The projective Wave front set.

math 212b

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Contents

1 Introduction and definitions.

If $X \to Y$ is a submersion, generalized functions pull back, and smooth functions pull back under any smooth f whether or not it is a submersion. These are extreme cases of the assertion that we can pull back a generalized function u if f is "transversal to the singularities of u" where this set of singularities is suitably defined as a subset of the projectivized cotangent bundle in accordance with the following definitions:

Let u be a generalized density on a manifold X and $0 \neq \ell \in T_x^*X,\ x \in X$. We say that u is **smooth at** ℓ if the following holds: Let S be an arbitrary manifold,

$$f : S \times X \to \mathbf{R}$$

an arbitrary smooth map with

$$d(f_s)_x = \ell \quad \text{where } f_s : X \to \mathbf{R},\ f_s(x) := f(s,x).$$

Define

$$F : S \times X \to S \times \mathbf{R} \text{ by } F(s,x) = (s, f(s,x)).$$

Then there exists a smooth function b with compact support defined on X (and hence on $S \times X$) such that

- $b(x) \neq 0$ and
- $F_*(bu)$ is smooth near $F(s, x)$.

If u is smooth in a neighborhood of x, then all covectors at x are smooth.

Since multiplication by $t \neq 0$ is a diffeomorphism of $\mathbf{R}$, if u is smooth at ℓ it is also smooth at $t\ell$. So we talk of a "smooth codirection $[\ell]$" where $[\ell]$ denotes the line through ℓ. The set of all lines through all non-zero covectors is denoted by PT^*X and is called the projectivized cotangent bundle.

The **projective wave front set** of a generalized density u is defined as the complement in PT^*X of the set of smooth codirections. It is denoted by $\mathrm{PWF}\, u$.

Proposition 1 *Let $g : X \to \mathbf{R}$ be a submersion, and v a generalized function on $\mathbf{R}$. Let*

$$u = Hg^*v$$

where H is a smooth density. Then

$$PWF(u) \subset \bigcup_{x|g(x)=c\in \mathrm{singsupp}(v)} P\mathcal{N}(g^{-1}(c)). \tag{1}$$

Proof. If $x \notin g^{-1}(\mathrm{singsupp}(v))$ then we can find a smooth function of compact support ϕ on $\mathbf{R}$ which is identically one in a neighborhood of $g(x)$ and such that ϕv is smooth. Then u will be smooth on $g^{-1}(\mathrm{supp}(\phi))$ which contains an open set containing x, and hence all codirections at x are smooth. So we may assume that $g(x) = c \in \mathrm{singsupp}\, v$. To say that $\ell \notin P\mathcal{N}(g^{-1}(c))$ means that

$$[\ell] \neq [dg_x].$$

Let $f : S \times X \to \mathbf{R}$ be as in the definition above for a smooth covector, so that $d(f_s)_x = \ell$. Then we can find local coordinates $x^1, \ldots, x^n$ about x such that $x^1 = g$ and $x^2 = f_s$. This coordinate system depends on S as a parameter space, but in the computations that follow we shall leave this dependence implicit. That is, if w is a smooth function on $\mathbf{R}$, we shall write $F^*w = w(x^2)$. We write the smooth density H as

$$H = hdx = h(x^1, \ldots, x^n)dx^1 \ldots dx^n$$

in terms of the coordinate system.

