Brownian Motion from Within

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Background: Given a family of $\mathbb{R}$-valued random variables $\{B(t) : t \geq 0\}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, we will say that $\{B(t) : t \geq 0\}$ is a *Brownian motion* if

(**a**) $B(0, \omega) = 0$ and $t \rightsquigarrow B(t, \omega)$ is continuous for each $\omega \in \Omega$.

(**b**) For all $(s, t) \in [0, \infty)^2$, $B(s + t) - B(s) \in N(0, t)$ is a centered Gaussian with mean value 0 and variance t which is independent of the sigma algebra $\sigma\big(\{B(\tau) : \tau \in [0, s]\}\big)$. Equivalently, for all $n \geq 1$ and $0 = t_0 < t_1 < \cdots t_n$, the $\mathbb{R}^n$ valued random variable $\big(B(t_1)-B(t_0), \dots, B(t_n)-B(t_{n-1})\big)$ is a centered Gaussian whose covariance is the diagonal matrix whose mth diagonal entry is $t_m - t_{m-1}$.

Lemma 1. *If $\{B(t) : t \geq 0\}$ is a family of $\mathbb{R}$-valued random variables which satisfies (**a**), then $\{B(t) : t \geq 0\}$ is a Brownian motion if and only if, for each compactly supported function $f : [0, \infty) \longrightarrow \mathbb{R}$ having bounded variation, the $\mathbb{R}$-valued random variable I_f given by the Riemann–Stieltjes integral*[1]

$$I_f(\omega) \equiv \int_0^\infty f(t)\, dB(t, \omega)$$

is a centered Gaussian with variance $\|f\|^2_{L^2([0,\infty))}$. Hence, if $\{B(t) : t \geq 0\}$ is a Brownian motion and $\{f_m\}_1^n$ is a family of compactly supported functions of bounded variation, then $\big(I_{f_1}, \dots, I_{f_n}\big)$ is a centered, $\mathbb{R}^n$-valued Gaussian with covariance

$$\big(\big((f_k, f_\ell)_{L^2([0,\infty))}\big)\big)_{1 \leq k, \ell \leq n}.$$

In particular, when $\{B(t) : t \geq 0\}$ is a Brownian motion, I_g will be independent of $\sigma\big(\{I_{f_1}, \dots, I_{f_n}\}\big)$ if and only if $(g, f_m)_{L^2([0,\infty))} = 0$ for each $1 \leq m \leq n$.

Proof. First suppose that $\{B(t) : t \geq 0\}$ is a Brownian motion. Given f, Riemann-Stietjes integrability implies that

$$I_f(\omega) = \lim_{N \to \infty} \sum_{m=1}^\infty f\left(\tfrac{m}{N}\right)\Big(B\big(\tfrac{m+1}{N}, \omega\big) - B\big(\tfrac{m}{N}, \omega\big)\Big).$$

Hence, since for each N, $\sum_{m=1}^\infty f\left(\frac{m}{N}\right)\Big(B\big(\frac{m+1}{N}, \omega\big) - B\big(\frac{m}{N}\big)\Big)$ is a centered Gaussian whose variance is $\frac{1}{N}\sum_{m=1}^\infty f\left(\frac{m}{N}\right)^2$, I_f is a centered Gaussian with variance $\|f\|^2_{L^2([0,\infty))}$. To prove the opposite implication, let $0 = t_0 < t_1 < \cdots < t_n$ and $\alpha_1, \dots, \alpha_n \in \mathbb{R}$ be given, set $f(t) = \sum_{m=1}^n \alpha_m \mathbf{1}_{[t_{m-1}, t_m)}(t)$, and conclude that

$$\mathbb{E}^{\mathbb{P}}\left[\exp\left(\sum_{m=1} \sqrt{-1}\alpha_m\big(B(t_m) - B(t_{m-1})\big)\right)\right] = \mathbb{E}^{\mathbb{P}}\big[e^{\sqrt{-1}I_f}\big] = \exp\left(-\frac{1}{2}\sum_{m=1}\alpha_m^2(t_m - t_{m-1})\right),$$

which shows that $\{B(t) : t \geq 0\}$ satisfies (**b**).

