Math 212a - Problem set 4

Convexity, arbitrage, and probability.

Thursday, October 21, 2004, Due Oct. 28

Contents

1 Farkas' lemma.

1.1 Caratheodory's theorem.

There are several variants of this theorem. Here is the one we will use:

Theorem 1 [Caratheodory.] *Let $\mathbf{a}_1, \ldots, \mathbf{a}_r$ be vectors in a real vector space V. Let $\mathbf{x}$ be a linear combination of the $\mathbf{a}_i$ with non-negative coefficients. Then $\mathbf{x}$ is a linear combination with non-negative coefficients of a linearly independent subset of the $\mathbf{a}_i$.*

1. Prove this theorem. [Hint: Write

$$\mathbf{x} = \sum_{i=1}^{k} \lambda_i \mathbf{x}_i, \quad \lambda_i > 0$$

where the $\mathbf{x}_i$ range over a subset of cardinality k of the $\mathbf{a}_i$ and we have chosen k minimal with this property. If $k = 0$ then $\mathbf{x} = 0$ and there is nothing to prove.

Otherwise, we wish to show that the $\mathbf{x}_i$ are linearly independent. Suppose not. Then there exist $\alpha_i \in \mathbb{R}$ not all zero with

$$\sum_i \alpha_i \mathbf{x}_i = 0.$$

We may assume that at least one of the α_i is positive. (Otherwise multiply them all by -1.) So $\alpha_i > 0$ for at least one i, and choose m so that λ_m/α_m is minimal among the positive α_i. Show that then we can write x as a combination with non-negative coefficients of the x_i, $i \neq m$.]

The set of all linear combinations of the $\mathbf{a}_i$ with non-negative coefficients is called the **cone** generated by the $\mathbf{a}_i$.

Corollary 2 *The cone generated by a finite number of vectors in a real (topological) vector space is closed.*

Proof. By Caratheodory's theorem, this cone is the union of the cones generated by the linearly independent subsets, and there are finitely many of these. So it is enough to prove the corollary under the additional assumption that the vectors in question are linearly independent. The subspace spanned by these vectors is closed (being finite dimensional) and we may identify this subspace with $\mathbb{R}^k$ with the vectors identified with the standard basis. If we do this, the cone becomes the closed (first) orthant consisting of all vectors with non-negative coordinates. $\square$

Since the cone generated by a finite number of vectors is convex, the corollary implies that the cone generated by a finite number of vectors is a closed convex set. So if A is an $m \times n$ matrix, then the set

$$\{Ax\} \quad x \in \mathbb{R}^n, \quad x \geq 0$$

is a closed convex subset of $\mathbb{R}^m$. Here we are using the notation $x \geq 0$ for $x \in \mathbb{R}^n$ to denote the assertion that all the coordinates of x are non-negative.

1.2   The separation theorem for closed convex bodies.

Let S be a closed convex non-empty subset of a real Banach space V. Let $\mathbf{x} \in V$, $\mathbf{x} \notin S$. Then the separation theorem asserts that

Theorem 3 [Separation.] *There exists a continuous linear function ℓ on V and a real number c such that $\ell(\mathbf{y}) \geq c$ for all $\mathbf{y} \in S$ and $\ell(\mathbf{x}) < c$.*

We will only need this theorem for finite dimensional vector spaces so we will only prove it in this case. We will therefore take $V = \mathbb{R}^n$ with its standard metric, and we will write vectors as x, y etc.

Lemma 1 *For any $x \in V$ there is a unique point $p(x, S) \in S$ which is closest to x, i.e. such that*

$$\|x - p(x, S)\| \leq \|x - y\| \quad \forall \; y \in S.$$

Indeed, if $B(x, r)$ denotes the closed ball centered at x then $B(x, r) \cap S$ is compact and is non-empty for large enough r. The function $y \mapsto \|x - y\|$ is continuous, and hence attains a minimum on such a non-empty $B(x, r) \cap S$, say at y_0, and clearly

$$\|x - y_0\| \leq \|x - y\| \ \ \forall \, y \in S.$$

We must show that y_0 is unique. Suppose that there is also a $y_1 \in S$ with

$$\|x - y_1\| \leq \|x - y\| \ \ \forall \, y \in S.$$

Then x, y_0, y_1 form an isosceles triangle with vertex at x and hence if we take $z = \frac{1}{2}(y_0 + y_1)$ to be the midpoint of the base then $z \in S$ since S is convex and