If b is a smooth function of compact support on X and if w is any smooth function of compact support on $\mathbf{R}$ we have

$$\begin{aligned}\langle F_* bu, w\rangle &= \langle F_* b(hdx)g^*(v), w\rangle \\ &= \langle g^*(v), (F^*w)bhdx\rangle \\ &= \langle v, g_*[(F^*w)bhdx]\rangle \\ &= \langle v, g_*[(hb)(\cdot, x^2, \ldots, x^n)w(x^2)dx^1dx^2\ldots dx^n]\rangle \\ &= \langle v, [\int (hb)(\cdot, x^2, \ldots, x^n)w(x^2)dx^2\ldots dx^n]dx^1\rangle \\ &= \int w(x^2)\left(\int \langle v, (hb)(\cdot, x^2, \ldots, x^n)dx^1\rangle dx^3\ldots dx^n\right)dx^2.\end{aligned}$$

This last expression exhibits $F_* bu$ as a smooth density on $\mathbf{R}$, namely $F_* bu$ when evaluated against any test function w is given by $\int w\sigma$ where σ is the smooth function given by the expression in parenthesis in the last line. QED

For the purposes of the next section, let us extract a little more information from the proof we have just given. Suppose that we have a family g_k of submersions where k ranges over some compact subset K of $\mathbf{R}^m$ and that the the generalized function v also depends on k. Suppose also that there is some neighborhood U of (x_0, ξ_0) which has empty intersection with

$$\bigcup_{k,\ x|g_k(x)=c\in \mathrm{singsupp}(v_k)} \mathcal{N}(g_k^{-1}(c)).$$

Then the expression

$$\sigma = \int \langle v, (hb)(\cdot, x^2, \ldots, x^n)dx^1\rangle dx^3\ldots dx^n$$

(which now depends implicitly on k as well as s) is smooth for all $k \in K$ and s in a fixed neighborhood of s_0 independent of k.

2 Using the Radon transform.

Let

$$u \in C_0^{-\infty}(\mathbf{R}^n).$$

Let S^{n-1} denote the unit sphere in $\mathbf{R}^n$. We may identify $\mathbf{R}^n \times S^{n-1}$ as the set of "unit covectors" in the cotangent bundle $T^*\mathbf{R}^n$. For each $w \in S^{n-1}$ we have the submersion

$$\rho_w : \mathbf{R}^n \to \mathbf{R}, \quad \rho_w(x) := w \cdot x$$

and hence the pushforward ρ_{w*} on generalized densities of compact compact support. The Radon transform is defined by

$$\mathcal{R}u(w, \cdot)dp = \rho_{w*}(udx). \tag{2}$$

Here dp on the left denotes the standard Lebesgue measure on $\mathbf{R}$ while dx on the right denotes Lebesgue measure on $\mathbf{R}^n$.

The operator I^r is defined on generalized functions of compact support on $\mathbf{R}$ by

$$I^r \nu := \mathcal{F}^{-1}(\tau_+^r \mathcal{F}(\nu))$$

where $\mathcal{F}$ denotes the Fourier transform in one variable. Then we can write the Radon inversion formula as

$$u = \frac{1}{(2\pi)^{n-1}} \int_{S^{n-1}} \rho_w^* \left(I^{n-1}[\mathcal{R}_* u(w, \cdot)] \right) dw. \tag{3}$$

It is an immediate consequence of the Fourier inversion formula in n dimensions.

Proposition 2 *u is smooth at (x_0, w_0) if and only if there exists a smooth density $H = hdx$ with $h \in C_0^\infty(\mathbf{R}^n)$ with $h(x_0) \neq 0$ and such that $\rho_{\pm w*}(Hu)$ is smooth as a density on $\mathbf{R}$ for all w near w_0.*

Proof. If (x_0, w_0) is a smooth codirection we may take $S = S^{n-1}$, $f = \rho.$ and $b = H$ in the definition of smoothness to conclude that the condition is necessary. To prove the sufficiency we use the Radon inversion formula (with u replaced by hu where $H = hdx$). We split the integral occurring in (3) into two parts: One consisting of an integral over an open neighborhood of $\pm w_0$. By hypothesis, the integrand is smooth for this integral, hence the push forward under any f_s will also be smooth. The remaining integral (over $K \subset S^{n-1}$ say) has as its integrand, generalized functions whose (projective) wave front sets lie in K by (1). Hence, if we are given f_s with $d(f_{s_0})_{x_0} = w_0$ we can find a neighborhood of s_0 such that the pushforwards

$$f_{s*} \rho_w^* \left(I^{n-1}[\mathcal{R}_* u(w, \cdot)] \right)$$

are all smooth, hence so is their integral over K. See the remarks at the end of the preceding section. QED

