities can be measured directly by observations made at large distances form the scattering object. They then prove that if the object is star-shaped, it is uniquely determined by the scattering matrix.