Math 212a Lecture 1.

The official title of this course is the

"Theory of functions of a real variable".

I am going to assume that you all know what the real numbers are - that they are the completion of the rational numbers and that they are complete in the sense that any Cauchy sequence of real numbers converges to a limit. In the notes we review the concept of "completion" in the more general context of the theory of metric spaces, but our treatment will depend on the existence of the real number system.

History: irrational numbers.

However a bit of history is in order: The Greeks knew that $\sqrt{2}$ is irrational: this was proved in a scholium to Euclid's "Elements". To the best of my knowledge no such statement was made about π in Greek mathematics. The great Islamic mathematician and astronomer al Biruni (937-1048) states in his al-Qanun al-Mas'udi, 3 vols (Hyderabad, 1954-56),vol 1,Part III, ch 5, p. 303: "... the perimeter of the circle to its diameter is a certain ratio; likewise, its number (meaning: the number of the perimeter) to its number (that of the diameter) is a ratio, which is surd (arabic: samm)...". I am indebted to Prof. Tony Levy for this translation.

The great Jewish jurist and philosopher Maimonides (1135-1205) in his commentary to the Mishnah (Eruvin I-5) also states that π is an irrational number. Neither of these authors offer a proof of this fact. The first correct proof that π is irrational was given by Lambert in 1761.

Transcendental numbers.

The first person to show that there exist numbers which are "transcendental",i.e. not solutions of an algebraic equation with rational coefficients, was Liouville, who showed in 1844 that the number

$$\sum_{n=1}^{\infty} 10^{-n!}$$

is transcendental.

In 1874 Cantor showed that "most" real numbers are transcendental by proving that the algebraic numbers are countable and that the real numbers are uncountable.

Hermite proved in 1873 that e is transcendental, and Lindemann proved in 1884 that π is transcendental using the methods developed by Hermite.

Definition.

It is therefore somewhat surprising that it was not until 1872 that a *definition* was given of the real numbers. In fact two definitions, one via "Dedekind cuts" given by Dedekind and one by Cantor via equivalence clases of Cauchy sequences of rational numbers.

What is the definition of a function?

The meaning of the word "function" however will not be settled in this course. Here is an example of the kind of trouble we will have to face: The first person to give a correct proof of the convergence of Fourier series to the function it should represent (for a reasonable class of functions) was Dirichlet in 1829. In the course of his analysis of what Fourier series mean, he introduced the so-called "function" $\mathbf{1}_{\mathbb{Q}}$ which takes on the value 1 on the rational numbers and the value 0 in the irrational numbers.

Is $1_{\mathbb{Q}}$ a function?

It can not be evaluated or graphed on any computer since computers use floating point arithmetic and see only rational numbers. From the point of view of a computer $\mathbf{1}_{\mathbb{Q}}$ is identically one.

From the point of view of Lebesgue integration theory which we will study later on this semester, two functions which agree except on a set of "measure zero" are to be regarded as the same. The rational numbers have measure zero in the Lebesgue theory. So from the point of view of Lebesgue's theory $\mathbf{1}_{\mathbb{Q}}$ is identically zero.

Can we actually apply the definition? For example, Euler introduced the number γ defined as the limit as $n \to \infty$ of $\sum_{j=1}^{n} \frac{1}{j} - \log n$. To this day we do not know if γ is rational or irrational. So what is $\mathbf{1}_{\mathbb{Q}}(\gamma)$?

Conclusion: We should not admit Dirichlet's $\mathbf{1}_{\mathbb{Q}}$ into our menagerie of functions.

How about Dirac's δ-function?

This was defined by Dirac to be the "function" such that $\delta(x) = 0$ for $x \neq 0$ but is "so infinite at 0" that

$$\int_{-\infty}^{\infty} f(x)\delta(x)dx = f(0)$$

for any continuous function f. From the point of view of Lebesgue's (or Riemann's) integration theory, this statement is nonsense: the value of a function at a single point (even if infinite) can not affect the value of an integral.

Yet Dirac's δ function is not only important in physics, we will be making a lot of use of it in this course. It is easily implemented on the computer. It is the rule which assigns to each function f the value $f(0)$. In other words, it is the map

$$f \mapsto f(0).$$

It is not a function of a real variable but rather a function of a function of a real variable.