Again assume that $\{B(t) : t \geq 0\}$ is a Brownian motion. To verify the assertion about $(I_{f_1}, \dots, I_{f_n})$, for each $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n$ use $f = \sum_{m=1}^n \alpha_m f_m$ to check that

$$\mathbb{E}^{\mathbb{P}}\left[\exp\left(\sum_{m=1}^n \sqrt{-1}\alpha_m I_{f_m}\right)\right] = \exp\big(-\tfrac{1}{2}\big(\boldsymbol{\alpha}, A\boldsymbol{\alpha}\big)_{\mathbb{R}^n}\big),$$

where $A = \big(\big((f_k, f_\ell)_{L^2([0,\infty))}\big)\big)_{1 \leq k\ell \leq n}$. Finally, to see that I_g is independent of $\sigma\big(\{I_{f_1}, \dots, I_{f_n}\}\big)$ when $f \perp \{f_1, \dots, f_n\}$ in $L^2\big([0, \infty)\big)$, note that the first row and column of the covariance matrix for $\big(I_g, I_{f_1}, \dots, I_{f_n}\big)$ will have 0 off-diagonal entries. □

[1]Recall that φ is Riemann–Stieltjes integrable with respect to ψ if and only if ψ is with respect to φ and that any continuous function is Riemann–Stieltjes integrable with respect to any compactly supported function of bounded variation.

Standard Gaussian Measure on a Hilbert Space: Given a separable, real Hilbert space $\mathbf{H}$, the Borel a probability measure μ is said to be the *standard Gaussian measure* on $\mathbf{H}$ and is denoted by $\mu_{\mathbf{H}}$ if its *characteristic function*

$$g \in \mathbf{H} \longmapsto \hat{\mu}(g) \equiv \int_{\mathbf{H}} e^{\sqrt{-1}(g,h)_H}\, \mu(dh) \in \mathbb{C}$$

is $g \rightsquigarrow \exp\big(-\frac{1}{2}\|g\|_{\mathbf{H}}^2\big)$. When $\mathbf{H}$ is finite dimensional and $\mathbf{H}$ is identified with $\mathbb{R}^{\dim(\mathbf{H})}$, then

$$\mu(dh) = \big((2\pi)^{\dim(\mathbf{H})}\det(A)\big)^{-\frac{1}{2}} e^{-\frac{1}{2}\|h\|_{\mathbf{H}}^2}\, dh,$$

where $A = \big(((e_k, e_\ell)_{\mathbf{H}})\big)_{1\le k,\ell\le n}$ and $\big(e_1, \dots, e_{\dim(\mathbf{H})}\big)$ is the standard Euclidean orthonormal basis for $\mathbb{R}^{\dim(\mathbf{H})}$. When $\mathbf{H}$ is infinite dimensional, $\mu_{\mathbf{H}}$ *cannot exist*. To see this, suppose it does, choose an orthonormal basis $\{g_m : m \ge 0\}$ in $\mathbf{H}$, and conclude that

$$\int_{\mathbf{H}} e^{-\|\mathbf{h}\|_{\mathbf{H}}^2}\, \mu_{\mathbf{H}}(dh) = \int_{\mathbf{H}} \exp\left(-\sum_{m=0}^{\infty}(g_m, h)_{\mathbf{H}}^2\right) \mu_{\mathbf{H}}(dh) = \prod_{m=0}^{\infty}\int_{\mathbf{H}} e^{-(g_m,h)_{\mathbf{H}}^2}\, \mu_{\mathbf{H}}(dh) = 0$$

since the random variables $\{X_m :\ge 0\}$ given by $X_m(h) = (g_m, h)_{\mathbf{H}}$ are mutually independent, standard centered Gaussian random variables with variance 1 and therefore

$$\int_{\mathbf{H}} e^{-(g_m,h)_{\mathbf{H}}^2}\, \mu_{\mathbf{H}}(dh) = (2\pi)^{-\frac{1}{2}} \int_{\mathbb{R}} e^{-\frac{3}{2}x^2}\, dx = 3^{-\frac{1}{2}}.$$

Next, consider the Hilbert space $\mathbf{H}_0^1$ of continuous $h : [0,\infty) \longrightarrow \mathbb{R}$ with $h(0) = 0$ and one derivative in $L^2\big([0,\infty);\mathbb{R}\big)$. That is, $h \in \mathbf{H}_0^1$ if and only if $h(t) = \int_0^t \dot{h}(\tau)\, d\tau$, $t \ge 0$ for some $\dot{h} \in L^2\big([0,\infty);\mathbb{R}\big)$. Next turn $\mathbf{H}_0^1$ into a separable Hilbert space by taking $(g,h)_{\mathbf{H}_0^1} = (\dot{g},\dot{h})_{L^2([0,\infty))}$. In spite of the preceding, suppose that $\mu_{\mathbf{H}_0^1}$ existed, and come to the conclusion that $\{h(t) : t \ge 0\}$ is a Brownian motion on the probability space $\big(\mathbf{H}_0^1, \mathcal{B}_{\mathbf{H}_0^1}, \mu_{\mathbf{H}_0^1}\big)$. To understand this, simply observe that $(g,h)_{\mathbf{H}_0^1} = \int_0^\infty \dot{g}(t)\, dh(t)$ when the derivative $\dot{g}$ of $g \in \mathbf{H}_0^1$ is compactly supported and has bounded variation, and so Lemma 1 says that $\{h(t) : t \ge 0\}$ would be a Brownian motion. Thus, if all this held together, the distribution of Brownian motion would be the distribution under $\mu_{\mathbf{H}_0^1}$ of

$$h \in \mathbf{H}_0^1 \longmapsto h = \sum_{m=0}^{\infty} X_m(h) g_m \in \mathbf{H}_0^1,$$

where $X_m(h) = (g_m, h)_{\mathbf{H}_0^1}$.