$$\|x - z\| < \|x - y_0\|$$

unless $y_1 = y_0$. $\quad \square$

Now suppose that $x \notin S$ so $p(x, S) \neq x$ so that we may form the unit vector

$$u(x, S) := \frac{1}{d(x, S)}(x - p(x, S))$$

where $d(x, S)$ denotes the distance from x to S so $d(x, S) = \|x - p(x, S)\|$. Let H denote the hyperplane through $p(x, S)$ orthogonal to $u(x, S)$. So H is defined by

$$H = \{z | u(x, S) \cdot (x - z) = u(x, S) \cdot (x - p(x, S)) = d(x, S).$$

The hyperplane H is a **support** hyperplane in the sense that $y = p(x, S) \in S \cap H$ and S lies entirely in one of the half spaces defined by H. Indeed, let H^- denote the closed halfspace bounded by H which does not contain x. We claim that S is completely contained in this halfspace. Indeed, suppose that there is some $y \in S$ not in H^-. Consider the line segment $[p(x.S), y]$ and let z be the point of this line segment closest to x. Then $z \in S$ and $\|x - z\| < \|x - p(x, S)\|$ contradicting the definition of $p(x, S)$. This shows that H is a support hyperplane and completes the proof of Theorem 3. $\quad \square$

Notice that a support hyperplane for a cone is a codimension one subspace (passing through the origin).We will actually need the existence of these support hyperplanes.

1.3 Farkas' lemma.

This says that for A an $n \times m$ matrix

$$(\exists x \in \mathbb{R}^m, x \geq 0, Ax = b) \Leftrightarrow (\forall \, y \in \mathbb{R}^n \ (y^\dagger A \geq 0 \Rightarrow y^\dagger b \geq 0)). \qquad (1)$$

2. Prove Farkas' lemma. [Hint: Use the result of the preceding section about support hyperplanes for cones.]

The Farkas Lemma is sometimes formulated as an alternative:

Theorem 4 [The Farkas Lemma.] *Either $\exists x \geq 0$ with $Ax = b$ or $\exists y \in \mathbb{R}^n$ with $y^\dagger A \geq 0$ and $y^\dagger b < 0$ but not both.*

2 De Finetti's arbitrage theorem.

Suppose that there is an experiment having m possible outcomes for which there are n possible wagers. That is, if you bet the amount y on wager i you win the amount $yr_i(j)$ if the outcome of the experiment is j. Here y can be positive, zero, or negative. A **betting strategy** is a vector $\mathbf{y} = (y_1, \ldots, y_n)^\dagger$ which means that you simultaneously bet the amount y_i on wager i for $i = 1, \ldots, n$. So if the outcome of the experiment is j, your gain (or loss) from the strategy $\mathbf{y}$ is

$$\sum_{i=1}^{n} y_i r_i(j).$$

Theorem 5 [De Finetti's arbitrage theorem.] *Exactly one of the following is true: Either*

- *there exists a probability vector* $\mathbf{p} = (p_1, \ldots, p_m)$ *so* $p_j \geq 0 \ \forall j, \ \sum_j p_j = 1$ *such that*

$$\sum_{j=1}^{m} p_j r_i(j) = 0 \ \ \forall \ i = 1, \ldots, m$$

or

- *there exists a betting strategy* $\mathbf{y}$ *such that*

$$\sum_{i=1}^{n} y_i r_i(j) > 0 \ \ \forall \ j = 1, \ldots m.$$

In other words either there exists a probability distribution on the outcome under which all bets have expected gain equal to zero, or else there is a betting strategy which always results in a positive win.

3. Prove De Finetti's theorem. [Hint: Consider the matrix

$$A := \begin{pmatrix} r_1(1) & \cdots & r_1(m) \\ \vdots & \cdots & \vdots \\ r_n(1) & \cdots & r_n(m) \\ -1 & \cdots & -1 \end{pmatrix}$$

and vector

$$b := \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ -1 \end{pmatrix}.$$

Use Farkas.]