Since many of the "functions" we will study involve infinite series, we start by looking at some infinite series from the view point of 18th century mathematicians.

18th century sums.

Let $a := 1 - 1 + 1 - \cdots$. Then

$$a = 1 - (1 - 1 + 1 - 1 + \cdots) = 1 - a$$

so $a = \frac{1}{2}$.

Let $s := 1 - 2 + 3 - 4 + 5 - 6 + \cdots$. Then

$$\begin{aligned} s &= 1 + (-2 + 3 - 4 + \cdots) \\ &= 1 - (2 - 3 + 4 - 5 + \cdots) \\ &= 1 - (1 - 1 + 1 - 1 + \cdots) - (1 - 2 + 3 - 4 + \cdots) \\ &= 1 - a - s \end{aligned}$$

so $s = 1 - \frac{1}{2} - s$ or

$$s = \frac{1}{4}.$$

Let $Z := 1 + 2 + 3 + 4 + 5 + \cdots$. Then

$$Z - s = 0 + 4 + 0 + 8 + 0 + 12 + \cdots = 4Z$$

so $3Z = -s$ or

$$Z = -\frac{1}{12}.$$

18th century philosophy.

Euler: "ich glaube, dass jede series einem bestimmten Wert haben müsse. Um aber allen Schwieregkeiten, welche dagegen gemacht worden, zu begegnen, so sollte dieser Wert nicht mit dem Namen der Summe belegt werden, weil man mit dieser Wort gemeiniglich eine solchen Begriff zu verknüpfen pflegt, als wenn die Summe durch eine wirkliche Summierung herausgebracht würde: welche Idee bei den seribus divergentibus nicht stattfindet..."

Hardy *Divergent series*:(p.5) " it does not occur to a modern mathematician that a collection of symbols should have a 'meaning' until one has been assigned to it by definition. [This] was not a triviality even to the greatest mathematicians of the eighteenth century. They had not the habit of definition: it was not natural to them to say, in so many words, 'by X we *mean* Y. ... mathematicians before Cauchy asked not 'How shall we *define* $1 - 1 + 1 - \cdots$' but ' What *is* $1 - 1 + 1 - \cdots$?', and this habit led them into unnecessary perplexities and controversies which were often really verbal."

Leonhard Euler

Born: **15 April 1707 in Basel, Switzerland**
Died: **18 Sept 1783 in St Petersburg, Russia**

Some justifications.

The geometric series

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + z^4 + \cdots$$

converges for $|z| < 1$. The *function* $z \mapsto \frac{1}{1-z}$ exists and is holomorphic for all $z \neq 1$. The value of this function as $z = -1$ is $\frac{1}{2}$. So if we define the **Abel sum** of a series $\sum_n a_n$ to be the limit (if it exists) of

$$\lim_{x \to 1} \sum_n a_n x^n$$

then the Abel sum of $a = 1 - 1 + 1 - 1 + \cdots$ is indeed $\frac{1}{2}$.

Differentiating the above geometric series (for $|z| < 1$) gives

$$\frac{1}{(1-z)^2} = 1 + 2z + 3z^2 + 4z^3 + \cdots .$$

Again the *function* on the left is holomorphic for all $z \neq 1$ and setting $z = -1$ gives

$$s = \frac{1}{4}$$

where this is interpreted as the Abel sum of $1 - 2 + 3 - 4 + \cdots$.

Justification of the formula $1+2+3+4+\cdots=-\frac{1}{12}$.

The justification of the formula $1+2+3+4+\cdots=-\frac{1}{12}$ lies a bit deeper: The series

$$1+\frac{1}{2^s}+\frac{1}{3^s}+\frac{1}{4^s}+\cdots$$

converges for Re $s>1$ and so defines a holomorphic function $\zeta(s)$ which has a singularity at $s=1$. It can be defined (and is holomorphic) for all $s\neq 1$ and

$$\zeta(-1)=-\frac{1}{12}.$$

In fact, Euler verifies this value by means of what is known today as the Euler-MacLaurin summation formula.

Toeplitz's theorem.