Construction of Brownian Motion: On the basis of the preceding, one might be tempted to turn the argument around and see whether one can use it to construct Brownian motion, albeit not on $\mathbf{H}_0^1$. To be precise, let $\{g_m : m \ge 0\}$ be an orthonormal basis in $\mathbf{H}_0^1$, let $\{X_m : m \ge 0\}$ be a sequence of mutually independent, centered Gaussian $\mathbb{R}$-valued random variables with variance 1 on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and attempt to define $\{B(t) : t \ge 0\}$ by $B = \sum_{m=0}^{\infty} X_m g_m$. Although we know that this series cannot be converging in $\mathbf{H}_0^1$, it may very well converge in some less stringent topology: for example, the one induced by uniform convergence on compact intervals. Because it simplifies the presentation, we will restrict our attention to Brownian motion on the time interval $[0,1]$. That is, we will construct $\{B(t) : t \in [0,1]\}$ so that (**a**) and (**b**) hold when the time parameter is restricted to $[0,1]$, and our construction will relie on our making a clever choice of orthonormal basis $\{g_m : m \ge 0\}$ in $\mathbf{H}_0^1\big([0,1];\mathbb{R}\big) \equiv \{h \restriction [0,1] : h \in \mathbf{H}_0^1\}$ such that, at least along some subsequence, $\sum_{m=0}^{\infty} X_m g_m$ $\mathbb{P}$-almost surely converges uniformly on $[0,1]$

The ease with which one can carry out the program just outlined depends on the care with which one chooses the orthornormal basis $\{g_m : m \ge 0\}$ in $\mathbf{H}_0^1$. As any reader of Norbert Wiener will attest, his choice was far from the best. Namely, as a harmonic analyst, he chose to take $\{\dot{g}_m : m \ge 0\}$ to be the orthornormal basis in $L^2([0,1];\mathbb{R})$ given by $\dot g_0 \equiv 1$ and $g_m(t) = \sqrt{2}\cos m\pi t$. Hence, Wiener[2] was left with the problem of showing that, along the dyadic subsequence $\{2^N : N \ge 0\}$, $\sum_{m=1}^\infty X_m \frac{\sin m\pi t}{m\pi}$ $\mathbb{P}$-almost surely converges uniformly with respect to $t \in [0,1]$. Both his own and his readers lives would have been greatly simplified had he instead chosen the $\{g_m : m \ge 0\}$ to be the Haar basis.[3] That is, take $\dot g_0 = 1$ and, for $N \ge 0$ and $0 \le \ell < 2^N$, take $\dot g_{2^N+\ell} = 2^{-\frac{N}{2}}\psi(2^N t - \ell)$, where $\psi(t) = 0$ when $t \notin [0,1)$, $\psi(t) = 1$ when $t \in [0,\frac12)$, and $\psi(t) = -1$ when $t \in [\frac12, 1)$. As is easily verified, $(\dot g_m, \dot g_{m'})_{L^2([0,1]} = \delta_{m,m'}$. In addition, a dimension counting argument shows that $\{g_m : m < 2^k\}$ spans the same subspace of $L^2\big([0,1];\mathbb{R}\big)$ as $\{\mathbf{1}_{J_{m,N}} : 0 \le m < 2^{N+1}\}$, where $J_{m,N} = [m2^{-N-1}, (m+1)2^{-N-1})$. This proves that $\{\dot g_m : m \ge 0\}$ is an orthonormal basis in $L^2\big([0,1];\mathbb{R}\big)$, and so $\{g_m : m \ge 0\}$ is an orthonormal basis in $\mathbf{H}^1_0\big([0,1];\mathbb{R}\big)$. Furthermore, for $2^N \le m < 2^{N+1}$, $\|g_m\|_{\mathrm{u}} = 2^{-N-1}$ and $g_m g_{m'} \equiv 0$ when $2^N \le m \ne m' < 2^{N+1}$. Hence, for any $q \in [1,\infty)$,[4]

$$\left\|\sum_{m=2^N}^{2^{N+1}-1} X_m g_m\right\|_{\mathrm{u}} \le 2^{-\frac{N+1}{2}} \max_{2^N \le m < 2^{N+1}} |X_m| \le 2^{-\frac{N+1}{2}} \left(\sum_{m=2^N}^{2^{N+1}-1} |X_m|^{2q}\right)^{\frac{1}{2q}}.$$