It is important to observe that De Finetti's theorem does *not* say that $\mathbf{p}$ is unique. But there are special circumstances in which uniqueness is obvious.

2.1 The price of a one period European call option.

Suppose that $m = 2$ and that there is exactly one wager. So the matrix A is given by

$$A = \begin{pmatrix} r(1) & r(2) \\ -1 & -1 \end{pmatrix}.$$

If $r(1) \neq r(2)$ this matrix is non-singular with inverse

$$A^{-1} = \frac{1}{r(2) - r(1)} \begin{pmatrix} -1 & -r(2) \\ 1 & r(1) \end{pmatrix}$$

so

$$A^{-1}b = \frac{1}{r(2) - r(1)} \begin{pmatrix} -1 & -r(2) \\ 1 & r(1) \end{pmatrix} \begin{pmatrix} 0 \\ -1 \end{pmatrix} = \frac{1}{r(2) - r(1)} \begin{pmatrix} r(2) \\ -r(1) \end{pmatrix}.$$

So if $r(2) > 0$ and $r(1) < 0$ both entries are positive and yield the unique probability vector

$$\mathbf{p} = \begin{pmatrix} 1-p \\ p \end{pmatrix} = \frac{1}{r(2) - r(1)} \begin{pmatrix} r(2) \\ -r(1) \end{pmatrix}.$$

Of course, if both $r(1)$ and $r(2)$ are positive any wager which assigns a positive bet to both is a guaranteed win, and if both are negative then any wager which assigns a negative bet to both is a guaranteed win.

Now suppose that another wager is allowed. Suppose this bet has the return a if 1 occurs and the return b if 2 occurs. Then according to De Finetti's theorem, unless

$$a(1 - p) + bp = 0,$$

there will some combination of the two wagers that has a guaranteed profit.

As an illustration, suppose that an asset (say a stock) has value $S(0)$ at the present time, and has only two possible values at time 1: Either

$$S(1) = uS(0) \quad \text{or} \quad S(1) = dS(0) \quad u > 1 > d.$$

In other words, either the stock can go up by a factor of u or down by a factor of d. Suppose also that if money is kept in the bank for this period it increases by a factor of $1 + r$. So the current value of M future dollars is $M(1 + r)^{-1}$.

So

$$r(2) = \frac{u}{1 + r} - 1, \quad r(1) = \frac{d}{1 + r} - 1$$

and

$$1 - p = \frac{u - 1 - r}{u - d},$$

$$p = \frac{1 + r - d}{u - d}. \tag{2}$$

Notice that these two values, p and $1 - p$ have nothing to do directly with any intuitive idea of how "probable" the stock is to go up or down. Of course, the

current market price will be influenced by what people believe the stock will do, so there is an indirect relation between p and intuitive probability. This is De Finetti's "market based" approach to the foundations of probability.

A **European call option** is the right to buy a number of shares of stock at time 1 at a specified **strike price** K. Let C be the (current) price of the call option. Let

$$K = kS(0).$$

Thus if the stock goes up by a factor of u then the gain per unit purchased is

$$\frac{u-k}{1+r} - C$$

since you can by the stock at time 1 for a price of $kS(0)$ and sell it immediately at the price $uS(0)$ and the option costs you C dollars today. If the stock goes down by a factor of d you lose C dollars. (I am assuming that $d \le k \le u$.) So unless

$$0 = -(1-p)C + p\left(\frac{u-k}{1+r} - C\right) = p \cdot \frac{u-k}{1+r} - C = \frac{1+r-d}{u-d} \cdot \frac{u-k}{1+r} - C$$

De Finetti's theorem guarantees the existence of a mixed strategy of buying or selling the stock and buying or selling the option with a sure profit. Since a fundamental law of economics says that "there is no free lunch" we must have

$$C = \frac{1+r-d}{u-d} \cdot \frac{u-k}{1+r}. \tag{3}$$

This is the "fair price" of the option in the sense that if the option were priced differently, an arbitrageur could make a guaranteed profit.

2.2 Odds.

In some situations the only type of wagers allowed are ones that choose one of the outcomes $i = 1, \ldots m$ and then bet that i is the outcome. An example: a horse race where each bet is on a single horse and one can only bet that horse wins or does not win. The return on such a bet is usually quoted in term of **odds**. If the odds against outcome i are o_i to 1 then a one unit bet will return o_i if i occurs and -1 if i is not the outcome.

