Problem set 1

Math 212

Sept. 23, 2004, due Tuesday Oct. 5

I plan to usually assign homework on Thursday due the following Thursday. But there will be **no class** next Thursday **Sept. 30** or the following Thursday **Oct. 7** due to the Jewish holidays. So things will be a bit different for the first two problem sets.

Contents

1 Fejer suppresses Gibbs.

1. Let f be a bounded Riemann integrable function on $\mathbf{T}$ and let $\|f\|_\infty$ denote its supremum. Show that

$$|C(f,n,x)| \leq \|f\|_\infty$$

so that, in particular, there is no overshoot in the Fejer approximations to the square wave.

2 Hurwitz's proof of the isoperimetric inequality.

Let $X = X(s), Y = Y(s)$ be a closed curve in the plane parametrized by arc length, where we assume that X and Y are differentiable. Let L denote the length of the curve, and A the area enclosed by the curve. The **isoperimetric inequality** says that

$$A \leq \frac{L^2}{4\pi}$$

with equality achieved only in the case of a circle.

Let $t := 2\pi s/L$ and $x(t) = X(2\pi s/L)$ (and similarly for Y) so $x(t)$ and $y(t)$ are periodic with period 2π and

$$x'(t)^2 + y'(t)^2 \equiv \left(\frac{L}{2\pi}\right)^2.$$

Also

$$A = \int_C xdy = \int_0^{2\pi} x(t)y'(t)dt.$$

2. Using the Fourier expansions $x(t) = \sum a_n e^{int}, \ y(t) = \sum b_n e^{int}$ and Parseval's identity, prove the isoperimetric inequality.

3 Euler's formula for the cotangent.

3. Compute the Fourier series of $e^{ax}, (-\pi < x < \pi, \ a \in \mathbf{C}, a \notin i\mathbf{Z})$. Compute the Fourier series of $\cos ax, a \notin \mathbf{Z}$. Derive Euler's formula

$$\pi \cot \pi z = \frac{1}{z} + 2z \sum_{n=1}^{\infty} \frac{1}{z^2 - n^2}, \quad z \notin \mathbf{Z}.$$

4 Category versus measure.

We say that a susbet S of $\mathbf{R}$ has *measure zero* if for any $\epsilon > 0$ we can find a countable set of intevals I_j such that

$$S \subset \bigcup I_j \ \text{ and } \sum_j |I_j| < \epsilon$$

where $|I|$ denotes the length of the interval I. If $S = S_1 \cup S_2 \cup \cdots$ is a countable union of measure zero sets, then taking $\epsilon/2^i$ for the ϵ of S_i and noting that a countable union of countably many intevals is again a countable union of intervals, we may conclude that a countable union of sets of measure zero is again countable. A property P of a real number is is said to hold **almost everywhere**, or to hold for **almost all** x if the set where it does not hold has measure zero. This is definition of the word “almost”. This formulation of “almost” is due to Borel and Lebesgue.

There is a competing notion which is purely topological due to Baire. We begin by reviewing the following material which is also in the notes:

The Baire category theorem. *In a complete metric space any countable intersection of dense open sets is dense.*

Proof. Let X be the space, let B be an open ball in X, and let $O_1,\ O_2 \dots$ be a sequence of dense open sets. We must show that

$$B \cap \left(\bigcap_n O_n \right) \neq \emptyset.$$

Since O_1 is dense, $B \cap O_1 \neq \emptyset$, and is open. Thus $B \cap O_1$ contains the closure $\overline{B_1}$ of some open ball B_1. Since B_1 is open and O_2 is dense, $B_1 \cap O_2$ contains the closure $\overline{B_2}$ of some open ball B_2, and so on. Since X is complete, the intersection of the decreasing sequence of closed balls we have constructed contains some point x which belong both to B and to the intersection of all the O_i. QED

A **Baire space** is defined as a topological space in which every countable intersection of dense open sets is dense. Thus Baire's theorem asserts that every metric space is a Baire space. A set A in a topological space is called **nowhere dense** if its closure contains no open set. Put another way, a set A is nowhere dense if its complement A^c contains an open dense set. A set S is said to be of **first category** if it is a countable union of nowhere dense sets. Then Baire's category theorem can be reformulated as saying that the complement of any set of first category in a complete metric space (or in any Baire space) is dense. A property P of points of a Baire space is said to hold **quasi-surely** or **quasi-everywhere** if it holds on an intersection of countably many dense open sets. In other words, if the set where P does not hold is of first category. This is the mathematical definition of the word “quasi”.

4.1 “Quasi” $\neq$ “Almost”.

On the real line $\mathbf{R}$ these notions are quite different. Indeed:

4. Show that there is a proposition P about real numbers which holds quasi-surely and whose negation holds almost everywhere. [Hint: Let a_n be an enumeration of the rationals, and let I_{ij} be the open interval of length $1/2^{i+j}$ centered at a_i. Consider the open sets $O_j := \bigcup_{i=1}^{\infty} I_{ij}\ (j = 1, 2, \dots)$.]

4.2 The Kuratowski-Ulam theorem.

This says:

Let X and Y be Baire spaces and suppose that Y is separable, that is, that it has a countable base of open sets. Let E be a subset of $X \times Y$ which is of first category. For each $x \in X$ let $E_x \subset Y$ be defined by

$$E_x := \{y|(x, y) \in E\}.$$

Then except for a set of first category in X, the sets E_x are of first category in Y. Let A be a nowhere dense subset of $X \times Y$. Then except for a set of first category in x, the set E_x is nowhere dense in Y.

