Problem set 6

Math 212a

November 4, 2004, Due Nov 11

In this problem set we present some standard counter-examples in the theory of integration and then return to the study of generalized functions.

Contents

1   Three standard counterexamples.

1.1   Non-dominated convergence.

Let $f_n := n\mathbf{1}_{(0,\frac{1}{n}]}$. This is similar to one of the examples we studied in class where $\int_{\mathbf{R}} f_n dx \equiv 1$ while $\lim f_n(x) = 0$ for all x. So we get strict inequality in Fatou's lemma, and can not move the limit past the integral sign. So we know in advance that this can not be a case of dominated convergence.

1. Let g be defined by $g(x) := \sup_k f_k(x)$. We know that $g \notin \mathcal{L}_1$. But compute g and show explicitly that its integral diverges.

1.2 Order of integration matters for non-integrable functions.

2. Take $X = Y = [0, 1]$ with standard Lebesgue measure. Let

$$f(x, y) = \begin{cases} \frac{1}{x^2} & \text{if } 0 < y < x < 1 \\ -\frac{1}{y^2} & \text{if } 0 < x < y < 1 \\ 0 & \text{otherwise} \end{cases} .$$

Compute $\int_X \left(\int_Y f(x, y) dy \right) dx$ and $\int_Y \left(\int_X f(x, y) dx \right) dy$ and verify that they are not equal. This implies that $f \notin \mathcal{L}_1(X \times Y, \mathbf{R})$ for otherwise Fubini would apply. Show directly that both f^+ and f^- have infinite integrals on $X \times Y$.

1.3 Order of integration matters in the non-σ-finite case.

Let $X = Y = [0, 1]$ as before. Take $\mathcal{F}$ to be the usual Lebesgue measurable sets and m to be Lebesgue measure. Take $\mathcal{G}$ to be "all" subsets, and take n to be

$$n(A) = \text{number of elements in } A$$

(in particular $n(A) = \infty$ if A is not a finite set). Since $[0, 1]$ is not the countable union of finite sets, it is not σ-finite. So we expect that Fubini might fail. Let $\Delta \subset X \times Y$ be the diagonal, i.e.

$$\Delta := \{(x, y) | x = y\}.$$

3. Show that $\Delta \in \mathcal{F} \times \mathcal{G}$. Let

$$f := \mathbf{1}_\Delta.$$

Evaluate $\int_X \left(\int_Y f(x, y) dy \right) dx$ and $\int_Y \left(\int_X f(x, y) dx \right) dy$ and verify that they are not equal.

2 The spaces $\mathcal{D}$ and $\mathcal{D}'$.

In the previous problem set we defined a generalized function to be a continuous linear function on the Schwartz space $\mathcal{S}$, and denoted the space of generalized functions by $\mathcal{S}'$. For various reasons (especially when we want to extend the theory to manifolds) it is convenient to study other spaces of "test functions" and "generalized functions".

Let $\mathcal{D}$ denote the space of infinitely differentiable functions each of which vanishes outside a compact set K (which may depend on the function). A sequence $f_n \in \mathcal{D}$ is said to converge to an $f \in \mathcal{D}$ if there is a fixed compact set K such that $\text{supp } f_n \subset K$ for all n, and then such that the f_n converge to

f uniformly, together with uniform convergence of all the derivatives. Then $\mathcal{D}'$ consists of all linear functions which are continuous relative to this notion of convergence.

We should really write $\mathcal{D}(\mathbf{R})$ if we are thinking of functions of one real variable, but we could equally well consider functions of n variables, in which case we would write $\mathcal{D}(\mathbf{R}^n)$ or write $\mathcal{D}(V)$ where V is any finite dimensional vector space.

For example, if Δ denotes the Laplacian on $\mathbf{R}^3$ (or $\mathbf{R}^n$), then one way of writing Green's theorem is

$$\int_G f\Delta\phi dx = \int_G (\Delta f)\phi dx + \int_{\partial G}\left(f\frac{\partial\phi}{\partial n} - \frac{\partial f}{\partial n}\phi\right)dS \qquad (1)$$

where f and ϕ are smooth functions, G is a region with smooth (or piecewise smooth) boundary ∂G, where dS denotes the surface measure on the boundary and $\partial/\partial n$ denotes normal derivative. If we take both f and ϕ to be in $\mathcal{D}$, and take G so large that both $\operatorname{supp} f$ and $\operatorname{supp}\phi$ lie in the interior of G, so that there are no boundary terms, then (1) becomes

$$\langle\Delta f,\phi\rangle = \langle f,\Delta\phi\rangle. \qquad (2)$$

For a generalized function f we take this as the *definition* of Δf. In other words, Δf is *defined* to be that generalized function given by

$$\langle\Delta f,\phi\rangle := \langle f,\Delta\phi\rangle \ \ \forall\phi\in\mathcal{D}. \qquad (3)$$