Let $M = (m_{nk})$ be an infinite matrix (indexed by the positive integers $\mathbb{N}$) which satisfies the following three conditions:

1. $\lim_{n\to\infty} m_{nk} = 0$ for all k (the terms in any column $\to 0$).
2. There is an H such that $\sum_k |m_{nk}| \leq H$ for every n. (The rows are uniformly absolutely summable.)
3. $\lim_n \sum_k m_{nk} = 1$. (The row sums tend to 1.)

Let $s_n \to s$ be a convergent sequence of real numbers. (For example s_n might be the partial sums of a convergent series.) Since the s_n are bounded, condition 2) implies that for each n the series

$$\sigma_n := \sum m_{n,k} s_n$$

converge for each n.

Toeplitz's theorem asserts that $\sigma_n \to s$.

Proof. Write $s_k - s := r_k$ so $\sigma_n = s\sum_k m_{nk} + \sum_k m_{nk} r_k$. By 3) $s\sum_k m_{nk} \to s$. Since $r_k \to 0$ we are reduced to proving the theorem for the case $s = 0$.

Hypotheses

1. $\lim_{n\to\infty} m_{nk} = 0$ for all k (the terms in any column $\to 0$).
2. There is an H such that $\sum_k |m_{nk}| \le H$ for every n. (The rows are uniformly absolutely summable.)
3. $\lim_n \sum_k m_{nk} = 1$. (The row sums tend to 1.)
4. $s_n \to 0$.

$$\sigma_n := \sum m_{n,k} s_n$$

To prove:

$$\sigma_n \to 0.$$

For any $\epsilon > 0$ choose $N = N(\epsilon)$ so large that $|s_k| < \epsilon/2H$ for $k > N$. By 1) we may then choose $P = P(\epsilon)$ so that

$$|m_{nk}| < \frac{\epsilon}{2\sum_{k=1}^{N} |s_k|} \quad \text{for } k = 1, 2, \ldots, N \quad \text{for } n > P(\epsilon).$$

Then

$$|\sigma_n| \le \sum_{k=1}^{N} |m_{nk} s_k| + \frac{\epsilon}{2H} \sum_{k>N} |m_{nk}| \le \frac{\epsilon}{2} + \frac{\epsilon}{2H} \sum_{1}^{\infty} |m_{nk}| \le \epsilon. \quad \square$$

Cesaro summability

Given any matrix M satisfying the conditions of Toeplitz's theorem, it is possible that the sequence Ms makes sense without s being convergent. That is, it is conceivable that the series $\sigma_n := \sum m_{n,k} s_n$ all converge. It is also possible that the σ_n converge to some limit s. Then we will say that $s_n \to s(M)$. For example, the matrix

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ 1/2 & 1/2 & 0 & 0 & \cdots \\ 1/3 & 1/3 & 1/3 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

clearly satisfies the conditions. Then

$$\sigma_n = \frac{1}{n}(s_1 + \cdots + s_n).$$

If $s_1 = 1,\ s_2 = 0,\ s_3 = 1$, etc. then $\sigma_n \to \frac{1}{2}$. These values of s_n are the partial sums of our series $a = 1 - 1 + 1 - 1 + \cdots$. We say that this series is **Cesaro summable** to $\frac{1}{2}$.

Abel summability

Let $0 \leq r_n < 1$, $r_n \to 1$. Let $a_{nk} := (1-r_n)r_n^k$. (Here we are indexing over the non-negative integers.) Clearly $\lim_{n\to\infty} a_{nk} = 0$ for each fixed k. Also by the geometric series $\sum_k |a_{nk}| = \sum_k a_{nk} = 1$. So

$$A = (a_{nk}) = ((1 - r_n)r_n^k)$$

satisfies the hypotheses of Toeplitz's theorem. For any sequence s_n

$$\sigma_n = (1-r_n)\sum_k s_k r_n^k = s_0 - s_1 r_n + s_2 r_n^2 + s_1 r_n^3 + \cdots - s_0 r_n - s_1 r_n^2 - s_2 r_n^3 - \cdots =$$

$$= s_0 + (s_1 - s_0)r_n + (s_2 - s_1)r_n^2 + (s_3 - s_2)r_n^3 + \cdots .$$

So if the s_n are partial sums: $s_0 = c_0$, $s_1 = c_0 + c_1$, $s_2 = c_0 + c_1 + c_2$ etc. of a series, we get

$$\sigma_n = \sum_{k\geq 0} c_k r_n^k.$$

If the σ_n approach a limit we have Abel summability relative to this subsequence. If the limit as $r \to 1$ of $\sum_k c_k r^n$ exists (not merely a subsequence) then we have we have what is known as **Abel summability**. (It is easy to check that Cesaro summability implies Abel summability.)