Using a partition of unity, we can give a local description of the wave front set via Prop. 2. It is clear that the set of smooth covectors as described by Prop. 2 is open. Hence we conclude that

Proposition 3 *The projective wave front set of any generalized function (or density) is a closed subset of PT^*.*

Here is an immediate and important consequence of the proof of Prop. 2: Suppose that we have found a smooth density H of compact support so that $\rho_{w*} Hu$ is smooth for all $w \in S^{n-1}$. Suppose that we multiply Hu by a smooth function g. Then

$$\rho_{w*}(gHu) = \frac{1}{(2\pi)^{n-1}} \rho_{w*} \int_{S^{n-1}} g \rho_w^* \left(I^{n-1}[\rho_{w*}(Hu)] \right) dw$$

by the Radon inversion formula. We can break this integral up into two pieces: one consisting of integration over a neighborhood of $\pm w_0$ where the integrand is smooth, whereas when we bring the ρ_{w*} under the integral sign in the second integral the result is smooth. Therefore

Proposition 4 *If g is a smooth function, then*

$$PWF(gu) \subset PWF(u) \tag{4}$$

As a second consequence, let $\pi : PT^*X \to X$ be the canonical projection. Then

$$\pi(PWF(u)) = \text{singsupp}(u). \tag{5}$$

Proof. If $x \notin \text{singsupp}\, u$ then u is smooth in a neighborhood of x and hence all covectors above x give smooth codirections. So the left hand side of the above equation is contained in the right hand side, as we already know. On the other hand, if $x \notin \pi(PWF(u))$ then there is a coordinate chart about u with the property that for each $w \in S^{n-1}$ there is an H such that $H(x) \neq 0$ and $\rho_{w*}(Hu)$ is smooth. This H works for a whole neighborhood of w, and since S^{n-1} is compact, we can cover it by finitely many such neighborhoods. Since, as we have just observed, multiplying H by a smooth function does not destroy the smoothness of $\rho_{w*}(Hu)$, we may use a partition of unity on the sphere to conclude the existence of an H such that $\rho_{w*}(Hu)$ is smooth for all H. Then, by the Radon inversion formula we conclude that Hu is smooth. So $x \notin \text{singsupp}(u)$. QED

3 The push forward.

Let

$$g : X \to Y$$

be a smooth map from one manifold to another. If

$$A \subset PT^*X$$

we define its pushforward

$$g_*A \subset PT^*Y$$

as follows: A point $(y, [\ell]) \in PT^*Y$ belongs to g_*A if and only if there is an $x \in X$ with $g(x) = y$ and either

$$dg_x^*\ell = 0, \quad \text{or} \quad (x, [dg_x^*\ell]) \in A.$$

In symbols,

$$g_*A := \left\{ (y, [\ell]) \in T^*Y \,\middle|\, \exists x \in X \text{ with } g(x) = y \text{ and } \begin{array}{l} dg_x^*\ell = 0 \\ \text{or} \\ (x, [dg_x^*\ell]) \in A \end{array} \right\} \tag{6}$$

Let u be a generalized density of compact support on X so that g_*u is a generalized density of compact support on Y. We claim that

$$PWF(g_*u) \subset g_*(PWF(u)). \tag{7}$$

Proof. Suppose that $(y_0, [\eta_0])$ does not belong to the right hand side. We wish to show that $(y_0, [\eta_0])$ is a smooth codirection for $g_* u$. So let $f : S \times Y \to \mathbf{R}$ satisfy