Let us illustrate this definition when f is *not* an element of $\mathcal{D}$: Let g be a smooth function, and let G be a domain with smooth boundary as in (1). Let

$$f = \mathbf{1}_G g.$$

As a *function,* f is equal to g inside G and to be zero outside G, and so the definition of f as a generalized function is

$$\langle f,\psi\rangle := \int_G g\psi dx, \quad \forall\ \psi\in\mathcal{D}.$$

Then this definition and (1) say that Δf is the sum of the function $(\Delta g)\mathbf{1}_G$ plus two other terms supported on the boundary:

$$\langle\Delta f,\phi\rangle = \int_G (\Delta g)\phi dx + \int_{\partial G} g\frac{\partial\phi}{\partial n}dS - \int_{\partial G}\frac{\partial g}{\partial n}\phi dS. \qquad (4)$$

In classical electrostatics these terms supported on the boundary are thought of as a distribution of electric charges and dipole moments on the boundary. For this reason Schwartz used the term "distribution" to denote an element of $\mathcal{D}'$.

In three dimensions, consider the function $1/r$ (where r is the distance from the origin). Since the function $1/r$ is integrable in three dimensions at the origin,

it defines an element of $\mathcal{S}'$ and of $\mathcal{D}'$ and so we can use (3) to define $\Delta(1/r)$. The *function* $1/r$ is smooth away from the origin, and (by direct computation) satisfies $\Delta(1/r) = 0$ there as a *function*. But $1/r$ is *not* a smooth function on all of $\mathbf{R}^3$. So we must use (3).

4. Show that

$$\Delta\left(\frac{1}{r}\right) = -4\pi\delta.$$

[Hint: Apply (1) to the region consisting of $\epsilon \leq r \leq R$ where R is chosen so large that $\operatorname{supp}\phi$ is contained in the open ball of radius R so that there are no terms coming from the outer boundary $r = R$. Compute the terms coming from the inner boundary and let $\epsilon \to 0$.]

3 Cauchy principal value.

Back to one dimension temporarily. The function $x \mapsto 1/x$ is not locally integrable in one dimension, but we do have $\log(|x|)' = 1/x$ at all $x \neq 0$ and $\log|x|$ is locally integrable and so defines a generalized function. We may therefore try to define a generalized function by taking the derivative of $\log|x|$ in the sense of generalized functions.

5. Show that $(\log|x|)' = \operatorname{pv}\left(\frac{1}{x}\right)$ where

$$\left\langle \operatorname{pv}\left(\frac{1}{x}\right), \phi\right\rangle = \int_0^\infty \frac{\phi(x) - \phi(-x)}{x}dx.$$

The symbol "pv" stands for principal value, a notion that was introduced by Cauchy. Also compute the second derivative of $\log|x|$.

4 Anti-derivatives of elements of $\mathcal{D}'$.

We want to study the differential equation

$$g' = f. \tag{5}$$

Here f is an element of $\mathcal{D}'$ and we look for a $g \in \mathcal{D}'$ which satisfies this equation (and differentiation is in the sense of generalized functions). We first examine the simplest case where $f = 0$:

6. Show that if $U \in \mathcal{D}'$ has derivative zero, then U is a constant, i.e. show that $\langle U, \phi\rangle = C\int_{\mathbf{R}}\phi dx$. [Hint: let $\mathcal{D}_0$ denote the set of all $\psi \in \mathcal{D}$ such that $\int_{\mathbf{R}}\psi dx = 0$. First show that U vanishes on $\mathcal{D}_0$. Then choose an element $\theta \in \mathcal{D}$ with $\int_{\mathbf{R}}\theta dx = 1$. Write every $\phi \in \mathcal{D}$ as $\phi = \psi + a\theta$ where $a = \int_{\mathbf{R}}\phi dx$.]