Some history.

Daniel Bernoulli(1753) proposed a trigonometric series as a solution for the problem of vibrating strings and Euler found the formula for the "Fourier coefficients"

$$f(x) = \sum a_n e^{inx}, \quad a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx$$

in 1757.

The problem of how "heat diffuses" in a continuous medium defied the mathematics of early 1800's. This problem was solved by Fourier using "intuitive methods". He submitted his manuscript *Théorie de la propagation de la chaleur* to the Institut de France on 21 December 1807. The prize committee consisted of Laplace, Lagrange, Lacroix and Monge, with Poisson acting as secretary. They rejected the manuscript. Lagrange was particularly firm, questioning the meaning of the equality sign in

$$f(x) \text{``} = \text{''} \sum a_n e^{inx}, \quad a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx.$$

For example for the "square wave"

$$f(x) := \begin{cases} -1, & -\pi < t < 0, \\ 1, & 0 < t < \pi \end{cases}$$

the series is

$$\frac{4}{\pi} \sum_{n \geq 1}^{\infty} \frac{\sin(2n-1)t}{2n-1}.$$

How could this series converge when the series of its coefficients

$$\frac{4}{\pi}\left(1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots\right)$$

diverges? Poisson summarized the rejection in 5 pages saying essentially that the paper was rejected on the grounds that it contained nothing new or interesting. Fourier submitted a revised version to essentially the same group of judges in 1811 as a candidate for the Grand prix de mathématiques for 1812. Although they awarded him the prize, they refused to publish the paper in the *Mémoires de l'Acadḿie des Sciences*. In 1824 Fourier became the Secretary of the Academy and he then had his 1811 paper published.

Fourier's influence.

In addition to stimulating the development of mathematics and physics for the next two centuries, the work of Fourier changed the notion of a function. Previously a function had been a natural object, such as a polynomial or exponential or trigonometric function. In short, a function was something given by a closed expression. They were like biological species to be discovered and classified. Now a function could be invented and be quite arbitrary. Also, there was some controversy as to the meaning of the definite integral, say as it appears in the formula above for the Fourier coefficients. In the 18th century, an integral was thought of as an anti-derivative. This is the way an operation like "int" is handled in a symbolic manipulation program like Maple today.

The fact of that the area under a curve is the difference of the anti-derivatives at the end-points was a *theorem*, not a definition. Since the anti-derivatives of some very important functions, such as $e^{-x^2/2}$ can not be expressed in terms of elementary known functions, what meaning does an expression such as $\int_a^b e^{-x^2/2}$ have? It was Fourier's important idea to bypass the anti-derivative, and *define* the definite integral as the area under the curve. Thus the definite integral becomes primary, and the indefinite integral or anti-derivative becomes secondary.

Jean Baptiste Joseph Fourier

Born: 21 March 1768 in Auxerre, Bourgogne, France
Died: 16 May 1830 in Paris, France

Dirichlet

Getting back to Fourier series, the problem of establishing their convergence was open. Poisson published an erroneous proof in 1820, and Cauchy published a flawed proof in 1826 - the error pointed out by Dirichlet in 1829 where he, Dirichlet, gave he first correct proof of sufficient conditions for convergence - that a piecewise smooth integrable function converges at every point to the average of its right and left hand limits. This result was extended by Jordan in 1881 to all functions of bounded variation.

Dirichlet also realized the once we extend the notion of a function, there will be certain functions for which "the area under the curve" definition of the definite integral makes no sense, such as his "function" $\mathbf{1}_{\mathbb{Q}}$.

Johann Peter Gustav Lejeune Dirichlet

Born: **13 Feb 1805 in Düren, French Empire (now Germany)**
Died: **5 May 1859 in Göttingen, Hanover (now Germany)**

Riemann

But the "area under the curve" did make sense for continuous functions, or functions which were continuous with only finitely many jumps. This was greatly extended by Riemann in his *Habilitationschrift* entitled *On the representability of a function by trigonometric series* written in 1854, but not published until after his death in 1867. He used his new concept of integration (today known as the Riemann integral) to conclude that the Fourier coefficients of any Riemann integrable function tend to zero as $n \to \pm\infty$. This was generalized by Lebesgue, to what is today known as the Riemann-Lebesgue lemma.