4. Show that unless

$$\sum_{i=1}^{m} \frac{1}{1+o_i} = 1$$

you can make a sure profit at the race track. Remember that at my race track you can bet a positive or negative amount on any horse to win or any combination of such bets, but there is no bet on a horse to place or to show.

5. Suppose that there are three horses and the odds are 1,2, and 3. By the previous problem a strategy exists for a sure win. Find such a strategy.

2.3 A multiperiod stock model.

Suppose that there are n consecutive periods and the bank interest rate is r per period. $S(0)$ denotes the initial price of the stock and $S(i)$ its price at time i. Suppose that $S(i)$ is either $uS(i-1)$ or $dS(i-1)$ where

$$d < 1 + r < u.$$

Stock may be purchased or sold at any one of the times $i = 0, 1, \ldots, n$. Let $X_i = 1$ if the price goes up by the factor u from time $i - 1$ to time i and $X_i = 0$ if it goes down. The succession of stock prices can be regarded as the outcome of an experiment whose possible values are given by the vector $X = (X_1, \ldots, X_n)$. According to De Finetti's theorem, there must be probabilities on these outcomes which make all bets fair. Otherwise there is a way of making a sure profit.

6. Show that to satisfy the De Finetti no sure profit condition, the X_i must be independent and have probability p for $X_1 = 1$ and $1 - p$ for $X_i = 0$ where p is given by (2). [HInt: Consider the following bet: Choose i and a specific vector $(x_1, \ldots, x_{i-1})$ of zeros and ones. Observe the stock market. If $X_j = x_j$ for every $j = 1, \ldots, i - 1$ buy a unit of stock at time $i - 1$ and sell it at time i.]

So

$$W = \sum_{i=1}^{n} X_i$$

is binomial with parameters n and p and so

$$S(n) = u^W d^{n-W} S(0). \tag{4}$$

Now suppose that you could also buy or sell an option to buy stock at time n at a price equal to $kS(0)$. The present value of the option is given by the random variable

$$(1 + r)^{-n} S(0) (u^W d^{n-W} - k)^+$$

per unit stock and so the fair price of the option per unit stock is

$$C = (1 + r)^{-n} E\left[(u^W d^{n-W} - k)^+ \right] S(0). \tag{5}$$

[I am using the notation f^+ to denote the function $x \mapsto \max(f(x), 0)$.

3 The Black-Scholes formula.

Suppose that in (4) and in (5) we take n large and p small in such a way that the central limit theorem applies. Also, replace $(1 + r)^n$ by e^{-rt} corresponding to a continuously compounded interest rate.

So the ratio of the stock price after t units of time to the current price will be given by
$$Y = e^X$$
where X is normal with mean μ and variance σ^2. Of course the central limit theorem holds under much more general hypotheses, so this model of the stock price ratio is more general than just taking the limit of (4).

We begin with some facts about Y.

3.1 The mean and variance of $Y = e^X$, X normal.

We claim that
$$E(Y) = e^{\mu + \frac{1}{2}\sigma^2}, \tag{6}$$
and
$$\operatorname{Var}(Y) = e^{2\mu + \sigma^2}\left(e^{\sigma^2} - 1\right). \tag{7}$$

7. Prove these formulas, first for $\mu = 0$ and then for the general case.

3.2 Determining the drift from De Finetti.

According to De Finetti, if A is any asset, whose value at time $t = 0$ is the number $V(A, 0)$ and whose value at time t is the random variable $V(A, t)$, and if r is the continuously compounded interest rate, then
$$V(A, 0) = e^{-rt}E(V(A, t)) \tag{8}$$

A universe where this holds is called a **risk neutral** universe. (Perhaps it should be called a De Finetti universe but I am following standard usage.)

Now suppose that
$$V(A, t) = V(A, 0)e^{\sigma t^{\frac{1}{2}}N + \nu t} \tag{9}$$

where N is the unit normal.