$$df_{s_0} = (y_0, \ell_0)$$

as in the definition of smooth codirection. Since $df_x^* \eta_0 \neq 0$ for any $x \in g^{-1}(y_0)$ we know that $f_{s_0} \circ g$ is a submersion for any $x \in g^{-1}(y_0)$. Since u has compact support, we conclude that $f_s \circ g$ is a submersion for all s in a neighborhood of s_0 and all x in $\operatorname{supp} u$ which lie in a neighborhood of $g^{-1}(y_0)$. We can find a b for each point in this set such that $(f_s \circ g)_*(bu)$ is smooth, because $[dg_x^* \eta_0] \notin PWF(u)$ for all $x \in g^{-1}(y_0)$. By compactness, finitely many such b will do, by further restriction, we may then assume that each $b \equiv 1$ in a neighborhood of each $x \in g^{-1}(y_0)$ and then use a partition of unity on a neighborhood of $g^{-1}(y_0)$ to conclude the existence of a single b defined on a neighborhood of $g^{-1}(y_0)$ and $\equiv 1$ on a slightly smaller neighborhood U of $g^{-1}(U)$ and such that $g(U)$ contains a neighborhood of y_0 and such that $(F \circ (\mathrm{id} \times g))_*(bu)$ is smooth. We may choose a function c which is supported in U and which is identically one in a neighborhood of y_0. Replacing b by $(g^* c)b$, we may assume that $b = g^* a$ for some function a which equals one at y_0 and is of compact support. But then $(f_s \circ g)_*(bu) = f_{s*}(a g_*(u))$ is smooth, showing that (y_0, η_0) is a smooth codirection for $g_* u$. QED

4 The pull back.

Let

$$B \subset PT^* Y$$

and let

$$f : X \to Y$$

be a smooth map. We say that f is **transversal** to B if, for all $x \in X$, if

$$y = f(x) \text{ and } (y, [\eta]) \in B \quad \Rightarrow \quad df_x^* \eta \neq 0.$$

For example, suppose that Z is a submanifold of Y and $B = \mathcal{PN}(Z)$. The condition of transversality becomes

$$z \in Z, \; z = f(x), \quad 0 \neq \eta \in T^* Y_z, \text{and } \iota_z^* \eta = 0, \quad \Rightarrow df_x^* \eta \neq 0,$$

in other words the induced map

$$df_x^* \, (TY_z / TZ_z)^* \to \left(TX_x / df_x^{-1}(TY_z) \right)^*$$

is injective which is the same as saying that the induced map

$$df_x : TX_x \to TY_z / TZ_z$$

is surjective. But this is the condition that f be transversal to Z (at x). So we have proved:

if $B = \mathcal{PN}(Z)$ then f is transversal to B if and only if f is transversal to Z.

In general, if f is transversal to a $B \subset PT^*Y$ we define the pull back $f^*B \subset PT^*X$ by

$$f^*B = \{(x, [\xi]) | \exists\ \eta \text{ with } (f(x), [\eta]) \in B \text{ and } df_x^*\eta = \xi\}. \tag{8}$$

Let B be a subset of PT^*Y. Consider the space

$$C_B^{-\infty}(Y) \subset C^{-\infty}(Y)$$

consisting of all generalized functions ν with

$$PWF(\nu) \subset B.$$

We will put a topology on this space which is stronger than the weak topology. It is defined by a family of seminorms. First, for each smooth density of compact support on Y introduce the semi-norm

$$||\nu||_\rho := |\langle \rho, \nu \rangle|.$$

These semi-norms define the weak topology. Next, choose a cover of Y by coordinate neighborhoods, and a partition of unity $\{\phi\}$ subordinate to this cover. Next, for each chosen coordinate chart U let dx denote the corresponding Lebesgue measure, and let $\{b\}$ be a collection of functions on U such that $\rho_{w*}(b\nu dx)$ is smooth on $S^{n-1} \times \mathbf{R}$ whenever $\nu \in C_B^{-\infty}(Y)$. Such functions b exist by the proof of Prop. 2. We can now consider the collection of semi-norms