Lebesgue

Riemann's notion of the integral prevailed until the begining of the twentieth century with the arrival of the Lebesgue integral. Lebesgue was also very interested in Fourier series, and the first application his new integral were to this subject. Lebesgue's notion of integral extended to a much wider class of functions - Riemann integrability is equivalent to continuity almost everywhere. There was much more freedom in taking limits under the integral sign. The theory of Lebesgue involves the concept of the "measure" of a set and the associated notion of an integral of a function. Lebesgue's notion of measure was extended by Caratheodory in 1905 to account for the area of k-dimensional subsets of n-dimensional space, where k is an integer, and this was generalized by Hausdorff in 1917 to allow the possibility of k being any real number, thus introducing the notion of "fractional dimension". This line of work was continued by a small group of researchers (mainly Besicovitch and his coworkers) but exploded into the public consciousness by the work of Mandelbrot in the 1980's with the advent of computer graphics. "Fractals" are now part of the everyday language.

Daniel and Stone

The integration aspect of Lebesgue's theory, starting with the idea that an integral is a linear function on a certain class of functions, and that this notion was primary with measure theory secondary was introduced by Daniell in 1911 and brought to perfection by Stone in 1942. In the meanwhile, Hilbert spaces such as the space L^2 of square integrable functions, and more generally the L^p spaces consisting of functions for which $\int |f|^p < \infty, \ p > 1$ were being studied. A key theorem proved by F. Riesz in the 1930's says that L^p is the dual space of L^q where p and q are related by

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Riesz

In other words, an element of L^p can be thought of (and is the most general) bounded linear function on L^q where we use the norm

$$||g||_q := \left(\int |g|^q\right)^{1/q}$$

on L^q. In other words, instead of thinking of an element of L^p as a function (on its domain of defintion, for example the real numbers, so assigning a complex number to every real number), we can think of it as a *functional*, a rule which assigns numbers to functions. For example, an element of L^2 can be thought of as assigning a number to any element of L^2 in a way which is continuous with respect to L^2 convergence. We will exploit this "duality", and the Riesz representation theorem in its various guises with be a key ingredient in this course.

Back to the square wave.

The "square wave" is defined as

$$s(x) := \begin{cases} -1, & -\pi < t < 0, \\ 1, & 0 < t < \pi \end{cases}$$

Its Fourier series can be written as

$$\frac{4}{\pi} \sum_{k \geq 1}^{\infty} \frac{\sin(2k-1)x}{2k-1}.$$

Indeed,

$$a_n = \frac{1}{2\pi} \left[\int_0^{\pi} e^{-inx} dx - \int_{-\pi}^{0} e^{-inx} dx \right] = \frac{1}{2\pi} \frac{2}{-in} \left[e^{-in\pi} - 1 \right].$$

The last expression in brackets vanishes when n is even, and equals -2 if n is odd. So the even terms of the Fourier series vanish, while the sum of the terms involving $e^{\pm inx}$ for odd n is

$$\frac{1}{2\pi} \frac{4}{i|n|} \left[e^{inx} - e^{-inx} \right] = \frac{4}{\pi|n|} \sin |n| x.$$

We let s_n denote the n-th partial sum. Here are the graphs of s_n for some values of n:

Notice that there is blip overshooting the square wave whose distance from the horizontal (of about .18) does not decrease as n increases. Rather the width of the blip appears to go to zero.