Lemma 2 *If V satisfies (9) then*
$$r = \nu + \frac{1}{2}\sigma^2. \tag{10}$$

8. Prove the lemma.

3.3 Black-Scholes

Let $\mathcal{E}$ denote the Gauss error function so
$$\mathcal{E}(t) := \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{t} e^{-x^2/2}dx.$$

Let $s(t) = V(A, t)$ as above and let $S = s(0)$. Let C be the value of a European call option with strike price K and time to expiration t. (**European** means that the call can only be exercised at the expiration date.) Let r be the continuously compounded bank (risk free) rate. The Black-Scholes equation asserts that

$$C = S\mathcal{E}\left(\frac{\log\frac{S}{K} + (r + \frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}}\right) - e^{-rt}K\mathcal{E}\left(\frac{\log\frac{S}{K} + (r - \frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}}\right). \qquad (11)$$

Proof. We know that

$$s(t) = Se^{\sigma t^{\frac{1}{2}}N + \nu t}$$

and that the value at time t is

$$\max(s(t) - K, 0).$$

So

$$C = e^{-rt}E(\max(s(t) - K, 0)).$$

Now

$$E(\max(s(t) - K, 0)) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \max(Se^{\sigma t^{\frac{1}{2}}y + \nu t} - K, 0)e^{-y^2/2}dy.$$

The maximum in the integrand equals 0 if

$$\sigma t^{\frac{1}{2}}y + \nu t \leq \log\frac{K}{S}$$

i.e. if

$$y \leq \frac{\log\frac{K}{S} - \nu t}{\sigma\sqrt{t}}.$$

So the expectation in question is

$$\frac{1}{\sqrt{2\pi}}\int_{\frac{\log\frac{K}{S} - \nu t}{\sigma\sqrt{t}}}^{\infty} Se^{\sigma t^{\frac{1}{2}}y + \nu t}e^{-y^2/2}dy - \frac{1}{\sqrt{2\pi}}\int_{\frac{\log\frac{K}{S} - \nu t}{\sigma\sqrt{t}}}^{\infty} Ke^{-y^2/2}dy \qquad (12)$$

The second integral is

$$K\left(1 - \mathcal{E}\left(\frac{\log\frac{K}{S} - \nu t}{\sigma\sqrt{t}}\right)\right) = K\mathcal{E}\left(-\frac{\log\frac{K}{S} - \nu t}{\sigma\sqrt{t}}\right) = K\mathcal{E}\left(\frac{\log\frac{S}{K} + \nu t}{\sigma\sqrt{t}}\right).$$

This accounts for the second term in (11). We can write the exponential in the first integral as

$$\sigma t^{\frac{1}{2}}y + \nu t - y^2/2 = -(y - \sigma t^{\frac{1}{2}})^2/2 + (\nu + \frac{1}{2}\sigma^2)t$$

so the first integral in (12) becomes

$$\frac{1}{\sqrt{2\pi}}\int_{\frac{\log\frac{K}{S} - \nu t}{\sigma\sqrt{t}}}^{\infty} Se^{\sigma t^{\frac{1}{2}}y + \nu t}e^{-y^2/2}dy = Se^{(\nu + \frac{1}{2}\sigma^2)t}\frac{1}{\sqrt{2\pi}}\int_{\frac{\log\frac{K}{S} - \nu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}}^{\infty} e^{-z^2/2}dz$$

as we make the change of variables $z = y - \sigma t^{\frac{1}{2}}$. The integral on the right of this equation evaluates as

$$\mathcal{E}\left(-\frac{\log\frac{K}{S} - \nu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right) = \mathcal{E}\left(\frac{\log\frac{S}{K} + \nu t}{\sigma\sqrt{t}} + \sigma\sqrt{t}\right)$$

$$\mathcal{E}\left(\log\frac{\frac{S}{K} + (\nu + \frac{1}{2}\sigma^2 + \frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}}\right) = \mathcal{E}\left(\log\frac{\frac{S}{K} + (r + \frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}}\right).$$

We must multiply this last expression by

$$e^{-rt}Se^{(\nu + \frac{1}{2}\sigma^2)t} = S.$$

This accounts for the first term in (11). $\quad\square$

For a fascinating account of how the Black Scholes equation, or rather the use of this equation together with a certain amount of hubris by its practitioners almost led to the collapse of the American financial system in September 1998 see the book *When Genius Failed* by Roger Lowenstein.