$$||\nu|| = ||\rho_{w*}(bu)||_k$$

where the norm on the right is the C^k norm on $S^{n-1} \times \mathbf{R}$. The proof of Prop. 2 shows in fact that we can make this definition even more implicit by throwing in a whole batch of seminorms which will not change the topology. Namely, let S be any parameter space and $g : S \times U \to \mathbf{R}$ where U is a coordinate chart on Y such that

$$[dg_s(x)] \notin B$$

for any $s \in S,\ x \in U$. Let b be any density of compact support in U, so $g_{s*}(bu)$ is smooth on $S \times U$. Let c be a smooth function of compact support on S. Thus $c(s)f_{s*}(b\nu)$ is smooth and of compact support on $S \times U$. Then for any non-negative integer k we can consider the semi-norm

$$||\nu||_{g,b,c,k} := ||cf_{\cdot *}(b\nu)||_k.$$

Via the inverse Radon transform we see that these semi-norms define the same topology as the more restricted family semi-norms introduced above using linear functions.

Proposition 5 *The space of smooth functions $C^\infty(Y)$ is dense in $C_B^{-\infty}$ in the above topology.*

Proof. By a partition of unity we can reduce the proof to the case where we are dealing with an open subset of Euclidean space. We will need a lemma relating the Radon transform to convolution. Recall that if ν is a generalized function of compact support on $\mathbf{R}^n$ and if h is a smooth function, their convolution $\nu \star h$ is defined as

$$(\nu \star h)(x) := \langle \nu, h_x dy \rangle$$

where

$$h_x(y) := h(x-y).$$

(I am dropping the factor $1/(2\pi)^{n/2}$ for the purposes of this proof so as not to clutter up the notation.) In case ν is smooth, this reduces to

$$(\nu \star h)(x) = \int \nu(y) h(x-y) dy.$$

It will be convenient to use the right hand side of this equation as notation for the left in general. We will let h_- denote the function

$$h_-(x) := h(-x)$$

then

$$\int \int \nu(y) h(x-y) g(x) dy dx = \int \left(\int h_-(y-x) g(x) dx \right) dy$$

or

$$\langle \nu \star h, g \rangle = \langle \nu, h_- \star g \rangle.$$

Thus if h converges to the delta function, then $\nu \star h$ converges to ν in the weak topology. We wish to obtain convergence in the stronger topology introduced above.

Lemma 1 *For smooth functions ν and h of compact support (and hence for ν a generalized function as well) we have*

$$\mathcal{R}_w(\nu \star h) = \mathcal{R}_w(\nu) \star \mathcal{R}_w(h) \tag{9}$$

where the convolution on the left is over $\mathbf{R}^n$ and the convolution on the right is over $\mathbf{R}$.

Proof. Without loss of generality we may assume that w is the unit vector in the x^1 direction. So $\mathcal{R}_w(\nu \star h) dx^1 =$

$$\begin{aligned}
\rho_{w*}((\nu \star h) dx) &= \left[\int (\nu \star h) dx^2 \cdots dx^n \right] dx^1 \\
&= \left[\int \left(\int \nu(x^1 - y^1, \ldots, x^n - y^n) h(y^1, \ldots, y^n) dy^1 \cdots dy^n \right) dx^2 \cdots dx^n \right] dx^1 \\
&= \left[\int \left(\nu(x^1 - y^1, \ldots, y^n - x^n) dx^2 \cdots dx^n \right) h(y^1, \ldots, y^n) dy^1 \cdots dy^n \right] dx^1 \\
&= \left[\int \left\{ \int \left(\nu(x^1 - y^1, \ldots, y^n - x^n) dx^2 \cdots dx^n \right) h(y^1, \ldots, y^n) dy^2 \cdots dy^n \right\} dy^1 \right] dx^1 \\
&= (\mathcal{R}_w \nu \star \mathcal{R}_w h) dx^1 \\
&= \text{QED}
\end{aligned}$$