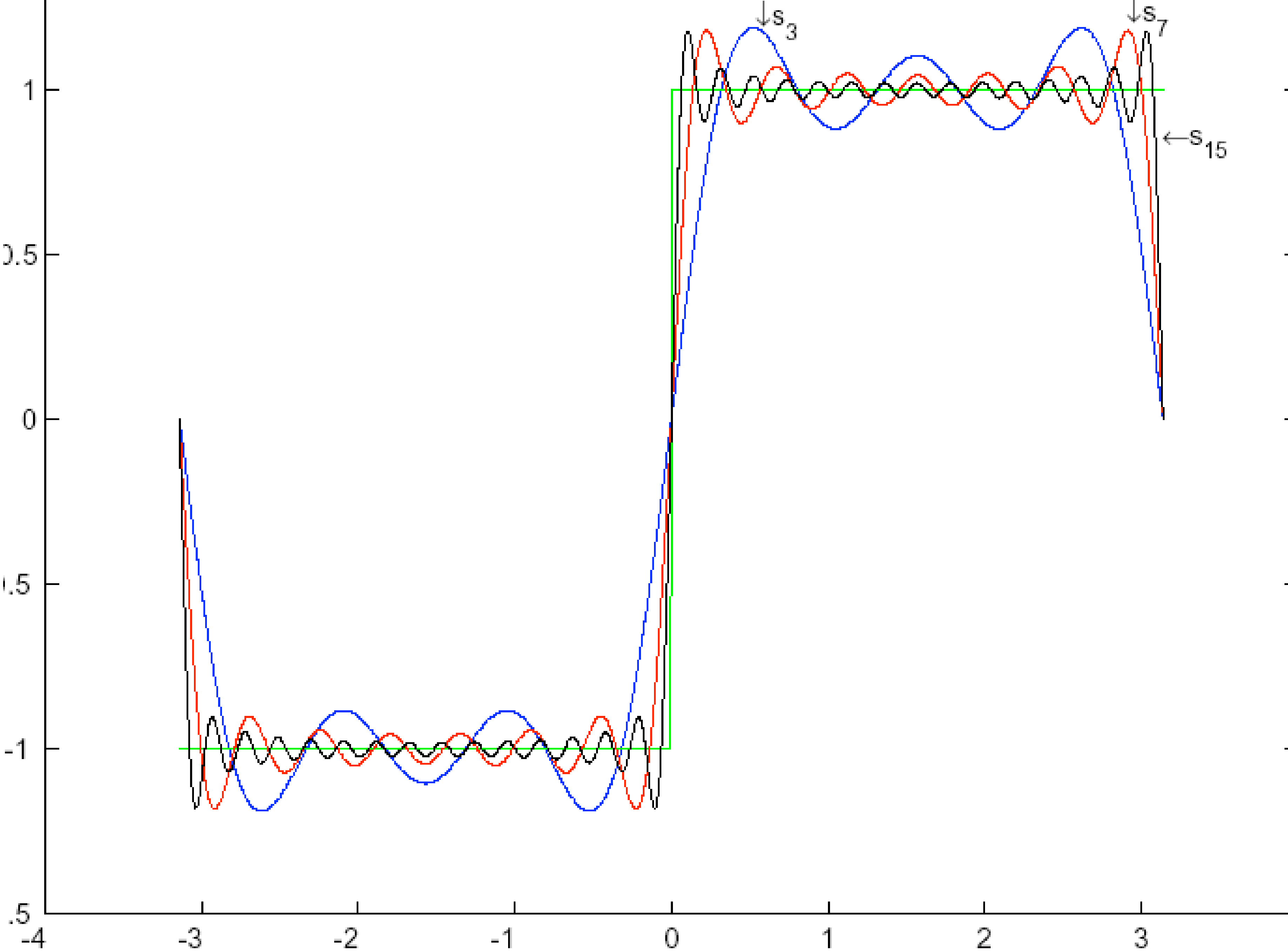


Figure 1: The graphs of s and s_3, s_7 and s_{15} over $[-\pi, \pi]$.

Verification.

Let us verify these facts. To locate the maxima we compute

$$\frac{ds_n}{dx} = \frac{4}{\pi}\sum_{k=1}^{n}\cos(2k-1)x$$

which we can sum via a geometric series (for $x \neq n\pi$) as

$$\frac{4}{\pi}\frac{1}{2}\left(e^{ix}\sum_{0}^{n-1}e^{2kix} + e^{-ix}\sum_{0}^{n-1}e^{-2kix}\right) = \frac{4}{\pi}\frac{1}{2}\left(e^{ix}\frac{1-e^{2inx}}{1-e^{2ix}} + e^{-ix}\frac{1-e^{-2inx}}{1-e^{-2ix}}\right) =$$