We can use a similar idea to show that a solution of (5) exists for a general $f \in \mathcal{D}'$. Indeed, (5) says that

$$-\langle g, \phi' \rangle = \langle f, \phi \rangle \quad \forall\ \phi \in \mathcal{D}. \tag{6}$$

This defines g on the set of all $\psi \in \mathcal{D}$ which are derivatives (in the usual sense) of elements of $\mathcal{D}$. But these are the elements of $\mathcal{D}_0$. So we wish to extend g (which is defined by (6) on $\mathcal{D}_0$) so as to be defined on all of $\mathcal{D}$. Choose θ as in Problem **6** and write a general $\phi \in \mathcal{D}$ as

$$\phi = a\theta + \phi_0, \quad a = \int_{\mathbf{R}} \phi dx.$$

The map

$$\phi \mapsto \phi_0$$

is a continuous linear map, call it π, from $\mathcal{D}$ to $\mathcal{D}_0$ which is the identity map when restricted to $\mathcal{D}_0$. So we can think of it as a "projection" from $\mathcal{D}$ to $\mathcal{D}_0$. (Of course, it depends on the choice of θ so we should (but won't) write π_θ.) Then define

$$\langle g, \phi \rangle := \langle g, \pi(\phi) \rangle = \langle g, \phi_0 \rangle.$$

Now g is defined on all of $\mathcal{D}$ and, by construction, (6) is satisfied. Problem **6** now implies that the solution g of (5) is unique up to "adding a constant".

7. Use the preceding discussion to show the following: Suppose that g is a continuous function whose derivative in the sense of generalized functions is a continuous function. I.e. assume that $-\int_{\mathbf{R}} g\phi' dx =: \langle g', \phi \rangle = \int_{\mathbf{R}} f\phi dx =: \langle f, \phi \rangle$ where f is a continuous function. Show that g is in fact continuously differentiable as a function and that $g' = f$ in the usual sense.

5 The tensor product of two generalized functions.

Let X and Y be two finite dimensional vector spaces over the real numbers, so that $X \times Y = X \oplus Y$ is again a finite dimensional vector space over the real number. We will write the typical point of $X \times Y$ as (x, y). We can consider the space $\mathcal{D}(X \times Y)$ which consists of all infinitely differentiable functions of compact of compact support on $X \times Y$. We can also consider the space $\mathcal{D}(X) \otimes \mathcal{D}(Y)$ which consists of all finite linear combinations of expressions of the form $\phi\psi$ where $\phi \in \mathcal{D}(X)$ and $\psi \in \mathcal{D}(Y)$. Any such expression defines a function on $X \times Y$ by the rule

$$(\phi\psi)(x, y) = \phi(x)\psi(y),$$

so we have an injection

$$\mathcal{D}(X) \otimes \mathcal{D}(Y) \mapsto \mathcal{D}(X \times Y)$$

which allows us to think of $\mathcal{D}(X) \otimes \mathcal{D}(Y)$ as a subspace of $\mathcal{D}(X \times Y)$. The Stone-Weierstrass theorem, or the original Weierstrass approximation theorem guarantees that $\mathcal{D}(X) \otimes \mathcal{D}(Y)$ is dense in $\mathcal{D}(X \times Y)$.

Now let $f \in \mathcal{D}'(X)$ and $g \in \mathcal{D}'(Y)$ be generalized functions. Let $\phi \in \mathcal{D}(X \times Y)$. For each fixed x, the function $\phi(x, \cdot) : y \mapsto \phi(x, y)$ belongs to $\mathcal{D}(Y)$ and so we can apply g to it to obtain the function $x \mapsto \langle g, \phi(x, \cdot) \rangle$. This is a function of x, and the continuity properties of g imply that this function belongs to $\mathcal{D}(X)$. We then may apply f to the function $x \mapsto \langle g, \phi(x, \cdot) \rangle$ to obtain a number. The notation is getting cumbersome, so we will shorten it and write the final result as

$$\langle f, \langle g, \phi \rangle \rangle.$$

We then define $f \otimes g$ to be this generalized function. In other words we define $f \otimes g \in \mathcal{D}'(X \times Y)$ by

$$\langle f \otimes g, \phi \rangle = \langle f, \langle g, \phi \rangle \rangle. \tag{7}$$

If $\phi = \tau \otimes \eta$ where $\tau \in \mathcal{D}(X)$ and $\eta \in \mathcal{D}(Y)$ then it is clear from the definition that

$$\langle f \otimes g, \tau \eta \rangle = \langle f, \tau \rangle \langle g, \eta \rangle.$$

This shows that on function of the form $\tau \eta$ it would not have made any difference in the definition of $f \otimes g$ had we done things in the reverse order, i.e. first apply f to the function $x \mapsto \phi(x, y)$ and then apply g to the resulting function of y. But since $\mathcal{D}(X) \otimes \mathcal{D}(Y)$ is dense in $\mathcal{D}(X \times Y)$ it follows that doing things in the reverse order yields the same answer on all of $\mathcal{D}(X \times Y)$. This is a sort of "generalized function version" of Fubini's theorem.