Let us now use the lemma to prove the proposition. Choose a smooth function h with compact support on $\mathbf{R}^n$ and such that

$$\int_{\mathbf{R}^n} h dx = 1.$$

Let h_r be defined by

$$h_r(x) = r^n h(rx)$$

so that

$$\int h_r(x)dx = r^n \int h(rx)dx = \int h(y)dy = 1$$

under the substitution $y = rx$ and the support of h_r is $(1/r)\,\mathrm{supp}(h)$. In other words, h_r approaches the δ function as $r \to \infty$. Also, we have

$$\mathcal{R}_w h_r = (\mathcal{R}_w h)_r$$

where the subscript r on the right hand side means performing the same operation in one dimension. Indeed, taking w to be the unit vector in the x^1 direction, the left hand side is

$$r^n \int (rx^1, rx^2, \ldots, rx^n)dx^2 \ldots dx^n = r \int (rx^1, y^2, \ldots, y^n)dy^2 \cdots y^n = (\mathcal{R}_w h)_r.$$

Thus

$$\rho_{w*}(h_r \star \nu dx) = (\mathcal{R}_w h)_r \star \mathcal{R}_w \nu$$

converges to $\mathcal{R}_w \nu$ in the C^k topology for any k if w is restricted to a compact set K such that $U \times [K]$ has empty intersection with the wave front set. QED

We now come to the main theorem of this section:

Theorem 1 *Let $f : X \to Y$ be a smooth map and g a generalized function on Y. Suppose that f is transversal to $W \subset PT^*Y$. Then then the pull back map*

$$f^* : C^\infty(Y) \to C^\infty(X), \quad f^*g = g \circ f$$

extends uniquely by continuity to a map

$$C_W^{-\infty}(Y) \to C^{-\infty}(X).$$

Furthermore

$$PWF(f^*g) \subset f^*\left(PWF(g)\right). \tag{10}$$

Proof. The uniqueness of the extension follows from the fact $C^\infty(Y)$ is dense in $C_W^{-\infty}(Y)$. To establish the existence, we will make some choices. The uniqueness then implies that the end result of our construction will be independent of our choices.

So let g be a generalized function on Y belonging to $C_W^{-\infty}(Y)$, and let μ be a smooth density of compact support on X. We wish to define

$$\langle u, f^*g \rangle.$$

We proceed as follows. Each $x \in X$ has a coordinate neighborhood U and a coordinate neighborhood V about $f(x)$ such that $f(U) \subset V$ and, regarding U and V as open subsets of the respective Euclidean spaces, there are disjoint closed subsets S_1 and S_2 of S^{n-1} (where $n = \dim Y$) invariant under multiplication by -1 such that

$$\pi^{-1} V \cap W \subset V \times S_1$$

and

$$\ker df^* \subset V \times S_2.$$

We can cover $\operatorname{supp} \mu$ by finitely many such U's with their corresponding V's. By a partition of unity, we may then replace X, Y by U, V as above. We may also choose a partition of unity, ψ_1, ψ_2 with

$$S_1 \subset \text{supp } \psi_1,\ S_2 \cap \text{supp } \psi_1 = \emptyset,\ S_1 \cap \text{supp } \psi_2 = \emptyset.$$

By the Radon inversion formula we have

$$(2\pi)^{n-1} g = \int \psi_1 \rho_w^* \left[I^{n-1}(\rho_{w*} g dy) \right] dw + \int \psi_2 \rho_w^* \left[I^{n-1}(\rho_{w*} g dy) \right] dw.$$

Since the projective wave front set of g has empty intersection with $[\operatorname{supp} \psi_2]$, the second integral is a smooth function, and hence its pull back is defined by the usual formula. In the first integral, we will try to define the pull back for each integrand, and then integrate. Now the fact that $df_x^* w \neq 0$ for $w \in \text{supp } \psi_1$ implies that

$$\rho_w \circ f \ \text{ is a submersion for all } \ w \in \text{supp } \psi_1.$$