$$\frac{2}{\pi}\frac{\sin 2nx}{\sin x}.$$

This extends by continuity to all x. This function vanishes at

$$x = \pm\frac{\pi}{2n}.$$

So these are the extrema nearest to the origin. Differentiating one more time we check that we have maximum to the right and a minimum to the left of the origin. Evaluating s_n at its maximum near the origin gives

$$s_n(\frac{\pi}{2n}) = \frac{4}{\pi}\sum_{k=1}^{n}\frac{\sin[(2k-1)\frac{\pi}{2n}]}{2k-1} = \frac{2}{\pi}\sum\frac{\sin[(2k-1)\frac{\pi}{2n}]}{(2k-1)\frac{\pi}{2n}}\cdot\frac{\pi}{n} \to \frac{2}{\pi}\int_0^{\pi}\frac{\sin t}{t}dt$$

$$s_n\left(\frac{\pi}{2n}\right) = \frac{4}{\pi}\sum_{k=1}^{n} \frac{\sin[(2k-1)\frac{\pi}{2n}]}{2k-1} = \frac{2}{\pi}\sum \frac{\sin[(2k-1)\frac{\pi}{2n}]}{(2k-1)\frac{\pi}{2n}} \cdot \frac{\pi}{n} \to \frac{2}{\pi}\int_0^{\pi} \frac{\sin t}{t} dt$$

as a Riemann integral. Evaluating of the integral gives approximately 1.18. Indeed, we have the power series expansion

$$\frac{\sin t}{t} = \sum_{r=0}^{\infty} \frac{(-1)^r t^{2r}}{(2r+1)!}$$

which is valid for all t and we may integrate term by term to get

$$\int_0^{\pi} \frac{\sin t}{t} dt = \pi \left(1 - \frac{\pi^2}{3!3} + \frac{\pi^4}{5!5} - \frac{\pi^6}{7!7} + \cdots \right).$$

As the series is an oscillating series with decreasing terms, the error involved in truncating the series is at most the first term neglected, and stopping at an odd term overshoots the mark while stopping at an even term undershoots the mark. Summing the first five terms gives 1.179384 while adding the next term gives 1.178957.

So this overshoot never disappears. This is the **Gibbs phenomenon**, Gibbs, 1899, first discovered by Wilbraham in 1848 and proved as a general phenomenon - that the overshoot at the jump is about 9% of the total jump by Bocher in 1906.

It appears from the figure that we have uniform convergence to the square wave as long as we stay a postive distance away from the jump. This is an illustration of Dirichlet's theorem. We also know (or will learn) that the square wave belongs to L_2 and that its Fourier series converges to it in the L_2 norm. So the Gibbs phenomenon shows something about the subtlety of L_2 convergence.

In the next two slides I will reproduce the original Gibbs paper which appeared in NATURE on April 27, 1899.

Gibbs worked with the zigzag function rather than the square wave function.

Fourier's Series.

I SHOULD like to correct a careless error which I made (NATURE, December 29, 1898) in describing the limiting form of the family of curves represented by the equation

$$y = 2\left(\sin x - \tfrac{1}{2}\sin 2x \;.\;.\;.\; \pm \frac{1}{n}\sin nx\right) \;.\;.\;. \quad (1)$$

as a zigzag line consisting of alternate inclined and vertical portions. The inclined portions were correctly given, but the vertical portions, which are bisected by the axis of X, extend beyond the points where they meet the inclined portions, their total lengths being expressed by four times the definite integral

$$\int_0^\pi \frac{\sin u}{u} du.$$

If we call this combination of inclined and vertical lines C, and the graph of equation (1) C_n, and if any finite distance d be specified, and we take for n any number greater than $100/d^2$, the distance of every point in C_n from C is less than d, and the distance of every point in C from C_n is also less than d. We may therefore call C the limit (or limiting form) of the sequence of curves of which C_n is the general designation.

But this limiting form of the graphs of the functions expressed by the sum (1) is different from the graph of the function expressed by the limit of that sum. In the latter the vertical portions are wanting, except their middle points.