Similarly, if we have three vector spaces X, Y and Z and $f \in \mathcal{D}(X)$, $g \in \mathcal{D}(Y)$, $h \in \mathcal{D}(Z)$ we can form $f \otimes (g \otimes h)$ and $(f \otimes g) \otimes h$ and verify that they give the same element of $\mathcal{D}(X \times Y \times Z)$.

It is easy to check directly from the definition that

$$\mathrm{supp}(f \otimes g) = \mathrm{supp}(f) \times \mathrm{supp}(g)$$

as a subset of $X \times Y$.

Suppose that $X = Y = Z$, and to fix the ideas (and hopefully to get the powers of 2π right, although I am not all that optimistic) that they all equal $\mathbf{R}$. If f and g were elements of L_1 we defined their convolution $f \star g$ as

$$(f \star g)(u) = \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} f(x)g(u-x)ds.$$

If we think of this as a generalized function and apply it to ϕ we get

$$\frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} \int_{\mathbf{R}} f(x)g(u-x)dx\phi(u)du = \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} \int_{\mathbf{R}} f(x)g(y)\phi(x+y)dxdy.$$

Therefore we would like to define the convolution $f \star g$ of any two elements of $\mathcal{D}'(\mathbf{R})$ by

$$\langle f \star g, \phi \rangle = \frac{1}{\sqrt{2\pi}} \langle f \otimes g, \phi(x+y) \rangle \tag{8}$$

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Here $\phi \in \mathcal{D}(\mathbf{R})$ and $\phi(x + y)$ denotes the function of two variables given by $(x, y) \mapsto \phi(x + y)$. The trouble is that $\phi(x + y)$ does not have compact support as a function of two variables. Indeed it is constant on any "anti-diagonal" line $x + y = a$. So the supp $\phi(x + y)$ is the anti-diagonal strip consisting of all (x, y) such that $x + y \in \text{supp}(\phi)$. So we can not use (8) in general. But by the remarks of the last section of the preceding exercise set, we can use this definition if we know that $\text{supp}(f) \times \text{supp}(g)$ intersects every anti-diagonal strip (of bounded width) in a compact set.

This will happen, for example, if

- Either f of g has compact support. For then $\text{supp}(f \otimes g)$ will be a horizontal or a vertical strip, which then meets any anti-diagonal strip of bounded width in a compact set. Or

- $\text{supp}(f) \subset [a, \infty)$ and $\text{supp}(g) \subset [b, \infty)$ for then $\text{supp}(f \otimes g)$ is contain in the (infinite) rectangle $[a, \infty) \times [b, \infty)$ which also intersects any bounded anti-diagonal strip in a compact set.

As an illustration of the first case, consider what happens if we take $f = \delta$. Then

$$\langle \delta \star g, \phi \rangle = \frac{1}{\sqrt{2\pi}} \langle \delta \otimes g, \phi(x + y) \rangle = \frac{1}{\sqrt{2\pi}} \langle g, \langle \delta, \phi(x + y) \rangle \rangle = \frac{1}{\sqrt{2\pi}} \langle g, \phi \rangle.$$

In other words,

$$(\sqrt{2\pi} \delta) \star g = g$$

for any $g \in \mathcal{D}'(\mathbf{R})$. Convolution with the element $\sqrt{2\pi} \delta$ is the identity operator.

Similarly, we can define the convolution of two generalized function in n dimensions, the unfortunate factor $\frac{1}{\sqrt{2\pi}}$ (which was determined by our conventions for the Fourier transform) being replaced by the factor $\frac{1}{(2\pi)^{n/2}}$. The second sufficient condition for the definition of the convolution is replaced by the condition that $\text{supp}(f)$ and $\text{supp}(g)$ are both contained in the same type of "orthant", for example in the "first quadrant" (or any translate thereof) in two variables.

Let D be any differential operator with constant coefficients. Define D^* to be the formal adjoint of D obtained by integration by parts and ignoring the boundary terms. So, for example, if D is a homogeneous differential operator of order k, then $D^* = (-1)^k D$. For instance $\Delta^* = \Delta$ for the Laplacian. The operator D is then defined on generalized functions by

$$\langle Df, \phi \rangle := \langle f, D^* \phi \rangle.$$

8. Show that
$$D(f \star g) = (Df) \star g = f \star Dg$$
whenever $f \star g$ is defined. In particular, in three dimensions, if f is any generalized function then
$$u = \left(\frac{c}{r} \right) \star f$$

is a solution to

$$\Delta u = f$$

where (I hope) $c = -\sqrt{\frac{\pi}{2}}$.