We know how to pull back generalized functions under submersions, so we define

$$\langle u, \psi_1(w) f^* \rho_w^* \left[I^{n-1}(\rho_{w*} g dy) \right] \rangle := \langle (\rho_w \circ f)_* u, I^{n-1} \mathcal{R}_w g \rangle.$$

We then integrate this expression with respect to w to give the pullback of the first integral. This process is clearly continuous with respect to our topology. We can apply Prop. 1 to conclude that

$$PWF(f^* g) \subset \bigcup_{x \in U} df_x^* \text{ supp } \psi_1$$

since the pull-back of the second integral above contributes notion to the wave front set. By choosing finer and finer neighborhoods U about each $x \in X$ of the type considered above, we conclude that (10) holds. This completes the proof of the theorem.

5 External products.

Let X and Y be manifolds. We have obvious embedding maps

$$C^\infty(X) \otimes C^\infty(Y) \to C^\infty(X \times Y)$$

where $f \otimes g$ goes into the function sending

$$(x, y) \mapsto f(x)g(y).$$

Similarly, given smooth densities μ on X and ν on Y we get a "product density" corresponding to $\mu \otimes \nu$ on $X \times Y$. In fact, we can even define the "external product" of two generalized densities as follows: If μ is a generalized density on X and ν a generalized density on Y, and h is a continuous function of compact support on $X \times Y$, then h_x defined by

$$h_x(y) := h(x, y)$$

is a smooth function of compact support on Y,and hence

$$\langle \nu, h_x \rangle$$

depends smoothly on x and vanishes outside a compact set. Thus we may evaluate μ on this function:

$$\langle \mu, \langle \nu, h_{(\cdot)} \rangle \rangle.$$

We will denote this generalized density by

$$\mu \diamond \nu.$$

On functions of the form $(f \otimes g)(x, y) = f(x)g(y)$ we clearly have

$$\langle \mu \diamond \nu, f \otimes g \rangle = \langle \mu, f \rangle \cdot \langle \nu, g \rangle,$$

and since linear combinations of such product functions are dense in $C^\infty(X \times Y)$, we see that in the definition of $\mu \diamond \nu$ we could have first held y fixed and evaluated μ and then evaluate ν. This interchange of order gives the same answer on the tensor product functions and hence on all smooth functions of compact support on $X \times Y$.

Because of the existence of covectors of the form $(x, y, \xi, 0)$ or $(x, y, 0, \eta)$ we do not have an identification of $PT^*(X \times Y)$ with $(PT^*(X)) \times (PT^*(Y))$ but as sets we can write

$$PT^*(X \times Y) = PT^*X \times PT^*(Y) \cup PT^*X \times 0_Y \cup 0_X \times PT^*Y$$

where 0_X denotes the set of zero covectors of X (which, as a set can be identified with X) and similarly for 0_Y.

We claim that

$$PWF(u \diamond v) \subset PWF(u) \times PWF(v) \cup PWF(u) \times 0_Y \cup 0_X \times PWF(v). \quad (11)$$

Indeed, this is a purely local assertion, so it is enough to prove the formula for the case that $X = \mathbf{R}^m$ and $Y = \mathbf{R}^n$. We need only show that if (x, y, ξ, η) does not belong to the right hand side of (11) then we can find a blip function h around (x, y) such that

$$\langle h(u \diamond v), e^{it\xi \cdot x + t\eta \cdot y} \rangle = O(|t|)^{-N}$$

(uniformly near (ξ, η)) for any positive integer N. We may choose $h(x, y) = h_1(x)h_2(y)$ so that the above expression becomes

$$\langle h_1\mu, e^{it\xi\cdot x}\rangle\langle h_2\nu.e^{it\eta\cdot y}\rangle$$

and the result follows.