In particular, if $B_N \equiv \sum_{m=0}^{2^N-1} X_m g_m$, then

$$\mathbb{E}^{\mathbb{P}}\big[\|B_{N+1} - B_N\|_{\mathrm{u}}\big] \le 2^{-\frac{N+1}{2}} \mathbb{E}^{\mathbb{P}}\left[\left(\sum_{m=2^N}^{2^{N+1}-1} |X_m|^{2q}\right)^{\frac{1}{2q}}\right] \le 2^{-\frac{N+1}{2}} \mathbb{E}^{\mathbb{P}}\left[\sum_{m=2^N}^{2^{N+1}-1} |X_m|^{2q}\right]^{\frac{1}{2q}} = G_{2q} 2^{-\frac{N}{2q'}},$$

where $q' = \frac{q}{q-1}$ and $G_{2q} \equiv \mathbb{E}^{\mathbb{P}}\big[|X_0|^{2q}\big]^{\frac{2}{q}} < \infty$. Thus, for $q > 1$,

$$\mathbb{E}^{\mathbb{P}}\left[\sup_{N>M} \|B_N - B_M\|_{\mathrm{u}}\right] \le C_q 2^{-\frac{M}{2q'}},$$

where $C_q \equiv \frac{4G_{2q}}{3}$, and starting from here it is an easy matter to conclude that there exists a measurable map $\omega \in \Omega \longmapsto B(\,\cdot\,,\omega) \in C\big([0,1];\mathbb{R}\big)$ such that

$$\mathbb{E}^{\mathbb{P}}\left[\|B - B_N\|_{\mathrm{u}}\right] \le C_q 2^{-\frac{N}{2q'}}. \tag{1}$$

Finally, to verify that $\{B(t) : t \in [0,1]\}$ is a Brownian motion, simply observe that, for any $g \in \mathbf{H}^1_0\big([0,1];\mathbb{R}\big)$ with $\dot g$ having bounded variation, $\omega \rightsquigarrow \big(g, B_N(\,\cdot\,,\omega)\big)_{\mathbf{H}^1_0([0,1])}$ is a centered Gaussian whose variance is $\sum_{m=0}^{2^N-1}(g, g_m)^2_{\mathbf{H}^1_0([0,1])}$, $\big(g, B_N(\,\cdot\,,\omega)\big)_{\mathbf{H}^1_0([0,1])} \longrightarrow \int_0^1 \dot g(t)\,dB(t,\omega)$ $\mathbb{P}$-almost surely, and therefore the random variable $\omega \longmapsto \int_0^1 \dot g(t)\,dB(t,\omega)$ is a centered Gaussian with variance $\|g\|^2_{\mathbf{H}^1_0([0,1])}$.

Remarks: Starting from (1), it is quite easy to show that, for each $\alpha \in (0,\frac12)$,

$$\mathbb{E}^{\mathbb{P}}\left[\sup_{0 \le s < t \le 1} \frac{|B(t) - B(s)|}{(t-s)^\alpha}\right] < \infty.$$

The trick is to use (1) with a q chosen so that $\alpha < \frac{1}{2q'}$.

Returning to the problem of constructing Brownian motion for all time, observe that it is sufficient to construct a sequence $\{B^{(n)} : n \ge\}$ of mutually independent Brownian motion for $[0,1]$ and to glue them together at integer times. That is, take $B(t) = B^0(t)$ for $t \in [0,1]$ and $B(t) = B(n) + B^{(n)}(t-n)$ for $t \in [n, n+1]$. Alternatively, one can contruct the Haar basis $\{g_{m,n} : (m,n) \in \mathbb{N}^2\}$ for $\mathbf{H}^1_0\big([0,\infty);\mathbb{R}\big)$ by taking $g_{m,n}(t) = 0$ if $t \notin [n, n+1)$ and $g_{m,n}(t) = g_m(t-n)$ if $t \in [n, n+1)$.

[2]This is only one of three constructions which Wiener suggests in *Differential Space*, J. Math. Phys. **2**, pp. 131–174 (1923). The extent to which Wiener himself completed any of them is debatable.

[3]Although Z. Cielieski was the first person to use the Haar basis, shortly after Wiener's paper appeared, Paul Lévy gave an appraoch which comes down to the same thing even though he did not phrase it in the terminology used here.

[4]We use $\|\cdot\|_{\mathrm{u}}$ to denote the uniform, or supremum, norm.