The second statement implies the first, since if $E = \bigcup A_i$ then $E_x = \bigcup (A_i)_x$ and the countable union of sets of first category is again of first category. So we need only prove the second assertion.

Proof of the second assertion. Let $O := X \times Y - \overline{A}$ so O is a dense open set of $X \times Y$. Let $\{V_n\}$ be a countable base for the open sets of Y and define $G_n \subset X$ by

$$G_n := \{x|(x, y) \in O \ \text{ for some } y \in V_n\}.$$

Suppose that $x \in G_n$ and that $y \in V_n$ is such that $(x, y) \in O$. Then there is an open set of the form $U \times V \subset O$ where $U \subset X$ is an open set containing x and V is an open set containing y with $V \subset V_n$. Thus $U \subset G_n$. In other words, G_n is an open subset of X. On the other hand, if W is an open subset of X, the set $O \cap W \times V_n$ is non-empty, since $W \times V_n$ is open in $X \times Y$ and O is dense. In other words, G_n is a dense open subset of X for each n. Hence the complement of $\bigcap G_n$ is a set of first category in X. But for any $x \in \bigcap G_n$, the set O_x contains points of V_n for every n. Hence G_x is dense. We have already seen that it is open. Its complement is thus nowhere dense. QED

We are going to apply the Kuratowski-Ulam theorem to the situation where $X = \mathcal{C}(\mathbf{T})$ is the space of continuous functions on the circle with the uniform norm $\|\cdot\|_\infty$ and $Y = \mathbf{T}$ to conclude that

5 Quasi all continuous functions on T have the property that their Fourier series diverge quasi-everywhere.

First some preliminaries: For any Riemann integrable function f on $\mathbf{T}$ we let

$$S_n(f, t) = \sum_{-n}^{n} a_k e^{ikt}, \quad a_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-int} dt.$$

In other words, S_n is the symmetric sum of the Fourier series of f from $-n$ to n. We will prove that the set of $t \in \mathbf{T}$ for which $\overline{\lim}_{n\to\infty} |S_n(f, t)| = \infty$ contains an intersection of countably many dense open sets for quasi all $f \in \mathcal{C}(\mathbf{T})$.

5.1 The Dirichlet kernel.

5. Show that

$$S_n(f,t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t-x)D_n(x)dx$$

where

$$D_n(x) := \frac{\sin(n+\frac{1}{2})t}{\sin\frac{t}{2}}.$$

[Hint: This is really just summing a geometric series.]

5.2 The Fourier kernel.

6. Show that

$$g(t) := \frac{1}{\sin t} - \frac{1}{t}, \qquad t \neq 0$$

extends to be a function which is continuous for all t if we set $g(0) := 0$. Let

$$S_n^*(f,t) := \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t-x)F_n(x)dx$$

where

$$F_n(x) := \frac{\sin(n+\frac{1}{2})x}{\frac{x}{2}}.$$

Show that for any continuous function f on $\mathbf{T}$

$$S_n(f,t) - S_n^*(f,t) \to 0$$

as $n \to \infty$ (uniformly in t for fixed f).

Let $G_0(M,n) \subset \mathcal{C}(\mathbf{T}) \times \mathbf{T}$ be defined by

$$G_0(M,n) := \{(f,t) |\ |S_n^*(f,t)| > M\}.$$

This is clearly an open set as are

$$G(M,N) := \bigcup_{n \geq N} G_0(M,n).$$

By Problem **6** we know that $\overline{\lim}|S_n(f,t)| = \infty$ on $\bigcap_{M,N} G_{M,N}$. To conclude the argument via the Kuratowski-Ulam theorem we must prove that

5.3 Each $G_{M,N}$ is dense in $\mathcal{C}(\mathbf{T}) \times \mathbf{T}$.

Let

$$g_n(x) := \mathbf{1}_{[0,\pi]} \sin(n+\frac{1}{2})x.$$

Then

$$S_n^*(g_n, 0) = \frac{1}{2\pi}\int_0^\pi \frac{sin^2(n+\frac{1}{2})x}{x/2}dx = 2\frac{1}{2\pi}\int_0^{(n+\frac{1}{2})\pi} \frac{sin^2 u}{u}du = \lambda_n \sim \log n.$$

Given any $(f, t_0) \in \mathcal{C}(\mathbf{T}) \times \mathbf{T}$ and any constant $h > M/\lambda_n$, at least one of the pairs $(f(\cdot + t_0) \pm hg_n, 0)$ belongs to $G_0(M, n)$. Hence one of the pairs $(f \pm hg_n(t - t_0), t_0)$ belongs to $G_0(M, n)$ and hence in $G(M, N)$ if $n > N$. But by choosing n large we can M/λ_n as small as we like, amll enough so that $(f \pm hg(t - t_0, t_0)$ is in an arbitrary given neighborhhod of $(f, t_0$ in $\mathcal{C}(\mathbf{T}) \times \mathbf{T}$. In other words, $G(M, N)$ is dense in $\mathcal{C}(\mathbf{T}) \times \mathbf{T}$. QED

The above argument was taken from the article “Baire's Category Theorem” by J. P. Kahane in *Journal d'Analyse Mathématique* **80** 2000, pp 143-182 where a lot of other beatiful applications of the Baire category theorem circle of ideas are given.

The result derived above should be contrasted with a famous result of Carleson which says that for any $f \in L_2(\mathbf{T})$ its Fourier series converges almost everywhere to $f(t)$. In view of Problem **4**, these results are not contradictory.