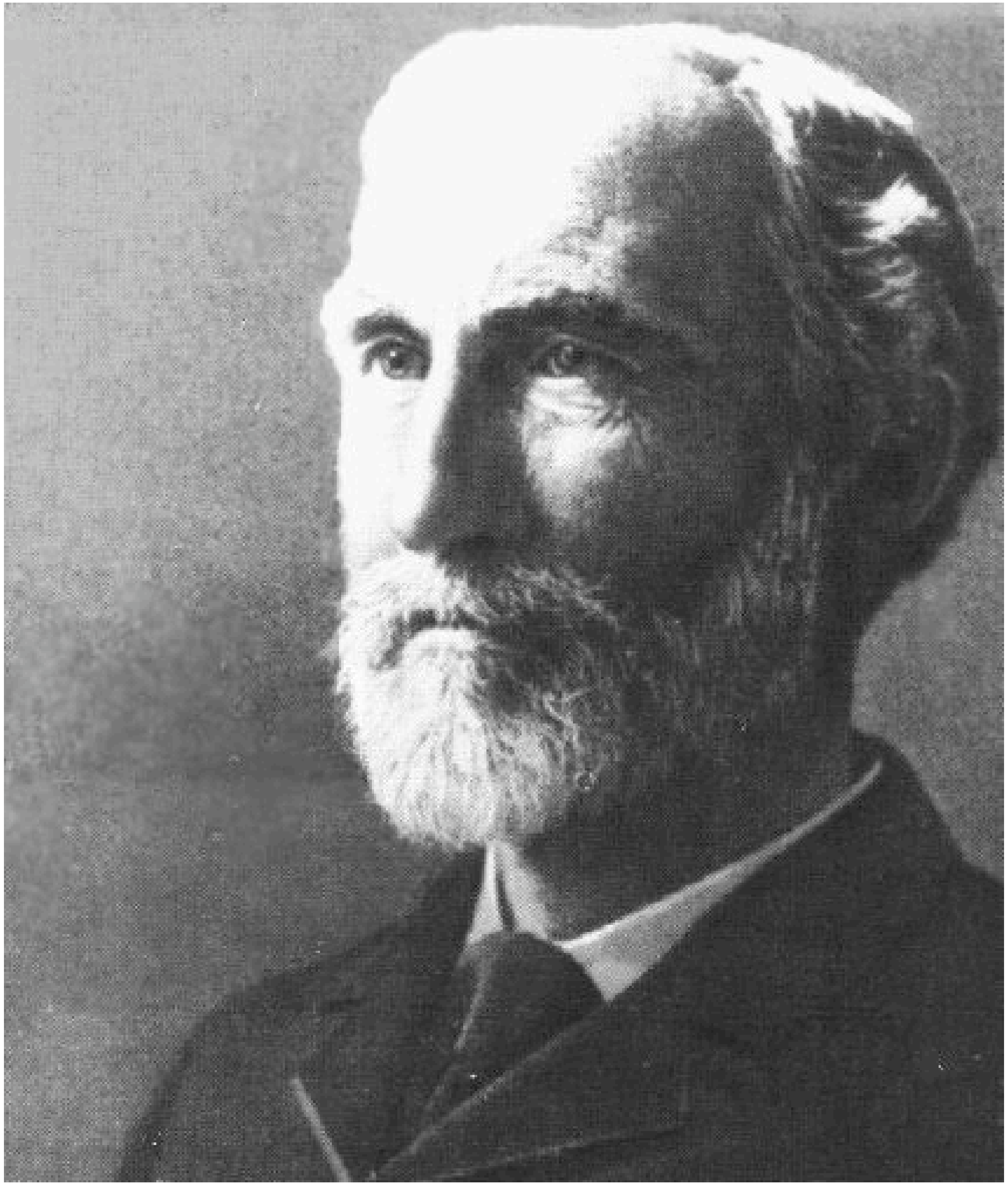
I think this distinction important; for (with exception of what relates to my unfortunate blunder described above), whatever differences of opinion have been expressed on this subject seem due, for the most part, to the fact that some writers have had in mind the *limit of the graphs*, and others the *graph of the limit* of the sum. A misunderstanding on this point is a natural consequence of the usage which allows us to omit the word *limit* in certain connections, as when we speak of the sum of an infinite series. In terms thus abbreviated, either of the things which I have sought to distinguish may be called the graph of the sum of the infinite series.

J. WILLARD GIBBS.

New Haven, April 12.

———

Josiah Willard Gibbs



Born: 11 Feb 1839 in New Haven, Connecticut, USA
Died: 28 April 1903 in New Haven, Connecticut, USA