6 Applications of the pushforward, pullback, and product formulas.

We have

$$PWF(f^*g) \subset f^*(PWF(g)) \text{ and } PWF(f_*\nu) \subset f_*(PWF(\nu))$$

when the appropriate pull backs or push forwards are defined, together with conditions as to when these operations are defined. We also have (11) for the external product. We will now derive various consequences of these formulas and conditions.

6.1 Multiplication

The product of two honest functions f and g on a manifold X is the function $x \mapsto f(x)g(x)$. In functorial language we can express this in terms of the external product $f \diamond g$ on $X \times X$ as

$$fg = \Delta^*(f \diamond g)$$

where

$$\Delta : X \to X \times X, \quad \Delta(x) = (x, x)$$

is the diagonal map. For this to make sense for generalized functions we need to know that Δ is transversal to $PWF(f \diamond g)$. Now

$$d\Delta_x^*(x, x, \xi, \eta) = (x, \xi + \eta).$$

We conclude:

Proposition 6 *Let f and g be generalized functions, and suppose that*

$$\xi + \eta \neq 0 \text{ for any } (x, [\xi]) \in PWF(f), \quad (x, \eta) \in PWF(g).$$

Then the product fg is defined (by continuity from smooth functions) and

$$PWF(fg) \subset (PWF(f) + PWF(g)) \cup PWF(f) \cup PWF(g) \tag{12}$$

where $PWF(f) + PWF(g)$ denotes the set of all $(x, [\xi + \eta])$ where $(x, [\xi]) \in PWF(f)$ and $(x, [\eta]) \in PWF(g)$.

6.2 Composition.

Let X, Y, and Z be manifolds, let K be a section of the bundle $|\wedge|Y$ pulled back to $X \times Y$, and let L be a section of $|\wedge|Z$ pulled back to $Y \times Z$. Their composition $K \circ L$ is defined as the section of $|\wedge|Z$ pulled back to $X \times Z$ by

$$(K \circ L)(x, z) = \int_Y K(x, y)L(y, z),$$

assuming this integral converges, for example if one or the other has compact support in Y for fixed (x, z). In functorial language this is

$$\pi_* \Delta^* (K \diamond L)$$

where $K \diamond L$ is a section of $|\wedge|Y \otimes |\wedge|Z$ over $X \times Y \times Y \times Z$, where

$$\Delta : \ X \times Y \times Z \to X \times Y \times Y \times Z, \quad (x, y, z) \mapsto (x, y, y, z)$$

and where

$$\pi : X \times Y \times Z \to X \times Z$$

is the obvious projection.

The problem of making sense of this for generalized sections is the the issue of the pullback under Δ. So we need Δ to be transversal to the projective wave front set of $K \diamond L$. Now

$$d\Delta_{(x,y,z)}(\xi, \eta_1, \eta_2, \zeta) = (\xi, \eta_1 + \eta_2, \zeta)$$

so we impose the condition

$$\nexists \eta \neq 0 \ \text{ such that } \ [(0, \eta)] \in PWF(K) \ \text{ and } [(-\eta, 0)] \in PWF(L). \tag{13}$$

Then

$$PWF(\Delta^*(K \diamond L)) \subset$$

$$\{[(x, y, z, \xi, \eta_1 + \eta_2, \zeta)] | [(x, y, \xi, \eta_1)] \in PWF(K) \text{ and } \ [(y, z, \eta_2, \zeta)] \in PWF(L)\}.$$

Assuming the appropriate compactness conditions, we can then "push forward under π" (i.e integrate over Y). The formula for the push forward then consists of $[(x, z, \xi, \zeta)]$ such that $[(x, y, z, \xi, 0, \zeta)]$ is of the above form for some y, i.e.

$$PWF(K \circ L) \subset$$

$$\{[(x, z, \xi, \zeta)] | \exists \ (y, \eta) | \ [(x, y, \xi, \eta)] \in PWF(K) \ \text{ and } [(y, z, -\eta, \zeta)] \in PWF(L)\}. \tag{14}$$